Localization of hidden oscillations in dynamical systems

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Outline

- Global attractor, limit cycles and homoclinic orbits in quadratic systems
  - One-dimensional case
  - Two-dimensional case (16th Hilbert problem)
  - Tree-dimensional case (Lorenz system)
- Harmonic Linearization and Describing Function Method
  - Justification of Describing Function Method
  - Analytically-numerical method for periodic solution localization
  - Aizerman's and Kalman's hypotheses
  - DFM in the critical case
  - Counterexample for Kalman's conjecture
  - Chua circuits
- Main publications

Quadratic systems: one-dimensional case

$$\dot{x} = ax + x^2 \quad x \in R^1$$

No global attractor. No limit cycles. No homoclinic orbits.

Quadratic systems: two-dimensional case

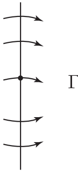
$$\dot{x} + a_1x^2 + b_1xy + c_1y^2 + \alpha_1 x + \beta_1 x$$
$$\dot{y} = a_2x^2 + b_2xy + c_2y^2 + \alpha_2 x + \beta_2 y$$

Without loss of generality

$$c_1 = \alpha_1 = 1, \ a_1 = b_1 = \beta_1 = 1$$
$$c_2 \neq 0, \ c_2 \neq -1, \ c_2 \neq b_2 - a_2$$

Positively invariant half plane

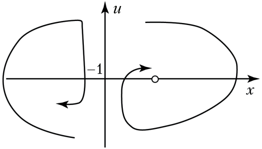
$$\Gamma = \{x > -1, \, r \in R^1\}$$

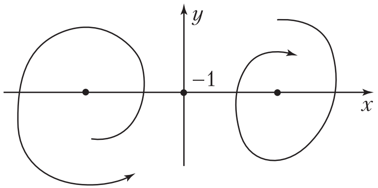


Transformation of Quadratic system to Lienard system

$$\dot{x} = u, \ \dot{u} = -f(x)u - g(x) \quad u = \left(y + \frac{x^2}{(x+1)}\right)|x+1|^q$$
$$f(x) = [(2c_2 - b_2 - 1)x^2 - (2 + b_2 + \beta_2) - \beta_2]|x+1|^{q-2}, \ q = -c_2$$
$$g(x) = [-x(x+1)^2(a_2x + \alpha_2) + x^2(x+1)(b_2x - \beta_2) - c_2x^4]\frac{|x+1|^{2q}}{(x+1)^3}$$

Theorem. Boundedness of $x(t), y(t)$ in $\Gamma \Leftrightarrow$ $c_2 \in (0,1), c_2 < b_2 - a_2$ and either $2c_2 > b_2 + 1$ or $2c_2 \leq b_2 + 1, \ 4a_2(c_2 - 1) > (b_2 - 1)^2$.





No global attractor in generic case of two-dimensional quadratic system

Theorem. If $c_2 \in (0,1)$, $b_2 > a_2 + c_2$, $2c_2 > b_2 + 1$ and only one unstable equilibria $x = y = 0$ in Γ, then there is limit cycle in Γ.

Theorem. If $c_2 \in (0,1)$, $b_2 > a_2 + c_2$, $2c_2 > b_2 + 1$, $4a_2(c_2 - 1) > (b_2 - 1)^2$ and only two unstable equilibriums — one in Γ and one in $R^2 \setminus \Gamma$, then there are two limit cycle — one in Γ and one in $R^2 \setminus \Gamma$.

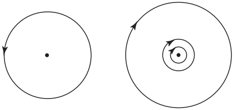
16th Hilbert Problem

$$\dot{x} + a_1x^2 + b_1xy + c_1y^2 + \alpha_1x + \beta_1x$$
$$\dot{y} = a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y$$
$$c_1 = \alpha_1 = 1,\ a_1 = b_1 = \beta_1 = 1$$
$$c_2 \neq 0,\ c_2 \neq -1,\ c_2 \neq b_2 - a_2$$

$c_2 \in (0,1),\ b_2 > a_2 + c_2,$
$2c_2 > b_2 + 1,\ 4a_2(c_2 - 1) > (b_2 - 1)^2$

One zero Lyapunov quantity $\Rightarrow$
2 small limit cycles (Bautin, 1952)

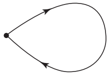
(1900) Number of limit cycles in
two-dimensional polynomial systems



Homoclinic orbit

(Bogdanov, 1976): For $\mu = -5/7$ there is homoclinic orbit

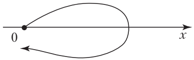
$$\dot{x} = y$$
$$\dot{y} = -y(-\mu + 1 + x) + x(x + 2)$$



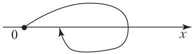
$$\ddot{x} + f(x)\dot{x} \neq g(x) = 0$$
$$f(x) = (-\beta_2(x+1)^2 + (2c_2 - 1)x^2 - 2x)|x+1|^{q-2}$$
$$g(x) = (-\alpha_2 x(x+1) - c_2 x^4)\frac{|x+1|^{2q}}{(x+1)^3}$$

$$c_1 = \alpha_1 = 0,$$
$$a_1 = b_1 = \beta_1 = 1$$
$$a_2 = b_2 = \beta_2,$$
$$c_2 < 0, \alpha_2 > 0$$

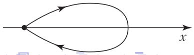
$$\beta_2 = 2c_2 - 1 : \ \ f(x) < 0 \ \forall x > 0$$



$$\beta_2 = 2c_2 - 2 : \ \ f(x) > 0, \ \ \forall x > 0$$



$$\exists \bar{\beta}_2 \in (2c_2 - 2, 2c_2 - 1))$$



Lorenz System

$$\dot{x} = \sigma(x - y), \ \dot{y} = rx - y - xz, \ \dot{z} = -bz + xy$$

Theorem. For any solution of Lorenz system we have

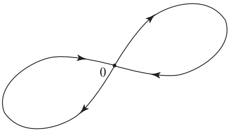
$$\limsup_{t \to +\infty} (y(t)^2 + (z(t) - r)^2) \le \ell^2 r^2$$

$$\limsup_{t \to +\infty} |x(t)| \le \ell r$$

$$\ell = \begin{cases} 1 & \text{if} \quad b \le 2 \\ \dfrac{b}{2\sqrt{b-1}} & \text{if} \quad b > 2 \end{cases}$$

Corollary. Global attractor of Lorenz system exists and locates in $|x| \le \ell r, \quad y^2 + (z-r)^2 \le \ell^2 r^2$
[Leonov's column formula (Lixin Tian et al., 2006)]

Theorem (Leonov, Chen). Given b and σ fixed, for the existence of $r \in (1, +\infty)$ corresponding to the homoclinic orbit it is necessary and sufficient that $2b + 1 < 3\sigma$
[Leonov-Chen inequality (Magnitskii&Sidorov, 2006)]



Lyapunov dimension of Lorenz system global attractor

$$X(t) = (x(t), y(t), z(t))$$

Lyapunov exponents: $\lambda_1(X(\cdot)) \leq \lambda_2(X(\cdot)) \leq \lambda_3(X(\cdot))$
$X(\cdot)$ $S(X(\cdot))$ $\quad \lambda_1 + \lambda_2 + S\lambda_3 = 0$

$$\dim_L K = \sup_K (2 + S(X(\cdot))), \quad X(t) \subset K.$$

Theorem. Suppose the inequalities

$$\sigma + 1 \geq b \geq 2,$$

$$r\sigma^2(4 - b) + 2\sigma(b - 1)(2\sigma - 3b) > b(b - 1)^2$$

are valid. Then

$$\dim_L K = 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma - 1) + 4r\sigma}}.$$

$$\dim_T K \leq \dim_H K \leq \dim_F K \leq \dim_L K.$$

Describing Function Method (DFM)

$$\dot{x} = Px + q\psi(r^*x), \ \psi(0) = 0,$$

$$W(p) = r^*(P - pI)^{-1}q$$

$$\mathrm{Im}W(i\omega_0) = 0$$

$$k = -(\mathrm{Re}W(i\omega_0))^{-1}$$

$$\dot{x} = (P + kqr^*)x + q\varphi(r^*x)$$

$$\varphi(\sigma) = \psi(\sigma) - k\sigma$$

$$P + kqr^* : \lambda_{1,2} = \pm i\omega_0$$

$$\mathrm{Re}\lambda_{j>2} < 0$$

Periodic solution: $\sigma(t) = r^*x(t) \approx a\cos\omega_0 t$

$$a : \int_0^{2\pi/\omega_0} \psi(a\cos\omega_0 t)\cos\omega_0 t dt = ka\int_0^{2\pi/\omega_0} (\cos\omega_0 t)^2 dt$$

Justification of Describing Function Method

$$\dot{x} = (P + kqr^*)x + q\varepsilon\varphi(r^*x)$$
$$\varphi(\sigma) = \psi(\sigma) - k\sigma, \ \psi(0) = 0$$
$$P + kqr^* : \ \lambda_{1,2} = \pm i\omega, \ \text{Re}\lambda_{j>2} < 0$$

$$\dot{x}_1 = -\omega_0 x_2 + b_1\varepsilon\varphi(x_1 + c^*x_3)$$
$$\dot{x}_2 = \omega_0 x_1 + b_2\varepsilon\varphi(x_1 + c^*x_3)$$
$$\dot{x}_3 = Ax_3 + b\varepsilon\varphi(x_1 + c^*x_3)$$

$\varphi(\sigma)$ – piecewise continuous function ν_j – break points

$\varepsilon_j = j/m, \ m : \ \varphi_0(\sigma) = \varepsilon_0\varphi(\sigma), \dots, \varphi_m(\sigma) = \varepsilon_m\varphi(\sigma).$

$$W(p) = r^*(P_0 - pI)^{-1}q = \frac{-b_1 p + b_2\omega_0}{p^2 + \omega_0^2} + c^*(A - pI)^{-1}b$$
$$K(a) = \int_0^{2\pi/\omega_0} \varphi(\cos(\omega_0 t)a)\cos(\omega_0 t)dt$$

Theorem [Leonov, 2008]. If $K(a) = 0, \ b_1\frac{dK(a)}{da} > 0, a \neq \pm\nu_j$, then for sufficiently small $\varepsilon > 0$ there is T-periodic solution such that $r^*x(t) = a\cos(\omega_0 t) + O(\varepsilon), \ T = \frac{2\pi}{\omega_0} + O(\varepsilon)$
This periodic solution is stable in the sense that there exists its certain neighborhood such that all solutions with initial data from this neighborhood remain in it with increasing time t.

Algorithm

$$\frac{dx}{dt} = (P + kqr^*)x + q\varphi(r^*x)$$

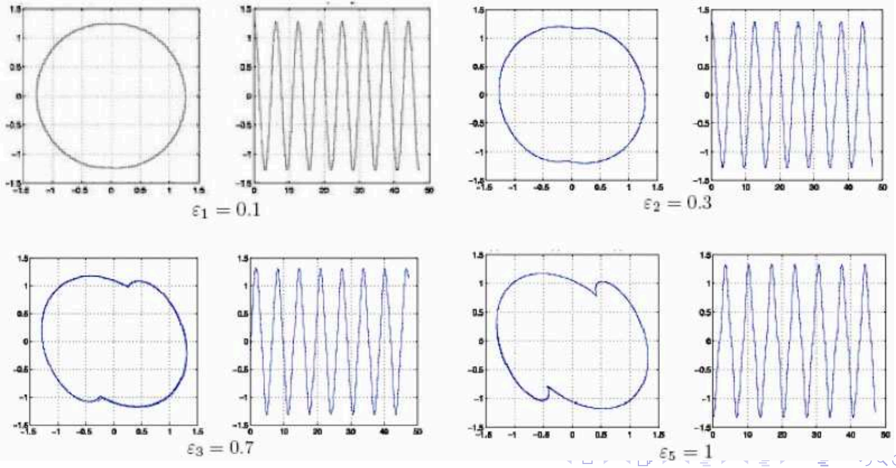
Consider finite sequence of the functions

$$\varphi_0(\sigma) = \varepsilon_0\varphi(\sigma), \varphi_1(\sigma) = \varepsilon_1\varphi(\sigma), \ldots, \varphi_m(\sigma) = \varepsilon_m\varphi(\sigma).$$

1) If periodic solution $x_0(t)$ is contained in the domain of attraction of the periodic solution $x_1(t)$ of the system with $\varphi_1(\sigma)$, then the solution of system with φ_1 can be sent out from $x_0(0)$ and after transient process the computational procedure "results" in computing the periodic solution $x_1(t)$.
2) Further we also compute the periodic solution $x_2(t)$, making use of the solution of system with $\varphi_2(\sigma)$ with the initial data $x_2(0)$. And so on up to $x_m(t)$ of system with $\varphi_m(\sigma) = \varphi(\sigma)$.
3) At a certain step the bifurcation of stability loss and the stopping of this algorithm is possible.

Example 1

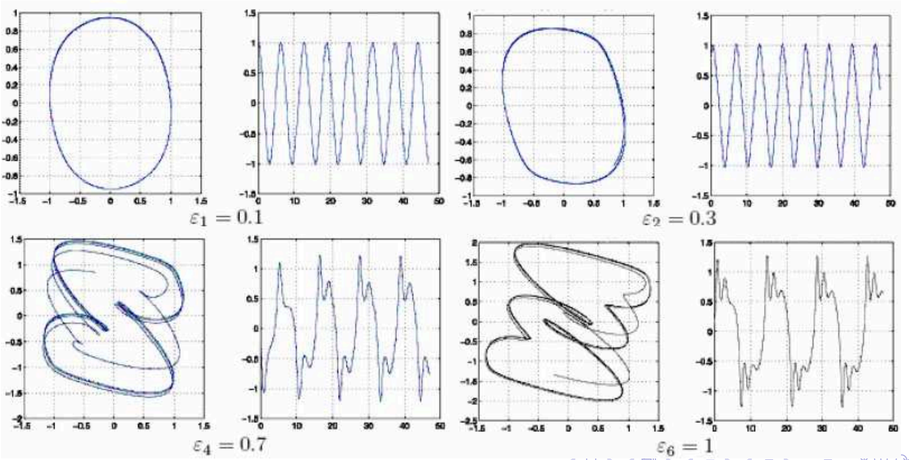
$\varphi(\sigma) = \sigma - \text{sign}\,\sigma$
transfer function $W(p) = \frac{p+1}{p^2+1} - \frac{1}{p+1}$



$\varepsilon_1 = 0.1$ $\varepsilon_2 = 0.3$ $\varepsilon_3 = 0.7$ $\varepsilon_5 = 1$

Example 2

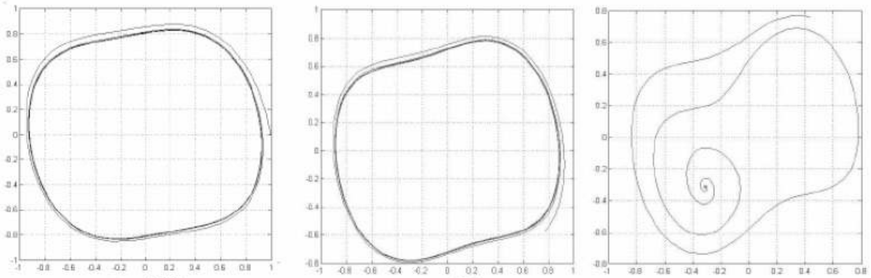
$\varphi(\sigma) = -3\sigma + 4\sigma^3$
transfer function $W(p) = \frac{p+1}{p^2+1} - \frac{1}{p+1}$



Example 3: periodic solution destroying

$\varphi(\sigma) = -3\sigma + 4\sigma^3$

transfer function $W(p) = \frac{p+1}{p^2+1} - \frac{1}{p+1}$



$\varepsilon = 0.25$ $\varepsilon = 0.3$ $\varepsilon = 0.35$

Aizerman's and Kalman's hypotheses

Let $\dot{x} = (A + kBC^*)x, \; x \in R^n \; \forall k \in (k_1, k_2)$, is stable

$$\dot{x} = Ax + B\varphi(C^*x), \quad \varphi(0) = 0$$

$$1949: \; k_1 < \varphi(\sigma)/\sigma < k_2$$

- N.P.Erugin, I.G.Malkin, N.N.Krasovsky (1952): proved for $n = 2$.
- V.A Pliss (1958): for $n = 3$ counterexample

$$1957: \; k_1 < \varphi'(\sigma) < k_2$$

- R.E. Fitts (1966): for $n = 4$ counterexample

DFM and Aizerman Problem

$$\dot{x} = Px + q\psi(r^*x), \ \psi(0) = 0, \qquad \dot{x} = (P + kqr^*)x + q\varphi(r^*x)$$

$$W(p) = r^*(P - pI)^{-1}q \qquad \varphi(\sigma) = \psi(\sigma) - k\sigma$$

$$\mathrm{Im}W(i\omega_0) = 0 \qquad P + kqr^* : \lambda_{1,2} = \pm i\omega_0$$

$$k = -(\mathrm{Re}W(i\omega_0))^{-1} \qquad \mathrm{Re}\lambda_{j>2} < 0$$

Periodic solution: $\sigma(t) = r^*x(t) \approx a\cos\omega_0 t$

$$a : \int_0^{2\pi/\omega_0} \psi(a\cos\omega_0 t)\cos\omega_0 t dt = ka\int_0^{2\pi/\omega_0} (\cos\omega_0 t)^2 dt$$

Aizerman: $\psi(\sigma) = \mu\sigma, \ \mu \in (\mu_1, \mu_2), \ \mu_1 < \psi(\sigma)/\sigma < \mu_2$
$k : k > \mu_2$ or $k < \mu_1 \Rightarrow \forall\sigma \neq 0 \ k\sigma^2 > \psi(\sigma)\sigma$ or $k\sigma^2 < \psi(\sigma)\sigma$

$$\forall a \neq 0 : \int_0^{2\pi/\omega_0} (\psi(a\cos\omega_0 t)\cos\omega_0 t - ka(\cos\omega_0 t)^2)dt \neq 0$$

DFM in the critical case

$$\dot{x}_1 = -\omega_0 x_1 + b_1 \varphi_0(x_1 + c^* x_3) \qquad \varphi_0(\sigma) = \mu\sigma,\ \forall\, \sigma \in (-\varepsilon, \varepsilon)$$
$$\dot{x}_2 = \omega_0 x_1 + b_2 \varphi(x_1 + c^* x_3) \qquad \varphi_0(\sigma) = M\varepsilon^3,\ \forall\, \sigma > \varepsilon$$
$$\dot{x}_3 = A x_3 + b\varphi(x_1 + c^* x_3) \qquad \varphi_0(\sigma) = -M\varepsilon^3,\ \forall\, \sigma < -\varepsilon$$

A — stable $(n-2) \times (n-2)$-matrix, b, c — $(n-2)$-vectors.

Theorem. [Leonov, 2008] If $b_1 < 0,\ \mu b_2 \omega_0 (c^* b + b_1) > -b_1 \omega_0^2$ then the system has a periodic solution with the initial data

$$x_1(0) = O(\varepsilon^2), \quad x_3(0) = O(\varepsilon^2),$$
$$x_2(0) = -\sqrt{\frac{\mu(b_2(c^* b + b_1)\mu + b_1\omega_0)}{3\omega_0 M(-b_1)}} + O(\varepsilon)$$

Counterexample for Kalman's conjecture

$$\dot{x}_1 = -x_2 - 0.5\varphi(x_1 + x_3 + 0.5x_4 + 2x_5)$$
$$\dot{x}_2 = x_1 - 2\varphi(x_1 + x_3 + 0.5x_4 + 2x_5)$$
$$\dot{x}_3 = -x_3 - 0.5\varphi(x_1 + x_3 + 0.5x_4 + 2x_5)$$
$$\dot{x}_4 = -4x_4 - \varphi(x_1 + x_3 + 0.5x_4 + 2x_5)$$
$$\dot{x}_5 = -2x_5 - 2\varphi(x_1 + x_3 + 0.5x_4 + 2x_5)$$

$$\varphi(\sigma) = 0.45\sigma, \ \forall\sigma \in (-\varepsilon, \varepsilon)$$
$$\varphi(\sigma) = 0.45\varepsilon^3, \ \forall\sigma > \varepsilon$$
$$\varphi(\sigma) = -0.45\varepsilon^3, \ \forall\sigma < -\varepsilon$$

Stability sector $(0, 0.46)$

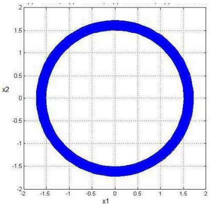


Figure : $\varepsilon = 0.1$

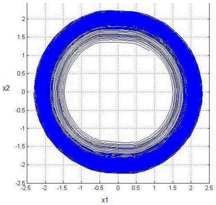


Figure : $\varepsilon = 0.4$

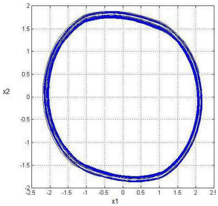


Figure : $\varepsilon = 0.7$

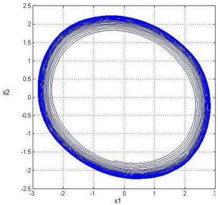


Figure : $\varepsilon = 1.0$

Leonov G.A., Bragin V.O., Algorithm for constructing counterexamples to the Kalman problem, Doklady math., 2010

Chua circuits

$$\dot{x} = a(y - x - \psi(x))$$
$$\dot{y} = x - y + z$$
$$\dot{z} = -by$$

$$\psi(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x + 1| - |x - 1|)$$

Chua circuits: road to chaos

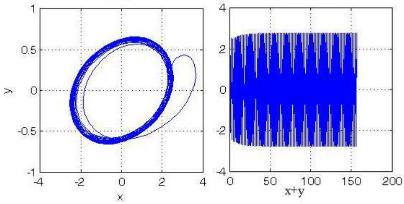


Figure : $\varepsilon = 0.1$

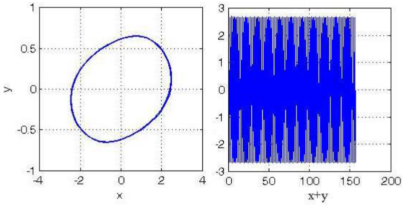


Figure : $\varepsilon = 0.4$

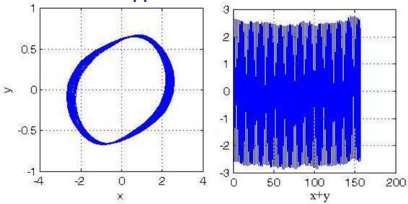


Figure : $\varepsilon = 0.7$

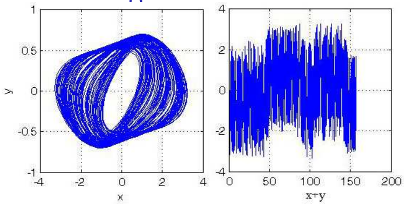


Figure : $\varepsilon = 1.0$

Example (Kuznetsov N.V.): $a = 15, b = 28, m_1 = -8/7, m_0 = -5/7 \Rightarrow \omega_0 = \sqrt{14}, k = -14/15$
zero stationary point can be stable (chaotic hidden oscillations)

Main publications

✓ Leonov G.A., Efficient Methods for Search of Periodical Oscillations in Dynamic Systems. Applied mathematics and mechanics, 2009 (survey) [in print]

✓ Leonov G.A., On the Method of Harmonic Linearization, Automation and Remote Control, 2009, Vol. 70, No. 5, pp. 800-810

✓ Leonov G.A., On the Harmonic Linearization Method, Doklady Mathematics, 2009, Vol. 79, No. 1, pp. 144-146.

✓ Leonov G.A., Limit Cycles of the Lienard Equation with Discontinuous Coefficients, Doklady Physics, 2009, Vol. 54, No. 5, pp. 238-241

✓ Leonov G.A., Kuznetsov N.V., and Kudryashova E.V., Cycles of Two-Dimensional Systems: Computer Calculations, Proofs, and Experiments, Vestnik St. Petersburg University. Mathematics, Vol. 41, No. 3, 2008, pp. 216-250

✓ Kuznetsov N.V., Leonov G.A., Lyapunov quantities, limit cycles and strange behavior of trajectories in two-dimensional quadratic systems, Journal of Vibroengineering, Vol. 10, Iss. 4, 2008, pp. 460-467

✓ Leonov G.A., Boichenko V.A., Leonov G.A., Reitmann V., Dimention Theory for Ordinary Differential Equation. Teubner, 2005

✓ Leonov G.A., Reitmann V. Attraktoreingrenzung fur nichtlineare Systeme Teubner Texte zur Mathematik. Leipzig, 1987