St. Petersburg Math. J.
13 (2001), .1 Vol. 13 (2002), No. 4

BROCKETT'S PROBLEM IN THE THEORY OF STABILITY OF LINEAR DIFFERENTIAL EQUATIONS

G. A. Leonov

Abstract. Algorithms for nonstationary linear stabilization are constructed. Combined with a nonstabilizabiity criterion, these algorithms result in the solution of the Brockett problem in a number of cases.

§1. Introduction

In the book [1], R. Brockett formulated the following problem.

For a triplet of matrices A, B, and C, what conditions ensure the existence of a matrix $K(t)$ such that the system

$$\frac{dx}{dt} = Ax + BK(t)Cx, \quad x \in \mathbf{R}^n, \tag{1}$$

is asymptotically stable.

The problem of stabilizing system (1) with the help of a constant matrix K is classical for automatic control theory [2, 3]. From this point of view, Brockett's problem can be reformulated as follows.

To what extent are the possibilities of classical stabilization extended by introducing matrices $K(t)$ that depend on time t?

Stabilizing mechanical systems often necessitates the invocation of a special class of stabilizing matrices $K(t)$. These matrices must be periodic and have zero mean on the period $[0, T]$:

$$\int_0^T K(t)\,dt = 0. \tag{2}$$

For example, consider a linear approximation near an equilibrium point for the pendulum with vertically oscillating suspension point:

$$\ddot{\theta} + \alpha\dot{\theta} + \bigl(K(t) - \omega_0^2\bigr)\theta = 0, \tag{3}$$

where α and ω_0 are positive numbers. Here, the most common choice for the function $K(t)$ is either $\beta \sin \omega t$ (see [4]), or

$$K(t) = \begin{cases} \beta, & t \in [0, T/2), \\ -\beta, & t \in [T/2, T) \end{cases} \tag{4}$$

(see [5, 6]). For such functions $K(t)$, the effect of stabilization of the upper equilibrium point is well known for large ω and, consequently, small T.

In this paper, we present certain algorithms enabling us to construct periodic piecewise constant functions $K(t)$ that solve the Brockett problem in a number of cases, and also periodic functions $K(t)$ that satisfy (2) and solve the stabilization problem. Moreover, we show that low-frequency stabilization ($T \gg 1$) is possible for the pendulum equation (3) with $K(t)$ of the form (4).

2000 *Mathematics subject Classification.* Primary 34D05.

Key words and phrases. Stability, transfer function, controllability, observability, invariant manifold.

§2. Conditions sufficient for stabilization

Suppose we have two matrices $K_j \quad (j = 1, 2)$ such that the systems

$$\frac{dx}{dt} = (A + BK_jC)x, \quad x \in \mathbf{R}^n, \tag{5}$$

possess stable linear manifolds L_j and invariant linear manifolds M_j. We assume that $M_j \cap L_j = \{0\}$ and $\dim M_j + \dim L_j = n$, and that λ_j, κ_j, α_j, and β_j are positive numbers satisfying the inequalities

$$|x(t)| \leq \alpha_j e^{-\lambda_j t}|x(0)|, \quad x(0) \in L_j, \tag{6}$$

$$|x(t)| \leq \beta_j e^{-\kappa_j t}|x(0)|, \quad x(0) \in M_j, \tag{7}$$

In what follows. We also assume that there exists a matrix $U(t)$ and a number $\tau > 0$ such that

$$Y(\tau)M_1 \subset L_2, \tag{8}$$

where $Y(t)$ is the fundamental matrix ($Y(0) = I$) of the following system:

$$\frac{dy}{dt} = (A + BU(t)C)y. \tag{9}$$

Theorem 1. *Suppose that*

$$\lambda_1\lambda_2 > \kappa_1\kappa_2 \tag{10}$$

and that (8) *is true.*

Then there exists a periodic matrix $K(t)$ *such that system* (1) *is asymptotically stable.*

Proof. Condition (10) implies that for every $T > 0$ there exist two numbers t_1 and t_2 such that

$$\begin{aligned} -\lambda_1 t_1 + \kappa_2 t_2 &< -T, \\ -\lambda_2 t_2 + \kappa_1 t_1 &< -T. \end{aligned} \tag{11}$$

We define the periodic matrix $K(t)$ as follows:

$$K(t) = \begin{cases} K_1, & t \in [0, t_1), \\ U(t - t_1), & t \in [t_1, t_1 + \tau), \\ K_2, & t \in [t_1 + \tau,\, t_1 + t_2 + \tau). \end{cases} \tag{12}$$

The period of $K(t)$ is equal to $t_1 = t_2 + \tau$. We show that if T is sufficiently large, then system (1) with such matrix $K(t)$ is asymptotically stable. For this, we introduce nonsingular matrices S_j bringing system (5) to a canonical form:

$$\begin{aligned} \frac{dz_j}{dt} &= Q_j z_j, \quad \dim z_j = \dim L_j, \\ \frac{dw_j}{dt} &= P_j w_j, \quad \dim w_j = \dim M_j. \end{aligned} \tag{13}$$

Here

$$S_j x = \begin{pmatrix} z_j \\ w_j \end{pmatrix}. \tag{14}$$

There is no loss of generality in assuming that

$$\begin{aligned} |z_j(t)| &\le e^{-\lambda_j t}|z_j(0)|, \\ |w_j(t)| &\le e^{-\kappa_j t}|w_j(0)|. \end{aligned} \tag{15}$$

Relations (12)–(14) show that

$$\begin{pmatrix} z_2(t_1+\tau) \\ w_2(t_1+\tau) \end{pmatrix} = S_2 Y(\tau) S_1^{-1} \begin{pmatrix} z_1(t_1) \\ w_1(t_1) \end{pmatrix}. \tag{16}$$

The inclusion (8) implies that the matrix $S_2Y(\tau)S_1^{-1}$ has the following structure:

$$S_2 Y(\tau) S_1^{-1} = \begin{pmatrix} R_{11}(\tau) & R_{12}(\tau) \\ R_{21}(\tau) & 0 \end{pmatrix}.$$

There fore, by (11) and (15),

$$\begin{aligned} |z_2(t_1+t_2+\tau)| &\le |R_{11}(\tau)|\, e^{-2T}|z_1(0)| + R_{12}(\tau)|\, e^{-T}|w_1(0)|, \\ |w_2(t_1+t_2+\tau)| &\le |R_{21}(\tau)|\, e^{-T}|z_1(0)|, \end{aligned}$$

which implies that for all sufficiently large values of T and for the initial values in the ball $|x(0)| \le 1$, we have

$$|x(t_1+t_2+\tau)| \le \frac{1}{2}.$$

Since the matrix $K(t)$ is periodic, it follows that system (1) is asymptotically stable. □

Now, we assume that the matrix $K(t)$ in (1) is a scalar function,

$$K_1 = K_2 = K_0, \quad \lambda_1 = \lambda_2 = \lambda, \quad \kappa_1 = \kappa_2 = \kappa, \quad U(t) \equiv U_0, \quad K_0 U_0 < 0,$$

the function $|Y(t)|$ is uniformly bounded on the interval $(0, +\infty)$, and there exists a sequence $\tau_j \to \infty$ such that

$$Y(\tau_j) M_1 \subset L_2. \tag{17}$$

Theorem 2. *If $\lambda > \kappa$ and (17) is fulfilled, then there exists a T-periodic function $K(t)$ such that (2) is true and system (1) is asymptotically stable.*

Proof. We define

$$K(t) = \begin{cases} K_0, & t \in [0, |U_0\tau_j/2K_0|), \\ U_0, & t \in [|U_0\tau_j/2K_0|, \tau_j + |U_0\tau_j/2K_0|), \\ K_0, & t \in [\tau_j + |U_0\tau_j/2K_0|, \tau_j + |U_0\tau_j/K_0|). \end{cases} \tag{18}$$

The period of $K(t)$ is equal to $T = \tau_j(1 + |U_0/K_0|)$.

Here, τ_j is a sufficiently large number satisfying condition (17). The rest of the proof repeats the arguments used in the proof of Theorem 1. □

We apply Theorem 2 to equation (3) with a function $K(t)$ of the form (4).

Suppose that

$$\alpha^2 < 4(\beta - \omega_0^2). \tag{19}$$

In this case, without loss of generality, we may assume that $\beta - \omega_0^2 - \alpha^2/4 = 1$.

We set $K_0 = -\beta$ and $U_0 = \beta$. Condition (19) implies that the characteristic polynomial of equation (3) with $K(t) = U_0$ has complex zeros, and, consequently, condition (17) is fulfilled for some $\tau_1 > 0$. Clearly, here we have $\tau_j = \tau_1 + 2j\pi$.

Since the zeros in question of the characteristic polynomial have negative real parts, we easily see that $|Y(t)|$ is uniformly bounded on $(0, +\infty)$.

For $K(t) = K_0 = -\beta$, the quantities λ and κ can easily be calculated:

$$\lambda = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + (\beta + \omega_0^2)}\ ,$$

$$\kappa = -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + (\beta + \omega_0^2)}\ .$$

Thus, all assumptions of Theorem 2 are fulfilled, and equation (3) with $K(t)$ of the form (18) is asymptotically stable for sufficiently large j. This result can be stated as follows.

Proposition 1. *Under condition* (19)*, for every τ there exists $T > \tau$ such that equation* (3) *with a function $K(t)$ of the form* (4) *is asymptotically stable.* □

In particular, this implies the possibility of stabilizing the upper equilibrium point of the pendulum for low-frequency vertical oscillations of the suspension point. Naturally, here the amplitude a of the oscillations is large:

$$a = \frac{lT^2\beta}{8}\,,$$

where l is the length of the pendulum and β is the absolute value of the acceleration divided by l.

The stabilization effect is well known for high-frequency oscillations (for small T); see [5, 6].

The following lemmas are often useful for checking condition (8).

Consider the system

$$\dot{z} = Qz, \quad z \in \mathbf{R}^n, \tag{20}$$

where Q is a constant nonsingular $(n \times n)$-matrix and h is a vector in $\mathbf{R}^n$.

Lemma. *Suppose that the solution $z(t)$ of system* (20) *has the form $z(t) = v(t) + w(t)$, where $v(t)$ is a periodic vector-valued function such that $h^* v(t) \not\equiv 0$, and $w(t)$ is a vector-valued function for which*

$$\int_0^{+\infty} |w(\tau)|\, d\tau < +\infty, \qquad \lim_{t \to +\infty} w(t) = 0.$$

Then there exist two numbers τ_1 and τ_2 such that

$$h^* z(\tau_1) > 0 \quad \text{and} \quad h^* z(\tau_2) < 0. \tag{21}$$

Proof. Assuming the contrary, we see that either $h^* z(t) \geq 0$ for any $t \geq 0$, or $h^* z(t) \leq 0$ for any $t \geq 0$. For definiteness, suppose that $h^* z(t) \geq 0$ for any $t \geq 0$. Then the relation $h^* v(t) \not\equiv 0$ implies that

$$\lim_{t \to +\infty} \int_0^t h^* z(\tau)\, d\tau = +\infty. \tag{22}$$

On the other hand, we have

$$int_0^t h^* z(\tau)\,d\tau = h^* Q^{-1}\big(z(t) - z(0)\big).$$

Since $z(t)$ is uniformly bounded on $(0, +\infty)$, the function

$$\int_0^t h^* z(\tau)\,d\tau$$

is uniformly bounded, which contradicts (22).This proves the lemma. □

Lemma 2. *Let $n = 2$ and let the matrix Q have complex eigenvalues. Then for any two nonzero vectors $h, u \in \mathbf{R}^2$ there exist numbers τ_1 and τ_2 such that*

$$h^* e^{Q\tau_1} u > 0 \quad \text{and} \quad h^* e^{Q\tau_2} u < 0. \tag{23}$$

This obvious assertion can be viewed as a consequence of Lemma 1.

Lemma 3. *Suppose that the matrix Q has two complex eigenvalues $\lambda_0 \pm i\omega_0$, and that the remaining eigenvalues $\lambda_j(Q)$ of Q satisfy the condition $\operatorname{Re}\lambda_j(Q) < \lambda_0$.*

Let $h, u \in \mathbf{R}^n$ be two vectors such that

$$\det(h, Q^* h, \dots, (Q^*)^{n-1} h) \neq 0, \tag{24}$$

$$\det(u, Qu, \dots, Q^{n-1} u) \neq 0. \tag{25}$$

Then there exist numbers τ_1 and τ_2 such that

$$h^* e^{Q\tau_1} u > 0 \quad \text{and} \quad h^* e^{Q\tau_2} u < 0. \tag{26}$$

We recall that conditions (24) and (25) are controllability conditions for the pair (Q, u) and observability conditions for the pair (Q, h).

Proof. It suffices to observe that the solution $z(t) = e^{Qt} u$ can be written as $z(t) = e^{\lambda_0 t}(v(t) + w(t))$, where $v(t)$ and $w(t)$ satisfy the assumptions of Lemma 1. The relation $h^* v(t) \not\equiv 0$ follows from the observability of (Q, h) and the controllability of (Q, u). □

Theorem1 and Lemma 2 readily imply the following statement.

Theorem 3. *Let $n = 2$. Suppose there exist matrices K_0 and U_0 satisfying the following conditions:*

1) $\det BK_0C = 0$, $\operatorname{Tr} BK_0C \neq 0$ *;*

2) *the matrix $A + BU_0C$ has complex eigenvalues.*

Then there exists a periodic matrix $K(t)$ such that system (1) *is asymptotically stable.*

Proof. It suffices to set $K_1 = K_2 = \mu K_0$, where $|\mu|$ is a sufficiently large number, and $\operatorname{Tr} \mu BK_0C < 0$. In this case, obviously, the assumptions of Theorem 1 are fulfilled. □

Now we consider the case where B is a column vector, C is a row vector, and $K(t) : \mathbf{R}^1 \to \mathbf{R}^1$ is a piecewise continuous function.

We introduce the transfer function of system (1):

$$W(p) = C(A - pI)^{-1}B,$$

where p is a complex variable. WE assume that the function $W(p)$ is nondegenerate. This means that the pair (A, B) is controllable and the pair (A, C) is observable.

Lemma 4. *If the hyperplane $\{h^*z = 0\}$ is an invariant manifold for the system*

$$\dot{x} = (A + \mu BC)x, \quad \mu \neq 0, \tag{27}$$

then the pair (A, h) is observable.

Proof. Suppose that (A, h) is not observable. In this case (see [7]), there exists a vector q and a number γ such that

$$h^*q = 0, \quad Aq = \gamma q, \quad q \neq 0.$$

The observability of the pair (A, C) implies the inequality $Cq \neq 0$.

Since $\{h^*z = 0\}$ is invariant with respect to (27), for all $z \in \{h^*z = 0\}$ we have

$$h^*(A + \mu BC)^k z = 0, \quad k = 1, 2, \dots .$$

Putting $z = q$ and $k = 1$, we obtain $h^*BCq = 0$, whence $h^*B = 0$. For $z = q$ and $k = 2$, using the preceding relation, we obtain $h^*AB = 0$. Continuing in this way, we obtain $h^*A^{k-1}B = 0$. The controllability of the pair (A, B) implies that $h = 0$, which contradicts our assumption that the pair (A, h) is not observable. The lemma is proved. □

Lemma 5. *Let $u \in \mathbf{R}^n$. If the line $\{\alpha u \mid \alpha \in \mathbf{R}^1\}$ is invariant with respect to system* (27)*, then the pair (a, u) is controllable.*

Proof. Invariance yields

$$(A + \mu BC)^k u = \gamma_k u, \quad k = 0, 1, \dots ,$$

where the γ_k are some numbers. The observability of (A, C) yields $Cu \neq 0$ (see [7]). Therefore, for $z \in \mathbf{R}^n$ satisfying $z^*u = 0$, $z^*Au = 0$, and $z^*A^{n-1}u = 0$ we have

$$z^*B = z^*AB = \cdots = z^*A^{n-1}B = 0,$$

whence $z = 0$ because the pair (A, B) is controllable.

Thus, the relations $z^*u = \cdots = z^*A^{n-1}u = 0$ imply that $z = 0$. Therefore, the pair (A, u) is controllable. □

The following result is a consequence of Theorem 1 and Lemmas 3-5.

Theorem 4. *Suppose that $B, C^* \in \mathbf{R}^n$, $\dim M_1 = 1$, $\dim L_2 = n - 1$, and the inequality* (10) *is fulfilled. Also, we assume that for some number $U_0 \neq K_j$ the matrix $A + U_0BC$ has two complex eigenvalues $\lambda_0 \pm i\omega_0$, and that the remaining eigenvalues λ_j satisfy the condition* $\operatorname{Re}\lambda_j < \lambda_0$.

Then there exists a periodic function $K(t)$ such that system (1) *is asymptotically stable.*

Proof. Combining Lemma 4 with the controllability of the pair (A, B), the observability of the pair (A, C), and the fact that $U_0 \neq K_j$, we deduce that the pair $(A + U_0BC, h)$ is observable, where h is a normal to L_2. Lemma 5 shows that the pair $(A + U_0BC, u)$ is controllable. Here $u \neq 0$ and $u \in M_1$. Consequently, by Lemma 3, there is a number τ such that

$$h^* \exp[(A + U_0BC)\tau]u = 0.$$

Since this implies (8, theorem 4 is proved. □

Lemmas 1-5 were obtained for checking relation (8) in the case where the rotation of the subspace M_1 is caused by complex eigenvalues. Another approach involves an impulse action $U(t) = \mu$ with large $|\mu|$ on a small time interval. In this case, the velocity vector $\dot{x}$ is often close to the vector γB, where γ is a number. We describe this approach in more detail.

We consider the system (27) with large parameter μ: $|\mu| \gg 1$.

Lemma 6. *Suppose that $CB = 0$ and that h, u are two vectors satisfying $h^*B \neq 0$ and $Cu \neq 0$. Then there exist numbers μ and $\tau(\mu) > 0$ such that $h^*x(\tau, u) = 0$ and*

$$\lim_{\mu\to\infty} \tau(\mu) = 0.$$

Proof. Introducing

$$t_0 = -\frac{h^*u}{\mu h^*BCu},$$

$$R = \frac{(1 + 2|\mu||B||C|t_0)|u|}{1 - (2|A|t_0 + 4|\mu||A||B||C|t_0^2)},$$

we fix μ so as to have $t_0 > 0$ and

$$2|A|t_0 + 4|\mu||A||B||C|t_0^2 < 1.$$

It is easily seen that

$$|C\dot{x}(t, u)| = |CAx(t, u)| \leq |A||C| \max_{t\in[0,2t_0]} |x(t, u)|$$

for all $t \in [0, 2t_0]$. Hence, for $t \in [0, 2t_0]$ we have

$$|Cx(t, u) - Cu| \leq 2|A||C|t_0 \max_{t\in[0,2t_0]} |x(t, u)|.$$

Combining this with (27), we obtain

$$|x(t, u) - u - \mu BCut| \leq (2|A|t_0 + 4|\mu||A||B||C|t_0^2) \max_{t\in[0,2t_0]} |x(t, u)|,$$

whence

$$\begin{aligned} |x(t, u)| &\leq R, & t &\in [0, 2t_0], \\ |h^*x(t, u) - h^*u - \mu h^*BCut| &\leq (2|A|t_0 + 4|\mu||A||B||C|t_0^2)R|h|, & t &\in [0, 2t_0]. \end{aligned}$$

It is easy to check that

$$(2|A|t_0 + 4|\mu||A||B||C|t_0^2)R|h| = O\left(\frac{1}{\mu}\right).$$

Therefore, for large $|\mu|$ there exists $\tau \in [0, 2t_0]$ such that $h^*x(\tau, u) = 0$. The lemma is proved. □

Lemma 7. *Suppose that $CB \neq 0$ and that h and u are two vectors satisfying $h^*B \neq 0$, $Cu \neq 0$, and*

$$\frac{h^*uCB}{h^*BCu} < 1.$$

Then there exist numbers μ and $\tau(\mu) > 0$ such that $h^* x(\tau, u) = 0$ and

$$\lim_{\mu \to \infty} \tau(\mu) = 0.$$

Proof. We define

$$t_0 = \frac{1}{\mu CB} \; \log \left(1 - \frac{h^* u CB}{h^* BCu} \right),$$

$$R = \frac{(1 + |B||C||CB|^{-1}(e^{2\mu CB t_0} + 1)_| u|}{1 - (2|A|t_0 + 2|A||B||C|t_0|CB|^{-1}(e^{2\mu CB t_0} + 1))} .$$

The number μ is taken in such a way that $t_0 > 0$ and

$$2|A|t_0 + 2|A||B||C|t_0|CB|^{-1}(e^{2\mu CB t_0} + 1) < 1.$$

It is easily seen that

$$|C\dot{x}(t, u) - \mu CBx(t, u)| \leq |A||C| \max_{t \in [0, 2t_0]} |x(t, u)|$$

for all $t \in [0, 2t_0]$. Hence, for $t \in [0, 2t_0]$ we have

$$|Cx(t, u) - e^{\mu CBt} Cu| \leq \frac{1 - e^{2\mu CB t_0|}}{-\mu CB} \, |A||C| \max_{t \in [0, 2t_0]} |x(t, u)|.$$

Combining this with (27), we obtain

$$\left| x(t, u) - u - \frac{BCu}{CB} \, (e^{\mu CBt} - 1) \right| \leq$$

$$\leq \big(2|A|t_0 + 2|A||B||C|t_0|CB|^{-1}(e^{2\mu CB t_0} + 1)\big) \max_{t \in [0, 2t_0]} |x(t, u)|,$$

whence

$$|x(t, u)| \leq R, \qquad t \in [0, 2t_0],$$

$$\left| h^* x(t, u) - h^* u - \frac{h^* BCu}{CB} \, (e^{\mu CBt} - 1) \right| \leq$$
$$\leq \big(2|A|t_0 + 2|A||B||C|t_0|CB|^{-1}(e^{2\mu CB t_0} + 1)\big) R|h|, \quad t \in [0, 2t_0].$$

It is easy to check that

$$\big(2|A|t_0 + 2|A||B||C|t_0|CB|^{-1}(e^{2\mu CB t_0} + 1)\big) R|h| = O\left(\frac{1}{\mu}\right).$$

There fore, for large $|\mu|$ there exists $\tau \in [0, 2t_0]$ such that $h^* x(\tau, u) = 0$. The lemma is proved. □

Theorem 5. *Suppose that $B, C^* \in \mathbf{R}^n$, $CB = 0$, $\dim M_1 = 1$, $\dim L_2 = n - 1$, and inequality* (10) *is fulfilled.*

Then there exists a periodic function $K(t)$ such that system (1) *is asymptotically stable.*

Proof. By Lemmas 4 and 5, the controllability of (A, B) and the observability of (A, C) imply the observability of $((A+KBC), h)$ for any $K \neq K_2$ and the controllability of $((A+KBC), u)$

for any $K \neq K_1$. Here h is a normal to the hyperplane L_2, and u is a nonzero vector in the subspace M_1. It follows that $h^*B \neq 0$ and $Cu \neq 0$. Then, by Lemma 6, there exist numbers μ and $\tau(\mu)$ such that if $U(t) \equiv \mu$, then system (8) satisfies condition (9). □

Theorem 6. *Suppose that $CB \neq 0$ and the matrix A has a positive eigenvalue κ and $n-1$ eigenvalues with real parts less that $-\lambda$, where $\lambda > \kappa$. We also assume that*

$$\frac{CB}{\lim\limits_{p\to\kappa}(\kappa - p)W(p)} < 1.$$

Then there exists a periodic function $K(t)$ such that system (1) *is asymptotically stable.*

Proof. Without loss of generality, we may assume that the matrix A and the vectors B and C are of the form

$$A = \begin{pmatrix} \kappa & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad C = (c_1, c_2),$$

where A_2 is an $(n-1)\times(n-1)$-matrix, $b_2 \in \mathbf{R}^{n-1}$, and $c_2^* \in \mathbf{R}^{n-1}$. In this case, the normal h to the subspace $L_1 = L_2$ and the vector $u \in M_1 = M_2$ are of the form $h = u = \binom{1}{u}$. Therefore, by Lemma 7, there exist numbers μ and $\tau(\mu)$ such that if $U(t) \equiv \mu$, then condition (9) is fulfilled for system (8) provided that

$$\frac{CB}{c_1 b_1} < 1.$$

We easily see that $c_1 b_1 \neq 0$ because (A, B) is controllable and (A, C) is observable, and

$$c_1 b_1 = \lim_{p\to\kappa}(\kappa - p)W(p).$$

Thus, all assumptions of Theorem 1 are fulfilled, and, consequently, system (1) is stabilizable. □

Lemma 8. *Suppose that an $(n-2)$-dimensional linear subspace L invariant for system* (27) *lies in the hyperplane $\{h^*z = 0\}$. If $h^*B = CB = 0$, then the pair (A, h) is observable.*

Proof. The controllability of the pair (A, B) and the invariance of L with respect to (27) imply that $B \overline{\in} L$ (see [7]). Therefore, the linear hull of B and L coincides with the hyperplane $\{h^*z = 0\}$.

Suppose that the pair (A, h) is not observable. Then there is a vector $q \neq 0$ and a number γ such that

$$h^*q = 0, \quad Aq = \gamma q$$

(see [7]). The observability of the pair (A, C) implies the relation $Cq \neq 0$. Since L is invariant, we have

$$h^*(A + \mu BC)^k z = 0, \quad k = 0, 1, \ldots, , \quad z \in L.$$

The above arguments imply the existence of a number ν and a vector $z \in L$ such that

$$q = z + \nu B.$$

If $\nu \neq 0$, then for $k = 1$ we have $\nu h^*AB = 0$, whence $h^*AB = 0$. Combining this with the relations $h^*B = CB = 0$ for $k = 2$, we obtain $\nu h^*A^2B = 0$. Successively continuing this procedure for $k = 3, \ldots$, we obtain the relations $h^*A^kB = 0$. By the controllability of (A, B), these relations imply $h = 0$, which contradicts the definition of (A, h) the vector $\nu \neq 0$.

Thus, we have proved the observability of (A, h) for $\nu \neq 0$.

If $\nu = 0$, then we use the same arguments as in the proof of Lemma 4. □

Theorem 7. *Suppose that* $B, C^* \in \mathbf{R}^n$*,* $CB = 0$*,* $\dim M_1 = 1$*,* $\dim L_2 = n - 2$*, and inequality* (10) *is true. If the assumptions of Theorem 4 are fulfilled for some number* $U_0 \neq K_j$*, then there exists a periodic function* $K(t)$ *such that system* (1) *is asymptotically stable.*

Proof. Observe that, as $\mu \to \infty$, the integral manifold $\Omega(\mu)$ consisting of the trajectories $x(t, x_0)$ of system (27) with initial values $x_0 in L_2$ approaches the hyperplane $\{h^* x = 0\}$. Here h is a normal to the linear hull of L_2 and B. Such convergence was described in the proof of Lemma 6.

Lemma 7 implies that the pair $((A + U_0 BC), h)$ is observable, while Lemma 5 implies that the pair $((A + U_0 BC), u)$ with $u \in M_1$, $u \neq 0$ is controllable. By Lemma 3, this shows that for system (9) with $U(t) = U_0$ the sign of the function $h^* y(t, u)$ changes for some values of t. Therefore, for sufficiently large $|\mu|$, there exists a number $\tau_0(\mu) > 0$ such that $y(\tau_0(\mu), u) \in \Omega(\mu)$.

Perturbing the right-hand side of (1) slightly, we can ensure that $Cy(\tau_0(\mu), u) \neq 0$ (asymptotic stability is preserved under small perturbations of the right-hand sides of periodic systems).

Next, if in (9) we set $U(t) = \mu$ (or $U(t) = -\mu$) on $(\tau_0, \tau]$, then, moving along the manifold $\Omega(\mu)$, we reach the set L_2 at the moment $t = \tau$. Here the sign of μ is chosen so that $\tau > \tau_0$ (see the proof of Lemma 6).

Now, we have $y(\tau, u) \in L_2$, and, consequently, condition (8) is fulfilled.

Thus, all assumptions of Theorem 1 are satisfied. □

§3. Necessary stabilization conditions

Now, we pass to conditions necessary for stabilization.

We consider the case where B is a column vector, C is a row vector, and $K(t) : \mathbf{R}^1 \to \mathbf{R}^1$ is a piecewise continuous function. This case in important for control theory.

As before, $W(p)$ is the transfer function of system (1):

$$W(p) = C^*(A - pI)^{-1} B = \frac{c_n p^{n-1} + \cdots + c_1}{p^n + a_n p^{n-1} + \cdots + a_1},$$

where the c_j and a_j are real numbers. If the transfer function $W(p)$ is nondegenerate, then system (1) can be written in the following scalar form (see [8]):

$$\begin{aligned} \dot{x}_1 &= x_2, \\ &\vdots \\ \dot{x}_{n-1} &= x_n, \\ \dot{x}_n &= -(a_n x_n + \cdots + a_1 x_1) - K(t)(c_n x_n + \cdots + c_1 x_1). \end{aligned} \tag{28}$$

Clearly, here we have

$$C = (c_1, \ldots, c_n), \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & 1 \\ -a_1 & \cdots & \cdots & -a_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}.$$

We recall that the transfer function $W(p)$ is nondegenerate if and only if the polynomials

$$\begin{array}{c} c_n P^{n-1} + \cdots + c_1, \\ p^n + a_n p^{n-1} + \cdots + a_1 \end{array}$$

have no common zeros.

In what follows, we assume that $c_n \neq 0$. In this case, without loss of generality we may put $c_n = 1$.

Theorem 8. *Assume that the following conditions are satisfied:*

1) *for $n > 2$, we have $c_1 \leq 0, \ldots, c_{n-2} \leq 0$;*

2)

$$\begin{array}{ll} c_1(a_n - c_{n-1}) & > a_1, \\ c_1 + (a_n - c_{n-1})c_2 & > a_2, \\ & \vdots \\ c_{n-2} + (a_n - c_{n-1})c_{n-1} & > a_{n-1}. \end{array}$$

Then there is no function $K(t)$ for which system (1) *is asymptotically stable.*

Proof. Consider the set

$$\Omega = \{x_1 \geq 0, \ldots, x_{n-1} \geq 0, \;\; x_n + c_{n-1}x_{n-1} + \ldots + c_1 x_1 \geq 0\} \subset \mathbf{R}^n.$$

We prove that Ω is positively invariant, i.e., if $x(t_0) \in \Omega$, then $x(t) \in \Omega$ for all $t \geq t_0$.

Observe that if $j = 1, \ldots, n-1$ and τ is such that

$$\begin{array}{l} x_j(\tau) = 0, \quad x_i(\tau) > 0, \quad i \neq j, \;\; i \leq n-1, \\ x_n(\tau) + c_{n-1}x_{n-1}(\tau) + \cdots + c_1 x_1(\tau) > 0, \end{array}$$

then

$$\dot{x}_j(\tau) > 0. \tag{29}$$

Indeed, for $j = 1, \ldots, n-2$ we have

$$\dot{x}_j(\tau) = x_{j+1}(\tau) > 0.$$

For $n = 2$, we have

$$\dot{x}_1(\tau) = x_2(\tau) > -c_1 x_1(\tau) = 0,$$

and for $n > 2$ we have

$$\dot{x}_{n-1}(\tau) = x_n(\tau) > -c_{n-2}x_{n-2}(\tau) - c_1 x_1(\tau) \geq 0.$$

Next, we observe that if $x_n(\tau) + c_{n-1}x_{n-1}(\tau) + \cdots + c_1 x_1(\tau) = 0$, and $x_j(\tau) > 0, \;\; j = 1, \ldots, n-1$, then

$$\big(x_n(\tau) + c_{n-1}x_{n-1}(\tau) + \cdots + c_1 x_1(\tau)\big)^{\bullet} > 0. \tag{30}$$

Indeed, we have

$$\begin{array}{l} \big(x_n(\tau) + c_{n-1}x_{n-1}(\tau) + \cdots + c_1 x_1(\tau)\big)^{\bullet} = \\ = (-a_{n-1} + c_{n-2} + (a_n - c_{n-1})c_{n-1})x_{n-1}(\tau) + \cdots \\ \cdots + (-a_2 + c_1 + (a_n - c_{n-1})c_2)x_2(\tau) + \\ +(-a_1 + (a_n - c_{n-1})c_1)x_1(\tau). \end{array}$$

and it remains to use condition 2) of the theorem.

Inequalities (29) and (30) imply that the boundary of the set Ω is transversal to the vector field of (28) almost everywhere, and the solutions of (28) pierce the boundary inward Ω almost everywhere. Since the solutions of (28) depend on the initial values continuously, we see that the set Ω is positively invariant, whence it easily follows that system (28) is not asymptotically stable. The theorem is proved. □

Another condition ensuring that system (1) is nonstable is well known (see [5]):

$$\mathrm{Tr}\,(A + BK(t)C) \geq \alpha > 0, \quad t \in \mathbf{R}^1.$$

§4. Stabilization of systems of order two and three

Now we show how the above results can be applied to the case where $n = 2$, B is a column vector, C is a row vector, and $K(t)$ is a scalar function.

We introduce the transfer function of system (1):

$$W(p) = C(A - pI)^{-1}B = \frac{\rho p + \gamma}{p^2 + \alpha p + \beta}\,.$$

Here p is a complex variable.

In what follows, we assume that $\rho \neq 0$. Furthermore, there is no loss of generality in assuming that $\rho = 1$. We also assume that the function $W(p)$ is nondegenerate, i.e., we have

$$\gamma^2 - \alpha\gamma + \beta \neq 0.$$

It is well known (see [8]) that in this case system (1) can be written as follows:

$$\begin{aligned} \dot{\sigma} &= \eta, \\ \dot{\eta} &= -\alpha\eta - \beta\sigma - K(t)(\eta + \gamma\sigma). \end{aligned} \tag{31}$$

It is easy to see that stabilization of system (31) with the help of a constant matrix $K(t) \equiv K_0$ is possible if and only if

$$\alpha + K_0 > 0, \quad \beta + \gamma K_0 > 0.$$

A number K_0 satisfying these inequalities exists if and only if $\gamma > 0$, or $\gamma \leq 0$ and $\alpha\gamma < \beta$.

We consider the case where stabilization with the help of a constant $K(t) \equiv K_0$ is impossible:

$$\gamma \leq 0, \quad \alpha\gamma > \beta.$$

We apply Theorem 3. Clearly, condition 1) of Theorem 3 is fulfilled, because $\det BK_0C = 0$ and $\mathrm{Tr}\,BK_0C = K_0CB = -K_0 \neq 0$.

Condition 2) of Theorem 3 is fulfilled if for some U_0 the polynomial

$$p^2 + \alpha p + \beta + U_0(p + \gamma)$$

has complex zeros. We easily see that such a number U_0 sexists if and only if

$$\gamma^2 - \alpha\gamma + \beta > 0. \tag{32}$$

Thus, if inequality (32) if fulfilled, then there exists a periodic function $K(t)$ such that system (32) is asymptotically stable.

The same result can be obtained with the help of Theorem 6.

For this, without loss of generality we assume that $\alpha > 0$. This can always be achieved if we properly choose K_0 in the expression

$$-(\alpha + K_0)\eta - (\beta + \gamma K_0)\sigma - \big(K(t) - K_0\big)(\eta + \gamma\sigma)$$

and change the notation: $\alpha + K_0 \to \alpha$, $\beta + \gamma K_0 \to \beta$, and $K(t) - K_0 \to K(t)$. The inequality $\alpha > 0$ implies that $\lambda > \kappa$. Here

$$\frac{CB}{\lim\limits_{p\to\kappa}(\kappa - p)W(p)} = \frac{\kappa + \lambda}{\kappa + \gamma}.$$

Therefore, all conditions of Theorem 6 are fulfilled if

$$(\lambda - \gamma)(\kappa + \gamma) = -\gamma^2 + \alpha\gamma - \beta < 0.$$

This inequality coincides with (32). It is easy to see that if

$$\gamma^2 - \alpha\gamma + \beta < 0, \tag{33}$$

then the conditions of Theorem 8 are also fulfilled.

Thus, the following result is true.

Theorem 9 [9]. *If inequality* (32) *is fulfilled, then there exists a periodic function* $K(t)$ *such that system* (31) *is asymptotically stable.*

If inequality (33) *is fulfilled, then there are no functions* $K(t)$ *for which system* (31) *is asymptotically stable.* □

This result was also obtained in [10] for a different class of stabilizing functions $K(t)$ of the form

$$K(t) = (k_0 + k_1\omega\cos\omega t), \quad \omega \gg 1,$$

with the help of averaging.

Now we consider systems (1) of order three with various transfer functions.

4.1. $W(p) = \dfrac{1}{p^3 + ap^2 + bp + c}$, where a, b, and c are some numbers.

If $a > 0$ and $b > 0$, then stationary stabilization is possible. Suppose $a > 0$ and $b \le 0$. In this case, stationary stabilization is impossible; we apply Theorem 7.

Clearly, if U_0 is sufficiently large, then the polynomial

$$p^3 + ap^2 + bp + U_0 + c$$

has one negative zero and two complex zeros $\lambda_0 \pm i\omega_0$, $\lambda_0 > 0$. We take K_1 so that

$$p^3 + ap^2 + bp + K_1 + c = (p - \kappa)(p^2 + \alpha_1 p + \beta_1)$$

with $\alpha_1 = a + \kappa_1$ and $\beta_1 = b + (a + \kappa_1)\kappa_1$. For κ_1 large, the polynomial $p^2 + \alpha_1 p + \beta_1$ has complex zeros with real part equal to $-(a + \kappa_1)/2$. We take K_2 so that

$$p^3 + ap^2 + bp + K_2 + c = (p + \lambda_2)(p^2 + \alpha_2 p + \beta_2)$$

with $\alpha_2 = a - \lambda_2$ and $\beta_2 = b - (a - \lambda_2)\lambda_2$. For λ_2 large, the polynomial $p^2 + \alpha_2 p + \beta_2$ has complex zeros with real part equal to $(\lambda_2 - a)/2$.

We have

$$\dim M_1 = \dim L_2 = 1, \quad \dim M_2 = \dim L_1 = 2,$$

$$\lambda_1 = \frac{a+\kappa_1}{2}\,, \quad \kappa_2 = \frac{\lambda_2 - a}{2}\,,$$

$$\lambda_1\lambda_2 - \kappa_1\kappa_2 = a(\lambda_2 + \kappa_1) > 0.$$

Thus, all conditions of Theorem 7 are fulfilled.

Since

$$\mathrm{Tr}\,(A + BK(t)C) = -a,$$

asymptotic stability is impossible for $a < 0$.

So, we can state the following result.

Theorem 10. *For $a > 0$ system* (1) *is stabilizable. For $a < 0$ stabilization is impossible.* □

4.2. $W(p) = \dfrac{p}{p^3 + ap^2 + bp + c}\,.$

If $a > 0$ and $c > 0$, stationary stabilization is possible. We consider the case where $a > 0$ and $c < 0$, and apply Theorem 5 with $K_1 = K_2$, $\lambda_1 = \lambda_2 = \lambda$, and $\kappa_1 = \kappa_2 = \kappa$. We take K_1 so that

$$(p-\kappa)(p^2 + \alpha p + \beta) = p^3 + ap^2 + (K_1 + b)p + c$$

with $\alpha = a + \kappa$ and $\beta = -c/\kappa$. If κ is small, the polynomial $p^2 + \alpha p + \beta$ has complex zeros with real part equal to $-(a+\kappa)/2$. Thus,

$$M_1 = M_2, \quad L_1 = L_2, \quad \dim M_1 = 1, \quad \dim L_2 = 2, \quad \text{and} \quad \lambda = (a+\kappa)/2.$$

Clearly, for small κ we have $\lambda > \kappa$. Since $\mathrm{Tr}\,(A + BK(t)C) = -a$ for $a < 0$, asymptotic stability is impossible.

This leads to the following result.

Theorem 11. *Suppose $a \neq 0$ and $c \neq 0$. Then system* (1) *is stabilizable if and only if $a > 0$.* □

4.3. $W(p) = \dfrac{p^2}{p^3 + ap^2 = bp + c}\,.$

If $b > 0$ and $c > 0$, stationary stabilization is possible. Theorem 8 shows that if $b < 0$ and $c < 0$, then stabilization is impossible.

We consider the case where $b >$) and $c < 0$ and apply Theorem 1.

Putting $K_1 = K_2$, $\lambda_1 = \lambda_2 = \lambda$, and $\kappa_1 = \kappa_2 = \kappa$, we take K_1 so that

$$(p-\kappa)(p^2 + \alpha p + \beta) = p^3 + (a + K_1)p^2 + bp + c.$$

Here

$$\alpha = \frac{-(c+\kappa b)}{\kappa^2}\,, \quad \beta = \frac{-c}{\kappa}\,.$$

Below it is assumed that κ is small. In this case, we define

$$\lambda = -\frac{(c+\kappa b)}{2\kappa^2} - \sqrt{\frac{(c+\kappa b)^2}{4\kappa^4} + \frac{c}{\kappa}}\,.$$

Obviously, $\lambda > \kappa$ for $b + \kappa^2 > 0$, and, consequently, condition (10) of Theorem 1 is fulfilled.

Now we construct $U(t)$ so as to ensure (8).

We introduce a number U_0 such that

$$(p-\nu)(p^2 + \alpha_1 p + \beta_1) = p^3 + (a + U_0)p^2 + bp + c.$$

Here

$$\alpha_1 = \frac{-(c+\kappa b)}{\nu^2}\,, \quad \beta_1 = \frac{-c}{\nu}\,.$$

Suppose that ν is sufficiently large. Without loss of generality, we assume that

$$A+U-)BC = \begin{pmatrix} \nu & 0 \\ 0 & Q \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad C = (c_1, c_2).$$

Here Q is a (2×2)-matrix, and b_2 and c_2 are two-dimensional vectors. For ν large, the matrix Q has complex eigenvalues. Therefore, for any nonzero vector $u \in M_1$ there exists $\tau_1 > 0$ such that the vectors

$$B, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [\exp(A+U_0BC)\tau_1]u$$

lie in one plane. It follows that for some ρ we have

$$[\exp(A+U_0BC)\tau_1]u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \rho \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

If $b_1 > 0$, then $\rho \in [-b_1^{-1}, 0]$, and if $b_1 < 0$, then $\rho \in [0, -b_1^{-1}]$.

The relations

$$\frac{CB}{c_1b_1} = \frac{-1}{\lim\limits_{p\to\nu}(\nu-p)W(p)} = 1 + \frac{\alpha_1\nu+\beta_1}{\nu^2} = 1 - \frac{2c}{\nu^3} - \frac{b}{\nu^2} < 1$$

imply that

$$\frac{CB(1+\rho b_1)}{c_1b_1} < 1 \tag{34}$$

for sufficiently large ν. By Lemma 8, the above inequality implies the existence of numbers μ and $\tau(\mu)$ such that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^* [\exp(A+(U_0+\mu)BC)\tau(\mu)][\exp(A+U_0BC)\tau_1]u = 0.$$

Since the planes L_2 and $\big\{x \mid big(\begin{smallmatrix}1\\0\end{smallmatrix}\big)^* x = 0\big\}$ intersect, and the matrix Q has complex eigenvalues, it follows that

$$[\exp(A+U_0BC)\tau_2][\exp(A+(U_0+\mu)BC)\tau(\mu)][\exp(A+U_0BC)\tau_1]u \in L_2,$$

for some $\tau_2 > 0$.

Thus, the inclusion (8) is valid for the function

$$U(t) = \begin{cases} U_0, & t \in [0, \tau_1), \\ U_0+\mu, & t \in [\tau_1, \tau_1+\tau(\mu)), \\ U_0, & t \in [\tau_1+\tau(\mu), \tau_1+\tau(\mu)+\tau_2, \end{cases}$$

where $\tau = \tau_1 + \tau(\mu) + \tau_2$.

We have proved the following result.

Theorem 12. *Suppose that $b \neq 0$ and $c < 0$. Then system* (1) *is stabilizable if and only if $b > 0$.* □

References

1. R. Brockett, *A stabilization problem*, Open Problems in Mathematical Systems and Control Theory. Springer, London, 1999, pp. 75-78.
2. L. Zadeh and Ch. Desoer, *Linear system theory*, McGraw-Hill Book Co., Inc., New York etc., 1963.
3. A. A. Pervozvanskii, *A course in automatic control theory*, "Nauka", Moscow, 1986. (Russian)
4. Yu. A. Mitropol'skii, *Averaging method in nonlinear mechanics*, "Naukova Dumka", Kiev, 1971. (Russian)
5. V. I. Arnol'd, *Ordinary differential equations*, "Nauka", Moscow, 1971; English transl. Of 3rd ed., Springer-Verlag, Berlin, 1992.
6. ——, *Mathematical methods of classical mechanics*, "Nauka", Moscow, 1979; English transl., 2nd ed., Grad. Texts in Math., vol. 60, Springer-Verlag, New York- Berlin, 1989.
7. G. A. Leonov, *Mathematical problems of control theory. Motivation for analysis*, S.-Peterburg. Gos. Univ., St. Petersburg, 1999. (Russian)
8. S. Lefschetz, *Stability of nonlinear control systems*, Academic Press, New York-London, 1965.
9. G. A. Leonov, *The Brockett stabilization problem*, International Conference Control of Oscillations and Chaos (July 2000, St. Petersburg), S.-Peterburg. Gos. Univ., St. Petersburg, 2000, pp. 38-39. (English)
10. L. Moreau and D. Aeyels, *Stabilization by means of periodic output feedback*, Conference of Decision and Control (Phoenix, Arizona USA, December 1999), Phoenix Univ., Phoenix, 1999, pp. 108-109.

St. Petersburg State University, Department of Mathematics and Mechanics, Bibliotechnaya pl., 2, Stary Peterhof, St. Petersburg, 198904, Russia

E-mail address: leonov @ math.spbu.ru

Received 27/MAR/2001

Translated by N.YU. NETSVETAEV