

# BOUNDS ON MEAN CYCLE TIME IN ACYCLIC FORK-JOIN QUEUEING NETWORKS

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## Abstract

Simple lower and upper bounds on mean cycle time in stochastic acyclic fork-join networks are derived using the (max, +)-algebra approach. The behaviour of the bounds under various assumptions concerning the service times in the networks is discussed, and related numerical examples are presented.

## 1 Introduction

One of the problems of interest in the analysis of stochastic queueing networks is to evaluate the mean cycle time of a network. Both the mean cycle time and its inverse which can be regarded as a throughput, present performance measures commonly used to describe efficiency of the network operation.

It is frequently rather difficult to evaluate the mean cycle time exactly, even though the network under study is quite simple. To get information about the performance measure in this case, one can apply computer simulation to produce reasonable estimates. Another approach is to derive bounds on the mean cycle time (see examples in [1, 2, 3]).

The paper is concerned with the derivation of a lower and upper bounds on the mean cycle time for stochastic acyclic fork-join queueing networks [4, 2]. A useful way to represent dynamics of the networks is based on the (max, +)-algebra approach [5, 2, 6]. We apply the (max, +)-algebra dynamic representation proposed in [7, 8] to get algebraic bounds and then exploit them to derive bounds in the stochastic case.

Compared to the techniques proposed in [1, 3], which mainly rely on results of the theory of large deviations as well as the Perron-Frobenius spectral theory, our approach is essentially based on pure algebraic manipulations combined with application of bounds on extreme values, obtained in [9, 10].

## 2 Algebraic Definitions and Results

The (max, +)-algebra is an idempotent commutative semiring (idempotent semifield) which is defined as  $\mathbb{R}_{\max} = (\mathbb{R}, \oplus, \otimes)$  with  $\mathbb{R} = \mathbb{R} \cup \{\varepsilon\}$ ,  $\varepsilon = -\infty$ , and binary operations  $\oplus$  and  $\otimes$  defined as

$$x \oplus y = \max(x, y), \quad x \otimes y = x + y \quad \forall x, y \in \mathbb{R}.$$

As it is easy to see, the operations  $\oplus$  and  $\otimes$  retain most of the properties of the ordinary addition and multiplication, including associativity, commutativity, and distributivity of multiplication over addition. However, the operation  $\oplus$  is idempotent; that is, for any  $x \in \mathbb{R}$ , one has  $x \oplus x = x$ .

There are the null and identity elements, namely  $\varepsilon$  and 0, to satisfy the conditions  $x \oplus \varepsilon = \varepsilon \oplus x = x$ , and  $x \otimes 0 = 0 \otimes x = x$ , for any  $x \in \mathbb{R}$ . The null element  $\varepsilon$  and the operation  $\otimes$  are related by the usual absorption rule involving  $x \otimes \varepsilon = \varepsilon \otimes x = \varepsilon$ .

Non-negative power of any  $x \in \mathbb{R}$  is defined as

$$x^{\otimes 0} = 0, \quad x^{\otimes q} = \underbrace{x \otimes \dots \otimes x}_{q \text{ times}}$$

for any integer  $q \geq 1$ . Clearly, the (max, +)-algebra power  $x^{\otimes q}$  corresponds to  $qx$  in ordinary notations.

### 2.1 Algebra of Matrices

The (max, +)-algebra of matrices is readily introduced in the regular way. Specifically, for any  $(n \times n)$ -matrices  $X = (x_{ij})$  and  $Y = (y_{ij})$ , the entries of  $U = X \oplus Y$  and  $V = X \otimes Y$  are calculated as

$$u_{ij} = x_{ij} \oplus y_{ij}, \quad \text{and} \quad v_{ij} = \sum_{\oplus, k=1}^n x_{ik} \otimes y_{kj},$$

where  $\sum_{\oplus}$  stands for the iterated operation  $\oplus$ . As the null element, the matrix  $\mathcal{E}$  with all entries equal to  $\varepsilon$  is taken in the algebra, whereas the diagonal matrix  $E = \text{diag}(0, \dots, 0)$  with the off-diagonal entries set to  $\varepsilon$  presents the identity.

For any square matrix  $X \neq \mathcal{E}$ , one can define

$$X^{\otimes 0} = E, \quad X^{\otimes q} = \underbrace{X \otimes \dots \otimes X}_{q \text{ times}}$$

for any integer  $q \geq 1$ . However, idempotency in this algebra leads, in particular, to the matrix identity

$$(E \oplus X)^{\otimes q} = E \oplus X \oplus \dots \oplus X^{\otimes q}.$$

For any matrix  $X$ , its norm is defined as

$$\|X\|_{\oplus} = \sum_{i,j} x_{ij} = \max_{i,j} x_{ij}.$$

The matrix norm possesses the usual properties. Specifically, for any matrix  $X$ , it holds  $\|X\|_{\oplus} \geq \varepsilon$ , and  $\|X\|_{\oplus} = \varepsilon$  if and only if  $X = \mathcal{E}$ . Furthermore,  $\|c \otimes X\|_{\oplus} = c \otimes \|X\|_{\oplus}$  for any  $c \in \mathbb{R}$ , and

$$\begin{aligned} \|X \oplus Y\|_{\oplus} &= \|X\|_{\oplus} \oplus \|Y\|_{\oplus}, \\ \|X \otimes Y\|_{\oplus} &\leq \|X\|_{\oplus} \otimes \|Y\|_{\oplus} \end{aligned}$$

for any two conforming matrices  $X$  and  $Y$ . Note that for any  $c > 0$ , we also have  $\|cX\|_{\oplus} = c \|X\|_{\oplus}$ .

Consider an  $(n \times n)$ -matrix  $X$  with its entries  $x_{ij} \in \mathbb{R}$ . It can be treated as an adjacency matrix of an oriented graph with  $n$  vertices, provided each entry  $x_{ij} \neq \varepsilon$  implies the existence of the edge  $(i, j)$  in the graph, while  $x_{ij} = \varepsilon$  does the lack of the edge.

It is easy to verify that for any integer  $q \geq 1$ , the matrix  $X^{\otimes q}$  has its the entry  $x_{ij}^{(q)} \neq \varepsilon$  if and only if there exists a path from vertex  $i$  to vertex  $j$  in the graph, which consists of  $q$  edges. Furthermore, if the graph associated with the matrix  $X$  is acyclic, we have  $X^{\otimes q} = \mathcal{E}$  for all  $q > p$ , where  $p$  is the length of the longest path in the graph. Otherwise, provided that the graph is not acyclic, one can construct a path of any length, lying along circuits, and then it holds that  $X^{\otimes q} \neq \mathcal{E}$  for all  $q \geq 0$ .

## 2.2 Further Algebraic Results

For any graph, its adjacency matrix  $G$  with the elements equal either to 0 or  $\varepsilon$  is said to be standard. It is easy to verify the next statement.

**Proposition 1** *For any matrix  $X$ , it holds*

$$X \leq \|X\|_{\oplus} \otimes G,$$

where  $G$  is the standard adjacency matrix of the graph associated with  $X$ .

In particular, for a matrix  $D = \text{diag}(d_1, \dots, d_n)$ , we have

$$D \leq \|D\|_{\oplus} \otimes E = \left( \sum_{i=1}^n d_i \right) \otimes E.$$

**Proposition 2** *Let matrices  $X_1, \dots, X_k$  have a common associated acyclic graph,  $p$  be the length of the longest path in the graph, and*

$$X = X_1^{\otimes m_1} \otimes \dots \otimes X_k^{\otimes m_k},$$

where  $m_1, \dots, m_k$  are nonnegative integers.

*If it holds that  $m_1 + \dots + m_k > p$ , then  $X = \mathcal{E}$ .*

**Proof:**

$$\begin{aligned} X &= X_1^{\otimes m_1} \otimes \dots \otimes X_k^{\otimes m_k} \\ &\leq \|X_1\|_{\oplus}^{\otimes m_1} \otimes \dots \otimes \|X_k\|_{\oplus}^{\otimes m_k} \otimes G^{\otimes m}, \end{aligned}$$

where  $m = m_1 + \dots + m_k$ .

Since the associated graph is acyclic, it holds that  $G^{\otimes q} = \mathcal{E}$  for all  $q > p$ . Therefore, if  $m > p$ , then  $G^{\otimes m} = \mathcal{E}$ , and we arrive at the inequality  $X \leq \mathcal{E}$  which leads us to the desired result. ■

**Lemma 1** *Let matrices  $X_1, \dots, X_k$  have a common associated acyclic graph, and  $p$  be the length of the longest path in the graph.*

*If  $\|X_i\|_{\oplus} \geq 0$  for all  $i = 1, \dots, k$ , then for any nonnegative integers  $m_1, \dots, m_k$ , it holds*

$$\left\| \prod_{i=1}^k (E \oplus X_i)^{\otimes m_i} \right\|_{\oplus} \leq \left( \sum_{i=1}^k \|X_i\|_{\oplus} \right)^{\otimes p},$$

where  $\prod_{\otimes}$  denotes the iterated operation  $\otimes$ .

**Proof:** Consider the matrix

$$X = \prod_{i=1}^k (E \oplus X_i)^{\otimes m_i},$$

and represent it in the form

$$\begin{aligned} X &= (E \oplus X_1)^{\otimes m_1} \otimes \dots \otimes (E \oplus X_k)^{\otimes m_k} \\ &= \sum_{i_1=0}^{m_1} X_1^{\otimes i_1} \otimes \dots \otimes \sum_{i_k=0}^{m_k} X_k^{\otimes i_k} \\ &= \sum_{i_1=0}^{m_1} \dots \sum_{i_k=0}^{m_k} X_1^{\otimes i_1} \otimes \dots \otimes X_k^{\otimes i_k} \\ &\leq \sum_{0 \leq i_1 + \dots + i_k \leq m} X_1^{\otimes i_1} \otimes \dots \otimes X_k^{\otimes i_k}, \end{aligned}$$

where  $m = m_1 + \dots + m_k$ . From Proposition 2 we may replace  $m$  with  $p$  in the last term to get

$$X \leq \sum_{0 \leq i_1 + \dots + i_k \leq p} X_1^{\otimes i_1} \otimes \dots \otimes X_k^{\otimes i_k}.$$

Proceeding to the norm, with its additive and multiplicative properties, we arrive at the inequality

$$\|X\|_{\oplus} \leq \sum_{0 \leq i_1 + \dots + i_k \leq p} \|X_1\|_{\oplus}^{\otimes i_1} \otimes \dots \otimes \|X_k\|_{\oplus}^{\otimes i_k}.$$

Since  $0 \leq \|X_i\|_{\oplus} \leq \|X_1\|_{\oplus} \oplus \dots \oplus \|X_k\|_{\oplus}$  for all  $i = 1, \dots, k$ , we finally have

$$\begin{aligned} \|X\|_{\oplus} &\leq \sum_{i=0}^p (\|X_1\|_{\oplus} \oplus \dots \oplus \|X_k\|_{\oplus})^{\otimes p} \\ &= (\|X_1\|_{\oplus} \oplus \dots \oplus \|X_k\|_{\oplus})^{\otimes p}. \end{aligned} \quad \blacksquare$$

### 3 Elements of Probability

In this section we present some probabilistic results associated with  $(\max, +)$ -algebra concepts introduced above. We start with a lemma which states properties of the expectation with respect to the operations  $\oplus$  and  $\otimes$ .

**Lemma 2** *Let  $\xi_1, \dots, \xi_k$  be random variables taking their values in  $\mathbb{R}$ , and such that their expected values  $\mathbb{E}[\xi_i]$ ,  $i = 1, \dots, k$ , exist. Then it holds*

1.  $\mathbb{E}[\xi_1 \oplus \dots \oplus \xi_k] \geq \mathbb{E}[\xi_1] \oplus \dots \oplus \mathbb{E}[\xi_k]$ ,
2.  $\mathbb{E}[\xi_1 \otimes \dots \otimes \xi_k] = \mathbb{E}[\xi_1] \otimes \dots \otimes \mathbb{E}[\xi_k]$ .

**Proof:** Clearly, it will suffice to prove the lemma only for  $k = 2$ , and then extend the proof by induction to the case of arbitrary  $k$ .

To verify the first inequality, first assume that one of the expectation on its right side, say  $\mathbb{E}[\xi_2]$ , is infinite; that is  $\mathbb{E}[\xi_2] = \varepsilon$ . Since it holds that  $\xi_1 \oplus \xi_2 \geq \xi_1$ , we have

$$\mathbb{E}[\xi_1 \oplus \xi_2] \geq \mathbb{E}[\xi_1] = \mathbb{E}[\xi_1] \oplus \mathbb{E}[\xi_2].$$

Suppose now that  $\mathbb{E}[\xi_1], \mathbb{E}[\xi_2] > \varepsilon$ , and consider the obvious identity

$$x \oplus y = \frac{1}{2}(x + y + |x - y|) \quad \forall x, y \in \mathbb{R}.$$

With ordinary properties of expectation, we get

$$\begin{aligned} \mathbb{E}[\xi_1 \oplus \xi_2] &\geq \frac{1}{2}(\mathbb{E}[\xi_1] + \mathbb{E}[\xi_2] + |\mathbb{E}[\xi_1] - \mathbb{E}[\xi_2]|) \\ &= \mathbb{E}[\xi_1] \oplus \mathbb{E}[\xi_2]. \end{aligned}$$

The second assertion of the lemma is trivial. ■

The next result [9, 10] provides an upper bound for the expected value of the maximum of independent and identically distributed (i.i.d.) random variables.

**Lemma 3** *Let  $\xi_1, \dots, \xi_k$  be i.i.d. random variables with  $\mathbb{E}[\xi_1] < \infty$  and  $\mathbb{D}[\xi_1] < \infty$ . Then it holds*

$$\mathbb{E} \left[ \sum_{i=1}^k \oplus \xi_i \right] \leq \mathbb{E}[\xi_1] + \frac{k-1}{\sqrt{2k-1}} \sqrt{\mathbb{D}[\xi_1]}.$$

Consider a random matrix  $X$  with its entries  $x_{ij}$  taking values in  $\mathbb{R}$ . We denote by  $\mathbb{E}[X]$  the matrix obtained from  $X$  by replacing each entry  $x_{ij}$  by its expected value  $\mathbb{E}[x_{ij}]$ .

**Lemma 4** *For any random matrix  $X$ , it holds*

$$\mathbb{E} \|X\|_{\oplus} \geq \|\mathbb{E}[X]\|_{\oplus}.$$

**Proof:** It follows from Lemma 2 that

$$\begin{aligned} \mathbb{E} \|X\|_{\oplus} &= \mathbb{E} \left[ \sum_{i,j} \oplus x_{ij} \right] \\ &\geq \sum_{i,j} \oplus \mathbb{E}[x_{ij}] = \|\mathbb{E}[X]\|_{\oplus}. \quad \blacksquare \end{aligned}$$

### 4 Acyclic Fork-Join Networks

We consider a network with  $n$  single-server nodes and customers of a single class. The network topology is described by an oriented acyclic graph with the set of vertices which represent the nodes of the network, and the set of edges which determine the transition routes of customers. It is assumed that there are vertices in the graph which have no either incoming or outgoing edges. Each vertex with no predecessors is assumed to represent an infinite external arrival stream of customers; provided that a vertex has no successors, it is considered as an output node intended to release customers from the network.

Each node of the network includes a server and infinite buffer which together present a single-server queue operating under the first-come, first-served queueing discipline. It is assumed that at the initial time, all the servers are free of customers, the buffers in the nodes with no predecessors have infinite number of customers, whereas the buffers in the other nodes are empty.

In addition to the usual service procedure, some special join and fork operations [4] may be performed in a node respectively before and after service of a customer. The join operation is actually thought to cause each customer which comes into a node not to enter the buffer at the server but to wait until at least one customer from every preceding node arrives. As soon as these customers arrive, they, taken one from each preceding node, are united to be treated as being one customer which then enters the buffer to become a new member of the queue.

The fork operation at a node is initiated every time the service of a customer is completed; it consists in giving rise to several new customers one for each succeeding nodes. These customers simultaneously depart the node, each being passed to separate node related to the first one. We assume that the execution of fork-join operations when appropriate customers are available, as well as the transition of customers within and between nodes require no time.

An example of an acyclic fork-join network with  $n = 5$  is shown in Fig. 1. Note that open tandem

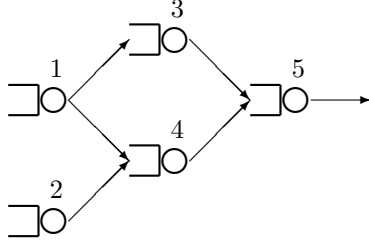


Figure 1: An acyclic fork-join network.

queueing systems (see Fig. 2) can be considered as trivial fork-join networks in which no fork and join operations are actually performed.

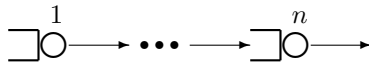


Figure 2: Open tandem queues.

For the queue at node  $i$ , we denote the  $k$ th departure epoch by  $x_i(k)$ , and the service time of the  $k$ th customer by  $\tau_{ik}$ . We assume that  $\tau_{ik} \geq 0$  are given parameters for all  $i = 1, \dots, n$ , and  $k = 1, 2, \dots$ , while  $x_i(k)$  are considered as unknown state variables. With the condition that the network starts operating at time zero, it is convenient to set  $x_i(0) = 0$  for all  $i = 1, \dots, n$ .

In order to describe the network dynamics, we apply the  $(\max, +)$ -algebra representation derived in [7, 8]. For the network model under study, the representation takes the form of the state equation

$$\mathbf{x}(k) = A(k) \otimes \mathbf{x}(k-1), \quad (1)$$

with  $\mathbf{x}(k) = (x_1(k), \dots, x_n(k))^T$  and

$$A(k) = (E \oplus \mathcal{T}_k \otimes G^T)^{\otimes p} \otimes \mathcal{T}_k, \quad (2)$$

where  $\mathcal{T}_k = \text{diag}(\tau_{1k}, \dots, \tau_{nk})$ , the matrix  $G$  is the standard adjacency matrix of the network graph, and  $p$  is the length of the longest path in the graph.

We consider the evolution of the system as a sequence of service cycles: the 1st cycle starts at the initial time, and it is terminated as soon as all the servers in the network complete their 1st service, the 2nd cycle is terminated as soon as the servers complete their 2nd service, and so on. Clearly, the completion time of the  $k$ th cycle can be represented as

$$\max_i x_i(k) = \|\mathbf{x}(k)\|_{\oplus}.$$

With the condition  $\mathbf{x}(0) = \mathbf{0}$ , we have from (1)

$$\|\mathbf{x}(k)\|_{\oplus} = \|A(k) \otimes \dots \otimes A(1)\|_{\oplus}.$$

The next lemma provides simple lower and upper bounds for the norm of the matrix

$$A_k = A(k) \otimes \dots \otimes A(1).$$

**Lemma 5** For all  $k = 1, 2, \dots$ , it holds

$$\left\| \sum_{i=1}^k \mathcal{T}_i \right\|_{\oplus} \leq \|A_k\|_{\oplus} \leq \sum_{i=1}^k \|\mathcal{T}_i\|_{\oplus} + p \left( \sum_{i=1}^k \|\mathcal{T}_i\|_{\oplus} \right).$$

**Proof:** To prove the left inequality note that

$$(E \oplus \mathcal{T}_i \otimes G^T)^{\otimes p} \otimes \mathcal{T}_i \geq \mathcal{T}_i$$

for each  $i = 1, \dots, k$ , and so  $A_k \geq \mathcal{T}_k \otimes \dots \otimes \mathcal{T}_1$ . Proceeding to the norm, and considering that  $\mathcal{T}_i$ ,  $i = 1, \dots, k$ , present diagonal matrices, we get

$$\begin{aligned} \|A_k\|_{\oplus} &\geq \|\mathcal{T}_k \otimes \dots \otimes \mathcal{T}_1\|_{\oplus} \\ &= \|\mathcal{T}_1 + \dots + \mathcal{T}_k\|_{\oplus}. \end{aligned}$$

With Proposition 1 we have

$$A_k \leq \prod_{i=1}^k \otimes \|\mathcal{T}_i\|_{\oplus} \otimes \prod_{i=1}^k (E \oplus \mathcal{T}_{k-i+1} \otimes G^T)^{\otimes p}.$$

By applying Lemma 1, we finally obtain

$$\begin{aligned} \|A_k\|_{\oplus} &\leq \prod_{i=1}^k \otimes \|\mathcal{T}_i\|_{\oplus} \otimes \left( \sum_{i=1}^k \|\mathcal{T}_i\|_{\oplus} \right)^{\otimes p} \\ &= \sum_{i=1}^k \|\mathcal{T}_i\|_{\oplus} + p \left( \sum_{i=1}^k \|\mathcal{T}_i\|_{\oplus} \right). \quad \blacksquare \end{aligned}$$

In many applications, one is normally interested in investigating the steady-state mean cycle time; that is, the limit of the ratio

$$\frac{1}{k} \|\mathbf{x}(k)\|_{\oplus} = \frac{1}{k} \|A_k\|_{\oplus}$$

as  $k$  tends to  $\infty$ . We will consider this problem with relation to stochastic networks in the next section.

## 5 Stochastic Networks

Suppose that for each node  $i = 1, \dots, n$ , the service times  $\tau_{i1}, \tau_{i2}, \dots$ , form a sequence of i.i.d. non-negative random variables with  $\mathbb{E}[\tau_{ik}] < \infty$  and  $\mathbb{D}[\tau_{ik}] < \infty$  for all  $k = 1, 2, \dots$ . With these conditions,  $\mathcal{T}_k$  are i.i.d. random matrices, whereas  $\|\mathcal{T}_k\|_{\oplus}$  present i.i.d. random variables with  $\mathbb{E}\|\mathcal{T}_k\|_{\oplus} < \infty$  and  $\mathbb{D}\|\mathcal{T}_k\|_{\oplus} < \infty$  for all  $k = 1, 2, \dots$ .

Furthermore, since the matrix  $A(k)$  depends only on  $\mathcal{T}_k$ , the matrices  $A(1), A(2), \dots$ , also present i.i.d. random matrices. It is easy to verify using (2) that  $0 \leq \mathbb{E}\|A(k)\|_{\oplus} < \infty$  for all  $k = 1, 2, \dots$ .

In the analysis of the mean cycle time of the system, one first has to convince himself that the limit

$$\lim_{k \rightarrow \infty} \frac{1}{k} \|A_k\|_{\oplus} = \gamma$$

exists. As a standard tool to verify the existence of the above limit, the next Subadditive Ergodic Theorem proposed in [11] is normally applied. One can find examples of the implementation of the theorem in the  $(\max, +)$ -algebra framework in [12, 2, 3].

**Theorem 1** *Let  $\{\xi_{lk} \mid l, k = 1, 2, \dots; l < k\}$  be a family of random variables which satisfy the following properties:*

*Subadditivity:*  $\xi_{lk} \leq \xi_{lm} + \xi_{mk}$  for all  $l < m < k$ ;

*Stationarity:* the joint distributions are the same for both families  $\{\xi_{l+1k+1} \mid l < k\}$  and  $\{\xi_{lk} \mid l < k\}$ ;

*Boundedness:* for all  $k = 1, 2, \dots$ , there exists  $\mathbb{E}[\xi_{0k}]$ , and  $\mathbb{E}[\xi_{0k}] \geq -ck$  for some finite number  $c$ .

Then there exists a constant  $\gamma$ , such that it holds

1.  $\lim_{k \rightarrow \infty} \xi_{0k}/k = \gamma$  with probability 1,
2.  $\lim_{k \rightarrow \infty} \mathbb{E}[\xi_{0k}]/k = \gamma$ .

In order to apply Theorem 1 to stochastic system (1) with transition matrix (2), one can define the family of random variables  $\{\xi_{lk} \mid l < k\}$  with

$$\xi_{lk} = \|A(k) \otimes \dots \otimes A(l)\|_{\oplus}.$$

Since  $A(i)$ ,  $i = 1, 2, \dots$ , present i.i.d. random matrices, the family  $\{\xi_{lk} \mid l < k\}$  satisfies the stationarity condition of Theorem 1. Furthermore, the multiplicative property of the norm endows the family with subadditivity. The boundedness condition can be readily verified using properties of the norm together with the condition that  $0 \leq \mathbb{E}[\tau_{ik}] < \infty$  for all  $i = 1, \dots, n$ , and  $k = 1, 2, \dots$

Now we are in a position to present our main result which offers bounds on the mean cycle time.

**Theorem 2** *In the stochastic dynamical system (1) the mean cycle time  $\gamma$  satisfies the double inequality*

$$\|\mathbb{E}[\mathcal{T}_1]\|_{\oplus} \leq \gamma \leq \mathbb{E} \|\mathcal{T}_1\|_{\oplus}. \quad (3)$$

**Proof:** Since Theorem 1 holds true, we may represent the mean cycle time as

$$\gamma = \lim_{k \rightarrow \infty} \frac{1}{k} \mathbb{E} \|A_k\|_{\oplus}.$$

Let us first prove the left inequality in (3). From Lemmas 5 and 4, we have

$$\begin{aligned} \frac{1}{k} \mathbb{E} \|A_k\|_{\oplus} &\geq \frac{1}{k} \mathbb{E} \left\| \sum_{i=1}^k \mathcal{T}_i \right\|_{\oplus} \\ &\geq \left\| \frac{1}{k} \sum_{i=1}^k \mathbb{E}[\mathcal{T}_i] \right\|_{\oplus} = \|\mathbb{E}[\mathcal{T}_1]\|_{\oplus}, \end{aligned}$$

independently of  $k$ .

With the upper bound offered by Lemma 5, we get

$$\frac{1}{k} \mathbb{E} \|A_k\|_{\oplus} \leq \mathbb{E} \|\mathcal{T}_1\|_{\oplus} + \frac{p}{k} \mathbb{E} \left[ \sum_{i=1}^k \|\mathcal{T}_i\|_{\oplus} \right].$$

From Lemma 3, the second term on the right-hand side may be replaced by that of the form

$$\frac{p}{k} \left( \mathbb{E} \|\mathcal{T}_1\|_{\oplus} + \frac{k-1}{\sqrt{2k-1}} \sqrt{\mathbb{D} \|\mathcal{T}_1\|_{\oplus}} \right),$$

which tends to 0 as  $k \rightarrow \infty$ . ■

## 6 Discussion and Examples

Now we discuss the behaviour of the bounds (3) under various assumptions concerning the service times in the network. First note that the derivation of the bounds does not require the  $k$ th service times  $\tau_{ik}$  to be independent for all  $i = 1, \dots, n$ . As it is easy to see, if  $\tau_{ik} = \tau_k$  for all  $i$ , we have  $\|\mathbb{E}[\mathcal{T}_1]\|_{\oplus} = \mathbb{E} \|\mathcal{T}_1\|_{\oplus}$ , and so the lower and upper bound coincide.

To show how the bounds vary with strengthening the dependency, we consider the network with  $n = 5$  nodes, depicted in Fig. 1. Let  $\tau_{i1} = \sum_{j=1}^5 a_{ij} \xi_{j1}$ , where  $\xi_{j1}$ ,  $j = 1, \dots, 5$ , are i.i.d. random variables with the exponential distribution of mean 1, and

$$a_{ij} = \begin{cases} a, & \text{if } i = j, \\ \frac{1}{4}(1-a), & \text{if } i \neq j, \end{cases}$$

where  $a$  is a number such that  $1 \leq a \leq 1/5$ .

It is easy to see that for  $a = 1$ , one has  $\tau_{i1} = \xi_{i1}$ , and therefore,  $\tau_{i1}$ ,  $i = 1, \dots, 5$ , present independent random variables. As  $a$  decreases, the service times  $\tau_{i1}$  become dependent, and with  $a = 1/5$ , we will have  $\tau_{i1} = (\xi_{11} + \dots + \xi_{51})/5$  for all  $i = 1, \dots, 5$ .

The next table presents estimates of the mean cycle time  $\hat{\gamma}$  obtained via simulation after performing 100000 service cycles, together with the corresponding lower and upper bounds calculated from (3).

$a$	$\ \mathbb{E}[\mathcal{T}_1]\ _{\oplus}$	$\hat{\gamma}$	$\mathbb{E} \ \mathcal{T}_1\ _{\oplus}$
1	1.0	1.005718	2.283333
1/2	1.0	1.002080	1.481250
1/3	1.0	1.000871	1.213889
1/4	1.0	1.000279	1.080208
1/5	1.0	1.000000	1.000000

Let us now consider the network in Fig. 1 under the assumption that the service times  $\tau_{i1}$  are

independent exponentially distributed random variables. We suppose that  $\mathbb{E}[\tau_{i1}] = 1$  for all  $i$  except for one, say  $i = 4$ , with  $\mathbb{E}[\tau_{41}]$  essentially greater than 1. One can see that the difference between the upper and lower bounds will decrease as the value of  $\mathbb{E}[\tau_{41}]$  increases. The next table shows how the bounds vary with different values of  $\mathbb{E}[\tau_{41}]$ .

$\mathbb{E}[\tau_{41}]$	$\ \mathbb{E}[\mathcal{T}_1]\ _{\oplus}$	$\hat{\gamma}$	$\mathbb{E}\ \mathcal{T}_1\ _{\oplus}$
1.0	1.0	1.005718	2.283333
2.0	2.0	2.004857	2.896032
3.0	3.0	3.004242	3.685531
4.0	4.0	4.003627	4.554525
5.0	5.0	5.003013	5.465368
6.0	6.0	6.002398	6.400835
7.0	7.0	7.001783	7.351985
8.0	8.0	8.001168	8.313731
9.0	9.0	9.000553	9.282968
10.0	10.0	10.000008	10.257692

Finally, let us discuss the effect of decreasing the variance  $\mathbb{D}[\tau_{i1}]$  on the bounds on  $\gamma$ . Note that if  $\tau_{i1}$  were degenerate random variables with zero variance, the lower and upper bounds in (3) would coincide. One can therefore expect that with decreasing the variance of  $\tau_{i1}$ , the accuracy of the bounds increases.

As an illustration, consider a tandem queueing system (see Fig. 2) with  $n = 5$  nodes. Suppose that  $\tau_{i1} = \xi_{i1}/r$ , where  $\xi_{i1}$ ,  $i = 1, \dots, 5$ , are i.i.d. random variables which have the Erlang distribution with the probability density function

$$f_r(t) = \begin{cases} t^{r-1}e^{-t}/(r-1)!, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Clearly,  $\mathbb{E}[\tau_{i1}] = 1$  and  $\mathbb{D}[\tau_{i1}] = 1/r$ . Related numerical results including estimates  $\hat{\gamma}$  evaluated by simulating 100000 cycles, and the bounds on  $\gamma$ , are shown below.

$r$	$\ \mathbb{E}[\mathcal{T}_1]\ _{\oplus}$	$\hat{\gamma}$	$\mathbb{E}\ \mathcal{T}_1\ _{\oplus}$
1	1.0	1.042476	2.928968
2	1.0	1.026260	2.311479
3	1.0	1.019503	2.045538
4	1.0	1.015637	1.890824
5	1.0	1.013110	1.787242
6	1.0	1.010864	1.711943
7	1.0	1.009920	1.654154
8	1.0	1.008409	1.608064
9	1.0	1.007726	1.570232
10	1.0	1.006657	1.538479

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