BROCKETT’S PROBLEM IN THE THEORY OF STABILITY OF LINEAR DIFFERENTIAL EQUATIONS

G. A. LEONOV

ABSTRACT. Algorithms for nonstationary linear stabilization are constructed. Combined with a nonstabilizability criterion, these algorithms result in the solution of the Brockett problem in a number of cases.

§1. INTRODUCTION

In the book [1], R. Brockett formulated the following problem.

For a triplet of matrices \( A, B, \) and \( C \), what conditions ensure the existence of a matrix \( K(t) \) such that the system

\[
\frac{dx}{dt} = Ax + BK(t)Cx, \quad x \in \mathbb{R}^n,
\]

is asymptotically stable.

The problem of stabilizing system (1) with the help of a constant matrix \( K \) is classical for automatic control theory [2, 3]. From this point of view, Brockett’s problem can be reformulated as follows.

To what extent are the possibilities of classical stabilization extended by introducing matrices \( K(t) \) that depend on time \( t \)?

Stabilizing mechanical systems often necessitates the invocation of a special class of stabilizing matrices \( K(t) \). These matrices must be periodic and have zero mean on the period [0, \( T \]):

\[
\int_0^T K(t) \, dt = 0.
\]

For example, consider a linear approximation near an equilibrium point for the pendulum with vertically oscillating suspension point:

\[
\ddot{\theta} + \alpha \dot{\theta} + (K(t) - \omega_0^2) \theta = 0,
\]

where \( \alpha \) and \( \omega_0 \) are positive numbers. Here, the most common choice for the function \( K(t) \) is either \( \beta \sin \omega t \) (see [4]), or

\[
K(t) = \begin{cases} 
\beta, & t \in [0, T/2), \\
-\beta, & t \in [T/2, T)
\end{cases}
\]

(see [5, 6]). For such functions \( K(t) \), the effect of stabilization of the upper equilibrium point is well known for large \( \omega \) and, consequently, small \( T \).

In this paper, we present certain algorithms enabling us to construct periodic piecewise constant functions \( K(t) \) that solve the Brockett problem in a number of cases, and also periodic functions \( K(t) \) that satisfy (2) and solve the stabilization problem. Moreover, we show that low-frequency stabilization (\( T \gg 1 \)) is possible for the pendulum equation (3) with \( K(t) \) of the form (4).

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§2. CONDITIONS SUFFICIENT FOR STABILIZATION

Suppose we have two matrices $K_j$ ($j = 1, 2$) such that the systems

$$\frac{dx}{dt} = (A + BK_j C)x, \quad x \in \mathbb{R}^n,$$

possess stable linear manifolds $L_j$ and invariant linear manifolds $M_j$. We assume that $M_j \cap L_j = \{0\}$ and $\dim M_j + \dim L_j = n$, and that $\lambda_j$, $\kappa_j$, $\alpha_j$, and $\beta_j$ are positive numbers satisfying the inequalities

$$|x(t)| \leq \alpha_j e^{-\lambda_j t}|x(0)|, \quad x(0) \in L_j,$$

$$|x(t)| \leq \beta_j e^{-\kappa_j t}|x(0)|, \quad x(0) \in M_j,$$

In what follows. We also assume that there exists a matrix $U(t)$ and a number $\tau > 0$ such that

$$Y(\tau)M_1 \subset L_2,$$

where $Y(t)$ is the fundamental matrix ($Y(0) = I$) of the following system:

$$\frac{dy}{dt} = (A + BU(t)C)y.$$

**Theorem 1.** Suppose that

$$\lambda_1 \lambda_2 > \kappa_1 \kappa_2$$

and that (8) is true.

Then there exists a periodic matrix $K(t)$ such that system (1) is asymptotically stable.

**Proof.** Condition (10) implies that for every $T > 0$ there exist two numbers $t_1$ and $t_2$ such that

$$-\lambda_1 t_1 + \kappa_2 t_2 < -T,$$

$$-\lambda_2 t_2 + \kappa_1 t_1 < -T.$$

We define the periodic matrix $K(t)$ as follows:

$$K(t) = \begin{cases} K_1, & t \in [0, t_1), \\ U(t - t_1), & t \in [t_1, t_1 + \tau), \\ K_2, & t \in [t_1 + \tau, t_1 + t_2 + \tau). \end{cases}$$

The period of $K(t)$ is equal to $t_1 = t_2 + \tau$. We show that if $T$ is sufficiently large, then system (1) with such matrix $K(t)$ is asymptotically stable. For this, we introduce nonsingular matrices $S_j$ bringing system (5) to a canonical form:

$$\frac{dz_j}{dt} = Q_j z_j, \quad \dim z_j = \dim L_j,$$

$$\frac{dw_j}{dt} = P_j w_j, \quad \dim w_j = \dim M_j,$$
Here

$$S_j x = \begin{pmatrix} z_j \\ w_j \end{pmatrix}. $$

There is no loss of generality in assuming that

$$|z_j(t)| \leq e^{-\lambda_j t}|z_j(0)|, \quad |w_j(t)| \leq e^{-\kappa_j t}|w_j(0)|. $$

Relations (12)–(14) show that

$$\left( z_2(t_1 + \tau) \atop w_2(t_1 + \tau) \right) = S_2 Y(\tau) S_1^{-1} \left( z_1(t_1) \atop w_1(t_1) \right). $$

The inclusion (8) implies that the matrix $S_2 Y(\tau) S_1^{-1}$ has the following structure:

$$S_2 Y(\tau) S_1^{-1} = \begin{pmatrix} R_{11}(\tau) & R_{12}(\tau) \\ R_{21}(\tau) & 0 \end{pmatrix}. $$

Therefore, by (11) and (15),

$$|z_2(t_1 + t_2 + \tau)| \leq |R_{11}(\tau)| e^{-2T}|z_1(0)| + |R_{12}(\tau)| e^{-T}|w_1(0)|, $$

$$|w_2(t_1 + t_2 + \tau)| \leq |R_{21}(\tau)| e^{-T}|z_1(0)|, $$

which implies that for all sufficiently large values of $T$ and for the initial values in the ball $|x(0)| \leq 1$, we have

$$|x(t_1 + t_2 + \tau)| \leq \frac{1}{2}. $$

Since the matrix $K(t)$ is periodic, it follows that system (1) is asymptotically stable. \(\square\)

Now, we assume that the matrix $K(t)$ in (1) is a scalar function,

$$K_1 = K_2 = K_0, \quad \lambda_1 = \lambda_2 = \lambda, \quad \kappa_1 = \kappa_2 = \kappa, \quad U(t) \equiv U_0, \quad K_0 U_0 < 0, $$

the function $|Y(t)|$ is uniformly bounded on the interval $(0, +\infty)$, and there exists a sequence $\tau_j \to \infty$ such that

$$Y(\tau_j) M_1 \subset L_2. $$

**Theorem 2.** If $\lambda > \kappa$ and (17) is fulfilled, then there exists a $T$-periodic function $K(t)$ such that (2) is true and system (1) is asymptotically stable.

*Proof.* We define

$$K(t) = \begin{cases} K_0, & t \in [0, |U_0 \tau_j/2K_0|], \\ U_0, & t \in [|U_0 \tau_j/2K_0|, \tau_j + |U_0 \tau_j/2K_0|], \\ K_0, & t \in [\tau_j + |U_0 \tau_j/2K_0|, \tau_j + |U_0 \tau_j/K_0|] \end{cases}. $$

The period of $K(t)$ is equal to $T = \tau_j(1 + |U_0/K_0|)$.

Here, $\tau_j$ is a sufficiently large number satisfying condition (17). The rest of the proof repeats the arguments used in the proof of Theorem 1. \(\square\)

We apply Theorem 2 to equation (3) with a function $K(t)$ of the form (4).

Suppose that

$$\alpha^2 < 4(\beta - \omega_0^2). $$
In this case, without loss of generality, we may assume that \( \beta - \omega_0^2 - \alpha^2/4 = 1. \)

We set \( K_0 = -\beta \) and \( U_0 = \beta. \) Condition (19) implies that the characteristic polynomial of equation (3) with \( K(t) = U_0 \) has complex zeros, and, consequently, condition (17) is fulfilled for some \( \tau_1 > 0. \) Clearly, here we have \( \tau_j = \tau_1 + 2j\pi. \)

Since the zeros in question of the characteristic polynomial have negative real parts, we easily see that \( |Y(t)| \) is uniformly bounded on \((0, +\infty)\).

For \( K(t) = K_0 = -\beta \), the quantities \( \lambda \) and \( \kappa \) can easily be calculated:

\[
\lambda = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + (\beta + \omega_0^2)}, \\
\kappa = -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + (\beta + \omega_0^2)}.
\]

Thus, all assumptions of Theorem 2 are fulfilled, and equation (3) with \( K(t) \) of the form (18) is asymptotically stable for sufficiently large \( j. \) This result can be stated as follows.

**Proposition 1.** Under condition (19), for every \( \tau \) there exists \( T > \tau \) such that equation (3) with a function \( K(t) \) of the form (4) is asymptotically stable. \( \square \)

In particular, this implies the possibility of stabilizing the upper equilibrium point of the pendulum for low-frequency vertical oscillations of the suspension point. Naturally, here the amplitude \( a \) of the oscillations is large:

\[
a = \frac{lT^2\beta}{8},
\]

where \( l \) is the length of the pendulum and \( \beta \) is the absolute value of the acceleration divided by \( l. \)

The stabilization effect is well known for high-frequency oscillations (for small \( T \)); see [5, 6].

The following lemmas are often useful for checking condition (8).

Consider the system

\[
\dot{z} = Qz, \quad z \in \mathbb{R}^n,
\]

where \( Q \) is a constant nonsingular \((n \times n)\)-matrix and \( h \) is a vector in \( \mathbb{R}^n. \)

**Lemma.** Suppose that the solution \( z(t) \) of system (20) has the form \( z(t) = v(t) + w(t), \) where \( v(t) \) is a periodic vector-valued function such that \( h^*v(t) \neq 0, \) and \( w(t) \) is a vector-valued function for which

\[
\int_0^{+\infty} |w(\tau)|\,d\tau < +\infty, \quad \lim_{t \to +\infty} w(t) = 0.
\]

Then there exist two numbers \( \tau_1 \) and \( \tau_2 \) such that

\[
h^*z(\tau_1) > 0 \quad \text{and} \quad h^*z(\tau_2) < 0.
\]

**Proof.** Assuming the contrary, we see that either \( h^*z(t) \geq 0 \) for any \( t \geq 0, \) or \( h^*z(t) \leq 0 \) for any \( t \geq 0. \) For definiteness, suppose that \( h^*z(t) \geq 0 \) for any \( t \geq 0. \) Then the relation \( h^*v(t) \neq 0 \) implies that

\[
\lim_{t \to +\infty} \int_0^t h^*z(\tau)\,d\tau = +\infty.
\]
On the other hand, we have

\[ \int^t_0 h^* z(\tau) \, d\tau = h^* Q^{-1}(z(t) - z(0)). \]

Since \( z(t) \) is uniformly bounded on \((0, +\infty)\), the function

\[ \int^t_0 h^* z(\tau) \, d\tau \]

is uniformly bounded, which contradicts (22). This proves the lemma.

Lemma 2. Let \( n = 2 \) and let the matrix \( Q \) have complex eigenvalues. Then for any two nonzero vectors \( h, u \in \mathbb{R}^2 \) there exist numbers \( \tau_1 \) and \( \tau_2 \) such that

\[ (23) \quad h^* e^{Q\tau_1} u > 0 \quad \text{and} \quad h^* e^{Q\tau_2} u < 0. \]

This obvious assertion can be viewed as a consequence of Lemma 1.

Lemma 3. Suppose that the matrix \( Q \) has two complex eigenvalues \( \lambda_0 \pm i\omega_0 \), and that the remaining eigenvalues \( \lambda_j(Q) \) of \( Q \) satisfy the condition \( \text{Re} \lambda_j(Q) < \lambda_0 \).

Let \( h, u \in \mathbb{R}^n \) be two vectors such that

\[ (24) \quad \det(h, Q^* h, \ldots, (Q^*)^{n-1} h) \neq 0, \]

\[ (25) \quad \det(u, Qu, \ldots, Q^{n-1} u) \neq 0. \]

Then there exist numbers \( \tau_1 \) and \( \tau_2 \) such that

\[ (26) \quad h^* e^{Q\tau_1} u > 0 \quad \text{and} \quad h^* e^{Q\tau_2} u < 0. \]

We recall that conditions (24) and (25) are controllability conditions for the pair \((Q, u)\) and observability conditions for the pair \((Q, h)\).

Proof. It suffices to observe that the solution \( z(t) = e^{Qt} u \) can be written as \( z(t) = e^{\lambda_0 t}(v(t) + w(t)) \), where \( v(t) \) and \( w(t) \) satisfy the assumptions of Lemma 1. The relation \( h^* v(t) \neq 0 \) follows from the observability of \((Q, h)\) and the controllability of \((Q, u)\). □

Theorem 1 and Lemma 2 readily imply the following statement.

Theorem 3. Let \( n = 2 \). Suppose there exist matrices \( K_0 \) and \( U_0 \) satisfying the following conditions:

1) \( \det BK_0 C = 0, \quad \text{Tr} BK_0 C \neq 0; \)
2) the matrix \( A + BU_0 C \) has complex eigenvalues.

Then there exists a periodic matrix \( K(t) \) such that system (1) is asymptotically stable.

Proof. It suffices to set \( K_1 = K_2 = \mu K_0 \), where \( |\mu| \) is a sufficiently large number, and \( \text{Tr} \mu BK_0 C < 0 \). In this case, obviously, the assumptions of Theorem 1 are fulfilled. □

Now we consider the case where \( B \) is a column vector, \( C \) is a row vector, and \( K(t) : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) is a piecewise continuous function.
We introduce the transfer function of system (1):

\[ W(p) = C(A - pI)^{-1}B, \]

where \( p \) is a complex variable. We assume that the function \( W(p) \) is nondegenerate. This means that the pair \((A, B)\) is controllable and the pair \((A, C)\) is observable.

**Lemma 4.** If the hyperplane \( \{h^*z = 0\} \) is an invariant manifold for the system

\[ (27) \quad \dot{x} = (A + \mu BC)x, \quad \mu \neq 0, \]

then the pair \((A, h)\) is observable.

**Proof.** Suppose that \((A, h)\) is not observable. In this case (see [7]), there exists a vector \( q \) and a number \( \gamma \) such that

\[ h^*q = 0, \quad Aq = \gamma q, \quad q \neq 0. \]

The observability of the pair \((A, C)\) implies the inequality \( Cq \neq 0 \).

Since \( \{h^*z = 0\} \) is invariant with respect to (27), for all \( z \in \{h^*z = 0\} \) we have

\[ h^*(A + \mu BC)^kz = 0, \quad k = 1, 2, \ldots. \]

Putting \( z = q \) and \( k = 1 \), we obtain \( h^*BCq = 0 \) whence \( h^*B = 0 \). For \( z = q \) and \( k = 2 \), using the preceding relation, we obtain \( h^*AB = 0 \). Continuing in this way, we obtain \( h^*A^{k-1}B = 0 \). The controllability of the pair \((A, B)\) implies that \( h = 0 \), which contradicts our assumption that the pair \((A, h)\) is not observable. The lemma is proved. \( \Box \)

**Lemma 5.** Let \( u \in \mathbb{R}^n \). If the line \( \{\alpha u \mid \alpha \in \mathbb{R}^1\} \) is invariant with respect to system (27), then the pair \((a, u)\) is controllable.

**Proof.** Invariance yields

\[ (A + \mu BC)^k u = \gamma_k u, \quad k = 0, 1, \ldots, \]

where the \( \gamma_k \) are some numbers. The observability of \((A, C)\) yields \( C^*u \neq 0 \) (see [7]). Therefore, for \( z \in \mathbb{R}^n \) satisfying \( z^*u = 0 \), \( z^*Au = 0 \), and \( z^*A^{n-1}u = 0 \) we have

\[ z^*B = z^*AB = \cdots = z^*A^{n-1}B = 0, \]

whence \( z = 0 \) because the pair \((A, B)\) is controllable.

Thus, the relations \( z^*u = \cdots = z^*A^{n-1}u = 0 \) imply that \( z = 0 \). Therefore, the pair \((A, u)\) is controllable. \( \Box \)

The following result is a consequence of Theorem 1 and Lemmas 3-5.

**Theorem 4.** Suppose that \( B, C^* \in \mathbb{R}^n \), \( \dim M_1 = 1 \), \( \dim L_2 = n - 1 \), and the inequality (10) is fulfilled. Also, we assume that for some number \( U_0 \neq K_j \), the matrix \( A + U_0BC \) has two complex eigenvalues \( \lambda_0 \pm i\omega_0 \), and that the remaining eigenvalues \( \lambda_j \) satisfy the condition \( \text{Re} \lambda_j < \lambda_0 \).

Then there exists a periodic function \( K(t) \) such that system (1) is asymptotically stable.

**Proof.** Combining Lemma 4 with the controllability of the pair \((A, B)\), the observability of the pair \((A, C)\), and the fact that \( U_0 \neq K_j \), we deduce that the pair \((A + U_0BC, h)\) is observable, where \( h \) is a normal to \( L_2 \). Lemma 5 shows that the pair \((A + U_0BC, u)\) is controllable. Here \( u \neq 0 \) and \( u \in M_1 \). Consequently, by Lemma 3, there is a number \( \tau \) such that

\[ h^* \exp[(A + U_0BC)\tau]u = 0. \]
Since this implies (8, theorem 4 is proved. □

Lemmas 1-5 were obtained for checking relation (8) in the case where the rotation of the subspace $M_1$ is caused by complex eigenvalues. Another approach involves an impulse action $U(t) = \mu$ with large $|\mu|$ on a small time interval. In this case, the velocity vector $\dot{x}$ is often close to the vector $\gamma B$, where $\gamma$ is a number. We describe this approach in more detail.

We consider the system (27) with large parameter $\mu$: $|\mu| \gg 1$.

**Lemma 6.** Suppose that $CB = 0$ and that $h$, $u$ are two vectors satisfying $h^*B \neq 0$ and $Cu \neq 0$. Then there exist numbers $\mu$ and $\tau(\mu) > 0$ such that $h^*x(\tau, u) = 0$ and

$$\lim_{\mu \to \infty} \tau(\mu) = 0.$$

**Proof.** Introducing

$$t_0 = -\frac{h^*u}{\mu h^*BCu},$$

$$R = \frac{(1 + 2|\mu||B||C|t_0)|u|}{1 - (2|A||t_0 + 4|\mu||A||B||C|t_0^2)},$$

we fix $\mu$ so as to have $t_0 > 0$ and

$$2|A||t_0 + 4|\mu||A||B||C|t_0^2 < 1.$$

It is easily seen that

$$|C\dot{x}(t, u)| = |CAx(t, u)| \leq |A||C| \max_{t \in [0, 2t_0]} |x(t, u)|$$

for all $t \in [0, 2t_0]$. Hence, for $t \in [0, 2t_0]$ we have

$$|Cx(t, u) - Cu| \leq 2|A||C|t_0 \max_{t \in [0, 2t_0]} |x(t, u)|.$$  

Combining this with (27), we obtain

$$|x(t, u) - u - \mu BCut| \leq (2|A||t_0 + 4|\mu||A||B||C|t_0^2) \max_{t \in [0, 2t_0]} |x(t, u)|,$$

whence

$$|x(t, u)| \leq R, \quad t \in [0, 2t_0],$$

$$|h^*x(t, u) - h^*u - \mu h^*BCut| \leq (2|A||t_0 + 4|\mu||A||B||C|t_0^2)R|h|, \quad t \in [0, 2t_0].$$

It is easy to check that

$$(2|A||t_0 + 4|\mu||A||B||C|t_0^2)R|h| = O \left( \frac{1}{|\mu|} \right).$$

Therefore, for large $|\mu|$ there exists $\tau \in [0, 2t_0]$ such that $h^*x(\tau, u) = 0$. The lemma is proved. □

**Lemma 7.** Suppose that $CB \neq 0$ and that $h$ and $u$ are two vectors satisfying $h^*B \neq 0$, $Cu \neq 0$, and

$$\frac{h^*uCB}{h^*BCu} < 1.$$
Then there exist numbers \( \mu \) and \( \tau(\mu) > 0 \) such that \( h^* x(\tau, u) = 0 \) and
\[
\lim_{\mu \to \infty} \tau(\mu) = 0.
\]

**Proof.** We define
\[
t_0 = \frac{1}{\mu C B} \log \left( 1 - \frac{h^* u C B}{h^* C B u} \right),
\]
\[
R = \frac{(1 + |B||C||CB|^{-1}(e^{2\mu CB t_0} + 1))|u|}{1 - (2|A|t_0 + 2|A||B||C|t_0|CB|^{-1}(e^{2\mu CB t_0} + 1))}.
\]
The number \( \mu \) is taken in such a way that \( t_0 > 0 \) and
\[
2|A|t_0 + 2|A||B||C|t_0|CB|^{-1}(e^{2\mu CB t_0} + 1) < 1.
\]
It is easily seen that
\[
|C x(t, u) - \mu C B x(t, u)| \leq |A||C| \max_{t \in [0, 2t_0]} |x(t, u)|
\]
for all \( t \in [0, 2t_0] \). Hence, for \( t \in [0, 2t_0] \) we have
\[
|C x(t, u) - e^{\mu C B t} C u| \leq \frac{1 - e^{2\mu CB t_0}}{-\mu CB} |A||C| \max_{t \in [0, 2t_0]} |x(t, u)|.
\]
Combining this with (27), we obtain
\[
\left| x(t, u) - u - \frac{BC u}{CB} (e^{\mu C B t} - 1) \right| \leq \left( 2|A|t_0 + 2|A||B||C|t_0|CB|^{-1}(e^{2\mu CB t_0} + 1) \right) \max_{t \in [0, 2t_0]} |x(t, u)|,
\]
whence
\[
|x(t, u)| \leq R, \quad t \in [0, 2t_0],
\]
\[
\left| h^* x(t, u) - h^* u - \frac{h^* B C u}{CB} (e^{\mu C B t} - 1) \right| \leq \left( 2|A|t_0 + 2|A||B||C|t_0|CB|^{-1}(e^{2\mu CB t_0} + 1) \right) R|h|, \quad t \in [0, 2t_0]
\]
It is easy to check that
\[
(2|A|t_0 + 2|A||B||C|t_0|CB|^{-1}(e^{2\mu CB t_0} + 1)) R|h| = O \left( \frac{1}{\mu} \right).
\]
There fore, for large \( |\mu| \) there exists \( \tau \in [0, 2t_0] \) such that \( h^* x(\tau, u) = 0 \). The lemma is proved.

\( \square \)

**Theorem 5.** Suppose that \( B, C^* \in \mathbb{R}^n \), \( C B = 0 \), \( \dim M_1 = 1 \), \( \dim L_2 = n - 1 \), and inequality (10) is fulfilled.
Then there exists a periodic function \( K(t) \) such that system (1) is asymptotically stable.

**Proof.** By Lemmas 4 and 5, the controllability of \( (A, B) \) and the observability of \( (A, C) \) imply the observability of \( ((A + KBC), h) \) for any \( K \neq K_2 \) and the controllability of \( ((A + KBC), u) \)
for any \( K \neq K_1 \). Here \( h \) is a normal to the hyperplane \( L_2 \), and \( u \) is a nonzero vector in the subspace \( M_1 \). It follows that \( h^*B \neq 0 \) and \( Cu \neq 0 \). Then, by Lemma 6, there exist numbers \( \mu \) and \( \tau(\mu) \) such that if \( U(t) \equiv \mu \), then system (8) satisfies condition (9). \( \square \)

**Theorem 6.** Suppose that \( CB \neq 0 \) and the matrix \( A \) has a positive eigenvalue \( \kappa \) and \( n - 1 \) eigenvalues with real parts less that \( -\lambda \), where \( \lambda > \kappa \). We also assume that

\[
\lim_{p \to \kappa} (\kappa - p)W(p) < 1.
\]

Then there exists a periodic function \( K(t) \) such that system (1) is asymptotically stable.

**Proof.** Without loss of generality, we may assume that the matrix \( A \) and the vectors \( B \) and \( C \) are of the form

\[
A = \begin{pmatrix} \kappa & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad C = (c_1, c_2),
\]

where \( A_2 \) is an \((n - 1)\times(n - 1)\)-matrix, \( b_2 \in \mathbf{R}^{n-1} \), and \( c^*_2 \in \mathbf{R}^{n-1} \). In this case, the normal \( h \) to the subspace \( L_1 = L_2 \) and the vector \( u \in M_1 = M_2 \) are of the form \( h = u = \left( \frac{1}{w} \right) \).

Therefore, by Lemma 7, there exist numbers \( \mu \) and \( \tau(\mu) \) such that if \( U(t) \equiv \mu \), then condition (9) is fulfilled for system (8) provided that

\[
\frac{CB}{c_1b_1} < 1.
\]

We easily see that \( c_1b_1 \neq 0 \) because \((A, B)\) is controllable and \((A, C)\) is observable, and

\[
c_1b_1 = \lim_{p \to \kappa} (\kappa - p)W(p).
\]

Thus, all assumptions of Theorem 1 are fulfilled, and, consequently, system (1) is stabilizable. \( \square \)

**Lemma 8.** Suppose that an \((n - 2)\)-dimensional linear subspace \( L \) invariant for system (27) lies in the hyperplane \( \{h^*z = 0\} \). If \( h^*B = CB = 0 \), then the pair \((A, h)\) is observable.

**Proof.** The controllability of the pair \((A, B)\) and the invariance of \( L \) with respect to (27) imply that \( B \in L \) (see [7]). Therefore, the linear hull of \( B \) and \( L \) coincides with the hyperplane \( \{h^*z = 0\} \).

Suppose that the pair \((A, h)\) is not observable. Then there is a vector \( q \neq 0 \) and a number \( \gamma \) such that

\[
h^*q = 0, \quad Aq = \gamma q
\]

(see [7]). The observability of the pair \((A, C)\) implies the relation \( Cq \neq 0 \). Since \( L \) is invariant, we have

\[
h^*(A + \mu BC)z = 0, \quad k = 0, 1, \ldots, \quad z \in L.
\]

The above arguments imply the existence of a number \( \nu \) and a vector \( z \in L \) such that

\[
q = z + \nu B.
\]

If \( \nu \neq 0 \), then for \( k = 1 \) we have \( \nu h^*AB = 0 \), whence \( h^*AB = 0 \). Combining this with the relations \( h^*B = CB = 0 \) for \( k = 2 \), we obtain \( \nu h^*AB = 0 \). Successively continuing this procedure for \( k = 3, \ldots \), we obtain the relations \( h^*AB = 0 \). By the controllability of \((A, B)\), these relations imply \( h = 0 \), which contradicts the definition of \((A, h)\) the vector \( \nu \neq 0 \).

Thus, we have proved the observability of \((A, h)\) for \( \nu \neq 0 \).

If \( \nu = 0 \), then we use the same arguments as in the proof of Lemma 4. \( \square \)
Theorem 7. Suppose that $B, C^* \in \mathbb{R}^n$, $CB = 0$, $\dim M_1 = 1$, $\dim L_2 = n - 2$, and inequality (10) is true. If the assumptions of Theorem 4 are fulfilled for some number $U_0 \neq K_j$, then there exists a periodic function $K(t)$ such that system (1) is asymptotically stable.

Proof. Observe that, as $\mu \to \infty$, the integral manifold $\Omega(\mu)$ consisting of the trajectories $x(t, x_0)$ of system (27) with initial values $x_0 \in L_2$ approaches the hyperplane $\{h^* x = 0\}$. Here $h$ is a normal to the linear hull of $L_2$ and $B$. Such convergence was described in the proof of Lemma 6.

Lemma 7 implies that the pair $((A + U_0 BC), h)$ is observable, while Lemma 5 implies that the pair $((A + U_0 BC), u)$ with $u \in M_1$, $u \neq 0$ is controllable. By Lemma 3, this shows that for system (9) with $U(t) = U_0$ the sign of the function $h^* y(t, u)$ changes for some values of $t$. Therefore, for sufficiently large $|\mu|$, there exists a number $\tau_0(\mu) > 0$ such that $y(\tau_0(\mu), u) \in \Omega(\mu)$.

Perturbing the right-hand side of (1) slightly, we can ensure that $Cy(\tau_0(\mu), u) \neq 0$ (asymptotic stability is preserved under small perturbations of the right-hand sides of periodic systems).

Next, if in (9) we set $U(t) = \mu$ (or $U(t) = -\mu$) on $(\tau_0, \tau)$, then, moving along the manifold $\Omega(\mu)$, we reach the set $L_2$ at the moment $t = \tau$. Here the sign of $\mu$ is chosen so that $\tau > \tau_0$ (see the proof of Lemma 6).

Now, we have $y(\tau, u) \in L_2$, and, consequently, condition (8) is fulfilled.

Thus, all assumptions of Theorem 1 are satisfied. \(\square\)

§3. Necessary stabilization conditions

Now, we pass to conditions necessary for stabilization.

We consider the case where $B$ is a column vector, $C$ is a row vector, and $K(t) : \mathbb{R}^1 \to \mathbb{R}^1$ is a piecewise continuous function. This case is important for control theory.

As before, $W(p)$ is the transfer function of system (1):

$$W(p) = C^*(A - pI)^{-1}B = \frac{c_n p^{n-1} + \cdots + c_1}{p^n + a_n p^{n-1} + \cdots + a_1},$$

where the $c_j$ and $a_j$ are real numbers. If the transfer function $W(p)$ is nondegenerate, then system (1) can be written in the following scalar form (see [8]):

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\vdots \\
\dot{x}_{n-1} &= x_n, \\
\dot{x}_n &= -(a_n x_n + \cdots + a_1 x_1) - K(t)(c_n x_n + \cdots + c_1 x_1).
\end{align*}
\]

Clearly, here we have

$$C = (c_1, \ldots, c_n), \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \ddots & \ddots \\ -a_1 & \cdots & -a_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$
We recall that the transfer function $W(p)$ is nondegenerate if and only if the polynomials
\[\frac{c_n p^{n-1} + \cdots + c_1}{p^n + a_n p^{n-1} + \cdots + a_1}\]
have no common zeros.

In what follows, we assume that $c_n \neq 0$. In this case, without loss of generality we may put $c_n = 1$.

**Theorem 8.** Assume that the following conditions are satisfied:
1) for $n > 2$, we have $c_1 \leq 0, \ldots, c_{n-2} \leq 0$;
2) $c_1(a_n - c_{n-1}) > a_1$, $c_1 + (a_n - c_{n-1})c_2 > a_2$,
   \[\vdots\]
   $c_{n-2} + (a_n - c_{n-1})c_{n-1} > a_{n-1}$.

Then there is no function $K(t)$ for which system (1) is asymptotically stable.

**Proof.** Consider the set
\[\Omega = \{x_1 \geq 0, \ldots, x_{n-1} \geq 0, x_n + c_{n-1}x_{n-1} + \cdots + c_1 x_1 \geq 0\} \subset \mathbb{R}^n.\]
We prove that $\Omega$ is positively invariant, i.e., if $x(t_0) \in \Omega$, then $x(t) \in \Omega$ for all $t \geq t_0$.

Observe that if $j = 1, \ldots, n-2$ and $\tau$ is such that
\[
x_j(\tau) = 0, \quad x_i(\tau) > 0, \quad i \neq j, \quad i \leq n-1,
\]
then
\[\dot{x}_j(\tau) > 0.\]
Indeed, for $j = 1, \ldots, n-2$ we have
\[\dot{x}_j(\tau) = x_{j+1}(\tau) > 0.\]
For $n = 2$, we have
\[\dot{x}_1(\tau) = x_2(\tau) > -c_1 x_1(\tau) = 0,\]
and for $n > 2$ we have
\[\dot{x}_{n-1}(\tau) = x_n(\tau) > -c_{n-2}x_{n-2}(\tau) - c_1 x_1(\tau) \geq 0.\]
Next, we observe that if $x_n(\tau) + c_{n-1}x_{n-1}(\tau) + \cdots + c_1 x_1(\tau) = 0$, and $x_j(\tau) > 0, \quad j = 1, \ldots, n-1$, then
\[(x_n(\tau) + c_{n-1}x_{n-1}(\tau) + \cdots + c_1 x_1(\tau))^{\bullet} > 0.\]
Indeed, we have
\[
(x_n(\tau) + c_{n-1}x_{n-1}(\tau) + \cdots + c_1 x_1(\tau))^{\bullet} = \\
= (-a_{n-1} + c_{n-2} + (a_n - c_{n-1})c_{n-1})x_{n-1}(\tau) + \cdots \\
\cdots + (-a_2 + c_1 + (a_n - c_{n-1})c_{n-2})x_2(\tau) + \\
+ (-a_1 + (a_n - c_{n-1})c_1)x_1(\tau).\]
and it remains to use condition 2) of the theorem.

Inequalities (29) and (30) imply that the boundary of the set \( \Omega \) is transversal to the vector field of (28) almost everywhere, and the solutions of (28) pierce the boundary inward \( \Omega \) almost everywhere. Since the solutions of (28) depend on the initial values continuously, we see that the set \( \Omega \) is positively invariant, whence it easily follows that system (28) is not asymptotically stable. The theorem is proved.

Another condition ensuring that system (1) is nonstable is well known (see [5]):

\[
\text{Tr} \left( A + BK(t)C \right) \geq \alpha > 0, \quad t \in \mathbb{R}^1.
\]

§4. Stabilization of systems of order two and three

Now we show how the above results can be applied to the case where \( n = 2 \), \( B \) is a column vector, \( C \) is a row vector, and \( K(t) \) is a scalar function.

We introduce the transfer function of system (1):

\[
W(p) = C(A - pI)^{-1}B = \frac{\rho p + \gamma}{p^2 + \alpha p + \beta}.
\]

Here \( p \) is a complex variable.

In what follows, we assume that \( \rho \neq 0 \). Furthermore, there is no loss of generality in assuming that \( \rho = 1 \). We also assume that the function \( W(p) \) is nondegenerate, i.e., we have

\[
\gamma^2 - \alpha \gamma + \beta \neq 0.
\]

It is well known (see [8]) that in this case system (1) can be written as follows:

\[
\begin{align*}
\dot{\sigma} &= \eta, \\
\dot{\eta} &= -\alpha \eta - \beta \sigma - K(t)(\eta + \gamma \sigma).
\end{align*}
\]

It is easy to see that stabilization of system (31) with the help of a constant matrix \( K(t) \equiv K_0 \) is possible if and only if

\[
\alpha + K_0 > 0, \quad \beta + \gamma K_0 > 0.
\]

A number \( K_0 \) satisfying these inequalities exists if and only if \( \gamma > 0 \), or \( \gamma \leq 0 \) and \( \alpha \gamma < \beta \).

We consider the case where stabilization with the help of a constant \( K(t) \equiv K_0 \) is impossible:

\[
\gamma \leq 0, \quad \alpha \gamma > \beta.
\]

We apply Theorem 3. Clearly, condition 1) of Theorem 3 is fulfilled, because \( \det BK_0 C = 0 \) and \( \text{Tr} BK_0 C = K_0 CB = -K_0 \neq 0 \).

Condition 2) of Theorem 3 is fulfilled if for some \( U_0 \) the polynomial

\[
p^2 + \alpha p + \beta + U_0(p + \gamma)
\]

has complex zeros. We easily see that such a number \( U_0 \) exists if and only if

\[
\gamma^2 - \alpha \gamma + \beta > 0.
\]

Thus, if inequality (32) if fulfilled, then there exists a periodic function \( K(t) \) such that system (32) is asymptotically stable.

The same result can be obtained with the help of Theorem 6.
For this, without loss of generality we assume that $\alpha > 0$. This can always be achieved if we properly choose $K_0$ in the expression

$$-(\alpha + K_0)\eta - (\beta + \gamma K_0)\sigma - (K(t) - K_0)(\eta + \gamma \sigma)$$

and change the notation: $\alpha + K_0 \to \alpha$, $\beta + \gamma K_0 \to \beta$, and $K(t) - K_0 \to K(t)$. The inequality $\alpha > 0$ implies that $\lambda > \kappa$. Here

$$\lim_{p \to \kappa} \frac{CB}{p - \kappa} W(p) = \frac{\kappa + \lambda}{\kappa + \gamma}.$$ 

Therefore, all conditions of Theorem 6 are fulfilled if

$$(\lambda - \gamma)(\kappa + \gamma) = -\gamma^2 + \alpha \gamma - \beta < 0.$$ 

This inequality coincides with (32). It is easy to see that if

$$(33) \quad \gamma^2 - \alpha \gamma + \beta < 0,$$

then the conditions of Theorem 8 are also fulfilled.

Thus, the following result is true.

**Theorem 9** [9]. If inequality (32) is fulfilled, then there exists a periodic function $K(t)$ such that system (31) is asymptotically stable.

If inequality (33) is fulfilled, then there are no functions $K(t)$ for which system (31) is asymptotically stable. \(\square\)

This result was also obtained in [10] for a different class of stabilizing functions $K(t)$ of the form

$$K(t) = (k_0 + k_1 \omega \cos \omega t), \quad \omega \gg 1,$$

with the help of averaging.

Now we consider systems (1) of order three with various transfer functions.

**4.1.** $W(p) = \frac{1}{p^3 + ap^2 + bp + c}$, where $a$, $b$, and $c$ are some numbers.

If $a > 0$ and $b > 0$, then stationary stabilization is possible. Suppose $a > 0$ and $b \leq 0$. In this case, stationary stabilization is impossible; we apply Theorem 7.

Clearly, if $U_0$ is sufficiently large, then the polynomial

$$p^3 + ap^2 + bp + U_0 + c$$

has one negative zero and two complex zeros $\lambda_0 \pm i \omega_0$, $\lambda_0 > 0$. We take $K_1$ so that

$$p^3 + ap^2 + bp + K_1 + c = (p - \kappa)(p^2 + \alpha_1 p + \beta_1)$$

with $\alpha_1 = a + \kappa_1$ and $\beta_1 = b + (a + \kappa_1)\kappa_1$. For $\kappa_1$ large, the polynomial $p^2 + \alpha_1 p + \beta_1$ has complex zeros with real part equal to $-(a + \kappa_1)/2$. We take $K_2$ so that

$$p^3 + ap^2 + bp + K_2 + c = (p + \lambda_2)(p^2 + \alpha_2 p + \beta_2)$$

with $\alpha_2 = a - \lambda_2$ and $\beta_2 = b - (a - \lambda_2)\lambda_2$. For $\lambda_2$ large, the polynomial $p^2 + \alpha_2 p + \beta_2$ has complex zeros with real part equal to $(\lambda_2 - a)/2$.

We have

$$\dim M_1 = \dim L_2 = 1, \quad \dim M_2 = \dim L_1 = 2,$$
\[ \lambda_1 = \frac{a + \kappa_1}{2}, \quad \kappa_2 = \frac{\lambda_2 - a}{2}, \]

\[ \lambda_1 \lambda_2 - \kappa_1 \kappa_2 = a(\lambda_2 + \kappa_1) > 0. \]

Thus, all conditions of Theorem 7 are fulfilled.
Since
\[ \text{Tr}(A + BK(t)C) = -a, \]

asymptotic stability is impossible for \( a < 0 \).

So, we can state the following result.

**Theorem 10.** For \( a > 0 \) system (1) is stabilizable. For \( a < 0 \) stabilization is impossible.

\[ \square \]

4.2. \( W(p) = \frac{p}{p^3 + ap^2 + bp + c} \).

If \( a > 0 \) and \( c > 0 \), stationary stabilization is possible. We consider the case where \( a > 0 \) and \( c < 0 \), and apply Theorem 5 with \( K_1 = K_2, \lambda_1 = \lambda_2 = \lambda, \) and \( \kappa_1 = \kappa_2 = \kappa \). We take \( K_1 \) so that
\[ (p - \kappa)(p^2 + \alpha p + \beta) = p^3 + ap^2 + (K_1 + b)p + c \]

with \( \alpha = a + \kappa \) and \( \beta = -c/\kappa \). If \( \kappa \) is small, the polynomial \( p^2 + \alpha p + \beta \) has complex zeros with real part equal to \( -(a + \kappa)/2 \). Thus,
\[ M_1 = M_2, \quad L_1 = L_2, \quad \dim M_1 = 1, \quad \dim L_2 = 2, \quad \text{and} \quad \lambda = (a + \kappa)/2. \]

Clearly, for small \( \kappa \) we have \( \lambda > \kappa \). Since \( \text{Tr}(A + BK(t)C) = -a \) for \( a < 0 \), asymptotic stability is impossible.

This leads to the following result.

**Theorem 11.** Suppose \( a \neq 0 \) and \( c \neq 0 \). Then system (1) is stabilizable if and only if \( a > 0 \). \( \square \)

4.3. \( W(p) = \frac{p^2}{p^3 + ap^2 + bp + c} \).

If \( b > 0 \) and \( c > 0 \), stationary stabilization is possible. Theorem 8 shows that if \( b < 0 \) and \( c < 0 \), then stabilization is impossible.

We consider the case where \( b > \) and \( c < 0 \) and apply Theorem 1.

Putting \( K_1 = K_2, \lambda_1 = \lambda_2 = \lambda, \) and \( \kappa_1 = \kappa_2 = \kappa \), we take \( K_1 \) so that
\[ (p - \kappa)(p^2 + \alpha p + \beta) = p^3 + (a + K_1)p^2 + bp + c. \]

Here
\[ \alpha = -\frac{c + \kappa b}{\kappa^2}, \quad \beta = \frac{-c}{\kappa}. \]

Below it is assumed that \( \kappa \) is small. In this case, we define
\[ \lambda = \frac{-(c + \kappa b)}{2\kappa^2} - \sqrt{\frac{(c + \kappa b)^2}{4\kappa^4} + \frac{c}{\kappa}}. \]

Obviously, \( \lambda > \kappa \) for \( b + \kappa^2 > 0 \), and, consequently, condition (10) of Theorem 1 is fulfilled.

Now we construct \( U(t) \) so as to ensure (8).

We introduce a number \( U_0 \) such that
\[ (p - \nu)(p^2 + \alpha_1 p + \beta_1) = p^3 + (a + U_0)p^2 + bp + c. \]
Here
\[ \alpha_1 = \frac{-(c + \kappa b)}{\nu^2}, \quad \beta_1 = \frac{-c}{\nu}. \]
Suppose that \( \nu \) is sufficiently large. Without loss of generality, we assume that
\[ A + U ) BC = \begin{pmatrix} \nu & 0 \\ 0 & Q \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad C = (c_1, c_2). \]
Here \( Q \) is a \((2 \times 2)\)-matrix, and \( b_2 \) and \( c_2 \) are two-dimensional vectors. For \( \nu \) large, the matrix \( Q \) has complex eigenvalues. Therefore, for any nonzero vector \( u \in M_1 \) there exists \( \tau_1 > 0 \) such that the vectors
\[ B, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \exp( A + U_0 BC ) \tau_1 | u \]
lie in one plane. It follows that for some \( \rho \) we have
\[ \exp( A + U_0 BC ) \tau_1 | u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \rho \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \]
If \( b_1 > 0 \), then \( \rho \in [-b_1^{-1}, 0] \), and if \( b_1 < 0 \), then \( \rho \in [0, -b_1^{-1}] \).

The relations
\[ \frac{CB}{c_1 b_1} = -\frac{1}{\lim_{\nu \to \nu} (\nu - p) W(p)} = 1 + \frac{\alpha_1 \nu + \beta_1}{\nu^2} = 1 - \frac{2c}{\nu^3} - \frac{b}{\nu^2} < 1 \]
imply that
\[ (34) \quad \frac{CB(1 + \rho b_1)}{c_1 b_1} < 1 \]
for sufficiently large \( \nu \). By Lemma 8, the above inequality implies the existence of numbers \( \mu \) and \( \tau(\mu) \) such that
\[ \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* \exp( A + (U_0 + \mu) BC ) \tau(\mu) | \exp( A + U_0 BC ) \tau_1 | u = 0. \]
Since the planes \( L_2 \) and \( \{ x \mid \text{big} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* x = 0 \} \) intersect, and the matrix \( Q \) has complex eigenvalues, it follows that
\[ \exp( A + U_0 BC ) \tau_2 | \exp( A + (U_0 + \mu) BC ) \tau(\mu) | \exp( A + U_0 BC ) \tau_1 | u \in L_2, \]
for some \( \tau_2 > 0 \).

Thus, the inclusion (8) is valid for the function
\[ U(t) = \begin{cases} U_0, & t \in [0, \tau_1), \\ U_0 + \mu, & t \in [\tau_1, \tau_1 + \tau(\mu)), \\ U_0, & t \in [\tau_1 + \tau(\mu), \tau_1 + \tau(\mu) + \tau_2, \end{cases} \]
where \( \tau = \tau_1 + \tau(\mu) + \tau_2 \).

We have proved the following result.

**Theorem 12.** Suppose that \( b \neq 0 \) and \( c < 0 \). Then system (1) is stabilizable if and only if \( b > 0 \). \( \Box \)


**St. Petersburg State University, Department of Mathematics and Mechanics, Bibliotechnaya pl., 2, Stary Peterhof, St. Petersburg, 198904, Russia**

*E-mail address: leonov @ math.spbu.ru*

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