

Unbiased Estimates for Gradients of Stochastic Network Performance Measures

Nikolai Krivulin

Faculty of Mathematics and Mechanics, St. Petersburg University
Bibliotechnaya Sq.2, Petrodvorets, St. Petersburg, 198904 Russia

Abstract

Three classes of stochastic networks and their performance measures are considered. These performance measures are defined as the expected value of some random variables and cannot normally be obtained analytically as functions of network parameters in a closed form. We give similar representations for the random variables to provide a useful way of analytical study of these functions and their gradients. The representations are used to obtain sufficient conditions for the gradient estimates to be unbiased. The conditions are rather general and usually met in simulation study of the stochastic networks. Applications of the results are discussed and some practical algorithms of calculating unbiased estimates of the gradients are also presented.

Key words. Stochastic network, stochastic optimization, gradient estimation, perturbation analysis.

1 Introduction

Stochastic network models are widely used in modern engineering, management, biology, etc., to investigate real systems. These models are often so complicated that they can hardly be studied with the help of the analytical methods only. A more fruitful way is to use computer simulation to analyze the networks [1, 2, 3]. By performing simulation experiments, one may get a large amount of information about the network behaviour.

Usually, the main aim of the analysis is to improve a network performance. In order to optimize a performance criterion with respect to network parameters, one needs to evaluate it. Simulation provides estimating the criterion as well as its sensitivity (or its gradient, when the parameters are continuous) in a rather simple way. It is generally not difficult to obtain the estimates provided that there exists a simulation model, although each simulation experiment may be very time consuming.

There are many stochastic optimization procedures which use the data obtained by simulation (see [1] and also a short survey in [4]). In many cases, the procedures that exploit gradient estimates are preferred to those using estimates of the objective function only. Specifically, the stochastic algorithms which apply unbiased estimates

of gradient are often highly efficient. As an example, one can compare the Robbins–Monro procedure with the Kiefer–Wolfowitz one. It is well known [4] that the first procedure based on the unbiased estimates of gradient, converges to the solution faster than the second one which approximates the gradient by the finite differences.

In this paper, we analyze the problem of an unbiased estimation of the gradient of stochastic network performance measures. The paper is based on the results obtained in [5, 6]. In Section 2, we describe three classes of stochastic networks and give some examples of the networks and their related optimization problems. We show that the sample performance functions of the networks of all three classes may be represented in a similar way. In fact, these functions are expressed through those given by using the operations of maximum, minimum and addition.

Section 3 includes technical results which provide a general representation for the sample performance functions of the networks.

In Section 4 we briefly discuss three methods of estimating gradients, based on simulation data.

The main results are presented in Section 5. First, we introduce a set of functions for which one may obtain unbiased estimates of their gradients. We prove some technical lemmata to state the properties of the set. In conclusion, we give the conditions that prove the gradient estimates to be unbiased. These conditions are rather general and usually fulfilled in simulation studies of the stochastic networks.

In Section 6 some algorithms of calculating the gradient estimates are described.

2 Stochastic Networks and Related Optimization Problems

In this section we present three classes of stochastic networks and discuss their related optimization problems. A performance criterion of the network is normally defined as the expected value of a random variable, $f(\theta, \omega)$, i.e.,

$$F(\theta) = E_{\omega}[f(\theta, \omega)] = E[f(\theta, \omega)],$$

where $\theta \in \Theta \subset R^n$ is a set of decision parameters and ω is a random vector representing the randomness of network behaviour. As a function of the parameters, $f(\theta, \omega)$ is usually called a sample performance function.

The problem is to optimize the performance measure $F(\theta)$ with respect to $\theta \in \Theta$. In practical problems, it is often very hard to evaluate the expectation analytically in closed form, even if there is an analytical formula available for $f(\theta, \omega)$. However, it is normally not difficult to obtain the value of $f(\theta, \omega)$ for any fixed $\theta \in \Theta$ and any realization of ω by using simulation. In that case, one can use the Monte Carlo approach to estimate the objective function $F(\theta)$ or its gradient.

The main purpose of this section is to show that for many optimization problems, the sample performance function $f(\theta, \omega)$ may be represented in similar algebraic forms. In other words, $f(\theta, \omega)$ is expressed in terms of some given random variables

by means of the operations \max , \min , and $+$. This representation offers the potential for analytical study of the estimates of performance measure gradients. It also provides a theoretical background for efficient algorithms of calculating the estimates.

2.1 Activity Network

We begin with stochastic activity network models widely used in corporate management in the scheduling of large projects. Consider a project consisting of some activities (or jobs) which must be done to complete it. Each activity is presumed to require a random amount of time for performing it. It is not permitted to begin each activity until some preliminary activities have been performed. One is normally interested in reducing the expected completion time of the whole project.

In order to describe the project as a network, we define an oriented graph (\mathbf{N}, \mathbf{A}) , where \mathbf{N} is the set of nodes and \mathbf{A} is the set of arcs. Each node $i \in \mathbf{N}$ represents the corresponding activity of the project. For some $i, j \in \mathbf{N}$, the arc (i, j) belongs to \mathbf{A} if and only if the i th activity must directly precede the j th one.

To simplify further formulae we define the set of the father nodes as $\mathbf{N}_F(i) = \{j \in \mathbf{N} | (j, i) \in \mathbf{A}\}$, and the set of the daughter nodes as $\mathbf{N}_D(i) = \{j \in \mathbf{N} | (i, j) \in \mathbf{A}\}$ for every $i \in \mathbf{N}$. In addition, we introduce the set of starting nodes $\mathbf{N}_S = \{i \in \mathbf{N} | \mathbf{N}_F(i) = \emptyset\}$ and the set of the end nodes $\mathbf{N}_E = \{i \in \mathbf{N} | \mathbf{N}_D(i) = \emptyset\}$ of the graph.

Now we have to define the duration of the activities, so that the network would be completely described. Denote the duration of the i th activity by τ_i , $i \in \mathbf{N}$. We assume τ_i to be a positive random variable, such that $\tau_i = \tau_i(\theta, \omega)$, where $\theta \in \Theta$ is a set of decision parameters and ω is a random vector which represents the random effects involved in realizing the project. The set $\mathbf{T} = \{\tau_i | i \in \mathbf{N}\}$ is presumed to be given.

The sample completion time of the i th activity may be expressed in the form

$$t_i(\theta, \omega) = \begin{cases} \max_{j \in \mathbf{N}_F(i)} t_j(\theta, \omega) + \tau_i(\theta, \omega), & \text{if } i \notin \mathbf{N}_S, \\ \tau_i(\theta, \omega), & \text{if } i \in \mathbf{N}_S. \end{cases} \quad (1)$$

For the sample completion time of the whole project, we have

$$t(\theta, \omega) = \max_{i \in \mathbf{N}_E} t_i(\theta, \omega).$$

In that case, the expected completion time is

$$T(\theta) = E[t(\theta, \omega)],$$

and the problem is to minimize $T(\theta)$ with respect to $\theta \in \Theta$.

It is easy to see from (1) that one can represent t as a function of $\tau \in \mathbf{T}$ by using the operations \max and $+$. To illustrate this, consider the simple network depicted in Figure 1.

For this network, applying (1) successively, we may write the sample completion time as

$$t = \tau_1 + \max\{\max\{\tau_2, \tau_3\} + \tau_4, \tau_3 + \tau_5\} + \tau_6.$$

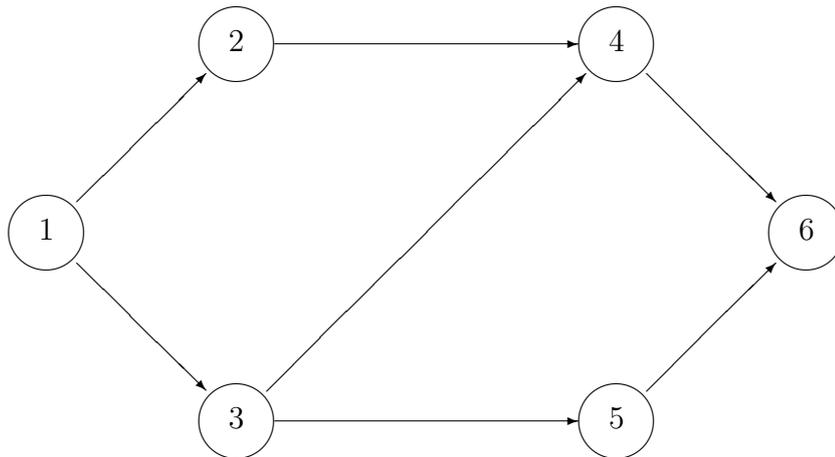


Figure 1: An activity network.

We will exploit the possibility of t being expressed in such a form in the discussion below.

We conclude this example with the remark about the main difficulty of the activity network optimization problem. It is easy to understand that in the case of general random variables $\tau \in \mathbf{T}$, it is usually very difficult or even impossible to obtain the expected completion time analytically, even if the network is as simple as that in Figure 1. To apply an optimization procedure in this situation, one normally estimates this function or its gradient by using the Monte Carlo approach. Notice, however, that the simulation models of such networks are generally rather simple.

2.2 Reliability Network

Another class of stochastic network models arises from the reliability investigation of complex interconnected systems in engineering, military research, biology etc. Consider a system of elements having bounded random lifetimes. Each element keeps in order until either this element has failed or all those directly supplying it have lost their working conditions. The whole system is presumed to be in order if at least one of the main elements that are supplied by some others but do not supply any element, keeps working. A usual problem in analyzing the system is to maximize its expected lifetime.

Let (\mathbf{N}, \mathbf{A}) be the directed graph describing the relations between the system elements. In the graph, the set of nodes \mathbf{N} corresponds to the set of system elements. If for some $i, j \in \mathbf{N}$, the i th element directly supplies the j th one, then $(i, j) \in \mathbf{A}$. For the graph, we retain the notations $\mathbf{N}_F(i)$, $\mathbf{N}_D(i)$, \mathbf{N}_S , and \mathbf{N}_E introduced above.

For each element $i \in \mathbf{N}$, we define the lifetime as the random variable $\tau_i(\theta, \omega)$ which depends on the set of decision parameters $\theta \in \Theta$. Assume the set $\mathbf{T} = \{\tau_i\}$

to be given. Now, we may represent the time for the i th element to be in order as

$$t_i(\theta, \omega) = \begin{cases} \min\{\max_{j \in \mathbf{N}_F(i)} t_j(\theta, \omega), \tau_i(\theta, \omega)\}, & \text{if } i \notin \mathbf{N}_S, \\ \tau_i(\theta, \omega), & \text{if } i \in \mathbf{N}_S. \end{cases} \quad (2)$$

The sample and expected lifetimes of the whole system may be written as

$$t(\theta, \omega) = \max_{i \in \mathbf{N}_E} t_i(\theta, \omega) \quad \text{and} \quad T(\theta) = E[t(\theta, \omega)],$$

respectively.

To illustrate this reliability network model, consider that depicted in Figure 2 ([1]).

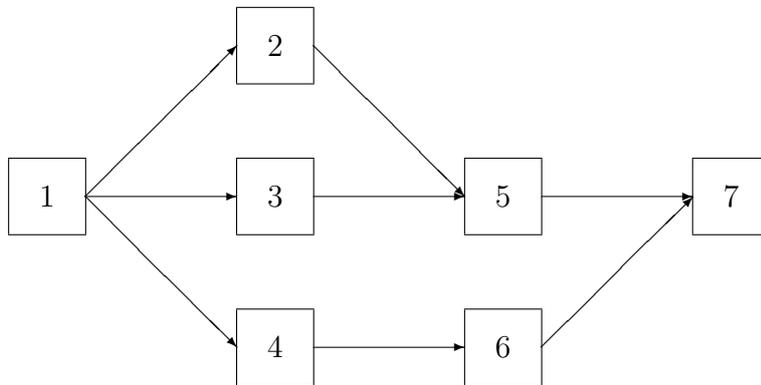


Figure 2: A reliability network.

For the sample lifetime of the system, we have from (2)

$$t = \min\{\tau_1, \max\{\min\{\tau_4, \tau_6\}, \min\{\max\{\tau_2, \tau_3\}, \tau_5\}\}, \tau_7\}.$$

We can see that the sample lifetime of such a network has one important property: it may be represented as a function of all $\tau \in \mathbf{T}$ by using only the operations max and min. Note that the difficulties in solving the problem of expected lifetime maximization are the same as in activity network optimization.

2.3 Queueing Network

Queueing networks provide a very rich class of stochastic network models. These models play the key role in simulation study of computer systems, communication networks, production lines, flexible manufacturing systems, etc. In this part of the section, a general class of queueing networks is considered and several performance measures of the network are defined.

The network which we describe consists of L single-server nodes. There are a server and a buffer with infinite capacity in each node i , $i = 1, \dots, L$. Once a customer arrives into node i , he occupies the server if it is free. The server keeps

busy a random amount of time until the service of the customer has been completed. Upon the completion of its service at node i , the customer goes to node j , chosen according to some routing procedure described below. We suppose that the customer arrives immediately into node j .

The customer may find the server of node i being busy. In that case, he joins the queue at the node and is placed into the buffer. The discipline in which queued customers are called forward for service is first-come, first-served. We assume that at the initial time, all servers of the network are free and there are n ($0 \leq n \leq \infty$) customers in the buffer at node i , $i = 1, \dots, L$. The customers are presumed to be of a single class.

For the network, define the set of random variables $\mathbf{T} = \{\tau_{ij}\}$, where $\tau_{ij}(\theta, \omega)$ is the service time of the customer that is the j th to initiate a service at node i . These random variables depend on the set of decision parameters $\theta \in \Theta$ and a random vector ω , and they are presumed to be given data.

Now, we discuss a routing mechanism of the network. A general routing procedure may be defined by means of the set of random variables, $\Sigma = \{\sigma_{ij}\}$, where $\sigma_{ij}(\omega)$ represents the next node to be visited by the customer who is the j th to depart from node i . Let s_{ij} be a realization of $\sigma_{ij}(\omega)$ for a fixed ω , $s_{ij} \in \{1, \dots, L\}$. The matrix

$$S = \begin{pmatrix} s_{11} & s_{12} & \dots \\ s_{21} & s_{22} & \dots \\ \dots & \dots & \dots \\ s_{L1} & s_{L2} & \dots \end{pmatrix}$$

is referred to as a routing table of the network. In addition, denote the set of all possible routing tables of the network by \mathbf{S} .

Firstly, we consider a special case of the network to demonstrate the relation between the queueing networks and those we have described above. Let us fix a routing table $S \in \mathbf{S}$ and consider the network with the deterministic routing procedure defined by S . One can state a lot of optimization problems of the network. The problems may differ in the performance criteria to be optimized. In order to produce useful representations of sample performance functions of the network, we introduce the following notations. For every node i , $i = 1, \dots, L$, let

- α_{ij} be the time of the j th arrival into the node,
- β_{ij} be the time of the j th initiation of a service,
- δ_{ij} be the time of the j th departure from the node.

Obviously, for each node i we may analytically represent the relationship between these variables as follows

$$\begin{cases} \delta_{ij} = \beta_{ij} + \tau_{ij}, \\ \beta_{ij} = \max\{\alpha_{ij}, \delta_{ij-1}\}, \quad j = 1, 2, \dots, \quad \delta_{i0} \equiv 0. \end{cases} \quad (3)$$

It should be noted that in the above identities each α_{ij} coincides with some δ_{mk} because the transition of customers from one node to another is immediate.

One of the performance measures of the network is the expected value of the M th service completion time at node K

$$D(\theta) = E[\delta_{KM}(\theta, \omega)]$$

that we wish to minimize with respect to $\theta \in \Theta$ for given K and M . Some other performance criteria one usually choose to optimize will be defined below.

The following fact is of great importance. For the network with a deterministic routing procedure, the sample completion time δ_{ij} may be represented as a function of $\tau \in \mathbf{T}$ using the operations \max , \min , and $+$ for every i, j provided that this service occurs.

To illustrate this, consider an example of network with three nodes and the routing table

$$S = \begin{pmatrix} 2 & 1 & 1 & 3 & \dots \\ 1 & 3 & 1 & 1 & \dots \\ 2 & 3 & 1 & 2 & \dots \end{pmatrix}.$$

Assume $n_1 = n_2 = n_3 = 1$ and choose a node of the network, say node 2. Since there is a customer at node 2 at the initial time, using (3) we may write

$$\alpha_{21} = 0, \quad \beta_{21} = 0, \quad \delta_{21} = \tau_{21}.$$

It is easy to see from S that the customer who is the second to arrive into node 2 will be one of those first serviced at both nodes 1 and 3. It will be just that customer who completes his service earlier. In that case we have

$$\alpha_{22} = \min\{\tau_{11}, \tau_{31}\}, \quad \beta_{22} = \max\{\alpha_{22}, \delta_{21}\}, \quad \delta_{22} = \beta_{22} + \tau_{22}.$$

Based on S , we also deduce that the other customer from these two will be the third to arrive into node 2. Therefore, we may write

$$\alpha_{23} = \max\{\tau_{11}, \tau_{31}\}, \quad \beta_{23} = \max\{\alpha_{23}, \delta_{22}\}, \quad \delta_{23} = \beta_{23} + \tau_{23}.$$

Finally, for δ_{23} we get

$$\delta_{23} = \max\{\max\{\tau_{11}, \tau_{31}\}, \max\{\min\{\tau_{11}, \tau_{31}\}, \tau_{21}\} + \tau_{22}\} + \tau_{23}.$$

As we can see, there is the representation of the sample performance function of the network like that we have pointed out in the previous examples. Notice that it may be very difficult to obtain such representations in practice. However, it is essential that the representation exists. We will give reason for this fact in the next section.

Now we define some traditional performance measures of the queueing networks. These definitions are also extended to the general network with the stochastic routing procedures.

Suppose that we observe the network until the M th service completion at node K , for given K, M . As sample performance functions for node K in the observation period, we consider the following:

$t(\theta, \omega) = \frac{1}{M} \sum_{j=1}^M (\delta_{Kj}(\theta, \omega) - \alpha_{Kj}(\theta, \omega))$, the average total time per customer,
 $w(\theta, \omega) = \frac{1}{M} \sum_{j=1}^M (\beta_{Kj}(\theta, \omega) - \alpha_{Kj}(\theta, \omega))$, the average waiting time per customer,

$u(\theta, \omega) = \sum_{j=1}^M \tau_{Kj}(\theta, \omega) / \delta_{KM}(\theta, \omega)$, the average utilization per unit time,

$c(\theta, \omega) = \sum_{j=1}^M (\delta_{Kj}(\theta, \omega) - \alpha_{Kj}(\theta, \omega)) / \delta_{KM}(\theta, \omega)$, the average number of customers per unit time,

$q(\theta, \omega) = \sum_{j=1}^M (\beta_{Kj}(\theta, \omega) - \alpha_{Kj}(\theta, \omega)) / \delta_{KM}(\theta, \omega)$, the average queue length per unit time.

Denote the expected values of these sample functions with respect to ω by $T(\theta)$, $W(\theta)$, $U(\theta)$, $C(\theta)$, and $Q(\theta)$, respectively. They are the performance measures that one usually chooses to optimize the network.

Note that the sample performance functions are described in terms of elements of the set $\{\alpha\}$, $\{\beta\}$, $\{\delta\}$, and $\{\tau\}$. It will be shown in the next section that each of the random variables α , β , and δ is expressed by a function of $\tau \in \mathbf{T}$, using the operations \max , \min , and $+$. This circumstance is of great importance and it will be necessary to study analytical properties of the performance measures and their gradient estimates in Section 5.

For the general routing procedure defined by the set Σ , any of the above sample performance functions may be represented as

$$f(\theta, \omega) = \sum_{S \in \mathbf{S}} 1_{[\Sigma(\omega)=S]} f_S(\theta, \omega). \quad (4)$$

Here $1_{[\Sigma(\omega)=S]}$ is the indicator of the event $\{\Sigma(\omega) = S\}$, and $f_S(\theta, \omega)$ is the sample performance function that coincides with $f(\theta, \omega)$ provided the routing procedure is deterministic and defined by the routing table S .

It should be noted in conclusion, that the sample performance functions of the queueing network are more difficult to take their expectation analytically than those of the networks we have considered above.

3 Algebraic Representations for the Networks

We have seen that the functions of network performance possess some algebraic properties. The point is that they may be expressed as a function of given random variables by means of the operations \max , \min , and $+$. For the activity networks and reliability, this follows directly from recursive equations (1) and (2) and does not require any special proofs. The possibility of such representations in describing of the sample performance functions of the queueing network is not obvious. In this section, we give the theorem that states the existence of the representation in the case of the network with a deterministic routing procedure. Two technical lemmatae are also included in the section.

In order to simplify further formulae, we introduce the notations \vee for maximum and \wedge for minimum. In addition, we will use the sign \bigvee (\bigwedge) to represent an iterated

maximum (minimum), i.e.,

$$\bigvee_{i=1}^n x_i = x_1 \vee \dots \vee x_n \quad \left(\bigwedge_{i=1}^n x_i = x_1 \wedge \dots \wedge x_n \right).$$

Let \mathbf{X} be a set supplied with the operations $+$, \vee , and \wedge . Without loss of generality, we may consider \mathbf{X} to be a set of real numbers. It is easy to extend the result of this section to various sets of real-valued functions and random variables. We assume that the traditional algebraic axioms are fulfilled in \mathbf{X} . In particular, we will use the following axioms.

Axiom 1 *Distributivity of maximum over minimum.*

$$\forall x, y, z \in \mathbf{X}, \quad (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z).$$

Axiom 2 *Distributivity of minimum over maximum.*

$$\forall x, y, z \in \mathbf{X}, \quad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z).$$

Axiom 3 *Distributivity of sum over maximum and minimum.*

$$\forall x, y, z \in \mathbf{X}, \quad (x \vee y) + z = (x + z) \vee (y + z), \quad (x \wedge y) + z = (x + z) \wedge (y + z).$$

The proof of the representation theorem for the queueing networks is based on the next result. Let $X = \{x_1, \dots, x_n\}$ be a finite set of real numbers. Suppose that we arrange its elements in order of increase, and denote the k th smallest element by $x_{(k)}$. If there are elements of an equal value, we count them repeatedly in an arbitrary order.

Lemma 1 *For each $k = 1, \dots, n$, the value of $x_{(k)}$ is given by*

$$x_{(k)} = \bigwedge_{I \in \mathfrak{S}_k} \bigvee_{i \in I} x_i, \tag{5}$$

where \mathfrak{S}_k is the set of all k subsets of the set $N = \{1, \dots, n\}$.

Proof. Denote the set of indices of the first k smallest elements by I^* . It is clear that $x_{(k)} = \bigvee_{i \in I^*} x_i$. Consider an arbitrary subset $I \in \mathfrak{S}_k$. Obviously, if $I \neq I^*$, there is at least one index $j \in I$ such that $x_j \geq x_{(k)}$. Therefore, we have $x_{(k)} \leq \bigvee_{i \in I} x_i$. It remains to take minimum over all $I \in \mathfrak{S}_k$ in the last inequality so as to get (5). \square

Theorem 2 *Let $S \in \mathbf{S}$ be a fixed routing table. For the network with the deterministic routing procedure defined by S , every α_{ij} , β_{ij} , and δ_{ij} ($i = 1, \dots, L$; $j = 1, 2, \dots$) is represented as a function of $\tau \in \mathbf{T}$, using the operations \max , \min and $+$, provided that its associated service occurs.*

Sketch of the proof. Consider a network of L nodes with a routing table S . The recursive equations of the network are written as

$$\begin{aligned}\delta_{ij} &= \beta_{ij} + \tau_{ij}, \\ \beta_{ij} &= \alpha_{ij} \vee \delta_{ij-1}, \quad j = 1, 2, \dots, \quad \delta_{i0} \equiv 0,\end{aligned}\tag{6}$$

for each node i , $i = 1, \dots, L$. The main idea of our proof is to reduce these equations to the one that expresses the departure time δ_{ij} through both the service time τ_{ij} and some other departure times δ_{km} ($k \in \{1, \dots, L\}$, $m \in \{1, 2, \dots\}$) only.

Let us first examine α_{ij} , i.e. the arrival time of the customer who is the j th to come into node i . It is plain that this customer is one of those who are to leave some other nodes to go to node i . Therefore, α_{ij} coincides with one of the departure times δ_{km} such that $s_{km} = i$.

Consider the customers who have to go to node i from some other, say node k . Clearly, we may restrict ourselves to the first j customers because this is enough to provide the j th customer to come into node i . Denote the set of the times at which customers depart from node k by $\Delta_k(i, j)$, $k \neq i$.

It may happen that $s_{im} = i$ for some m . This means that a customer who is the m th serviced at node i should join the queue of the same node again. In this case, we have to take account of such customers being served before the j th one only. The corresponding set of the departure times is $\Delta_i(i, j) = \{\delta_{im} | s_{im} = i \text{ and } m < j\}$.

Let $\Delta(i, j) = \bigcup_{k=1}^L \Delta_k(i, j)$. Note that if there are $n_i > 0$ customers in the buffer of node i at the initial time, then $\alpha_{i1} = \dots = \alpha_{in_i} = 0$. It is plain that for $j > n_i$ the arrival time α_{ij} coincides with the $(j - n_i)$ th smallest element of $\Delta(i, j)$. It follows from Lemma 1 that α_{ij} is represented as a function of elements of the set $\Delta(i, j)$ by using the operations max and min. In addition, using (6) we may get such a representation for β_{ij} as a function of departure times $\delta \in \Delta(i, j) \cup \{\delta_{ij-1}\}$.

Finally, it follows from (6) that there exists a representation of δ_{ij} as the function of both τ_{ij} and the elements of the set $\Delta(i, j) \cup \{\delta_{ij-1}\}$ which is a superposition of the operations max, min, and $+$. One can resolve this equation and get δ_{ij} as a function of the service times $\tau \in \mathbf{T}$ only. This produces the representation that the theorem requires. \square

We conclude this section with the following technical lemma that offers a general form of the representation.

Lemma 3 *Let $\varphi(x_1, \dots, x_p)$ be a function of the variables x_1, \dots, x_p taking their values in \mathbf{X} , φ is defined as a composition of the operations \vee , \wedge , and $+$. Then φ can be represented as*

$$\varphi(x_1, \dots, x_p) = \bigvee_{i \in I} \bigwedge_{j \in J_i} \sum_{k=1}^p \alpha_{ij}^k x_k,$$

where I and J_i for all $i \in I$ are finite sets of indices, and all α_{ij}^k are integers.

Proof. Without loss of generality we suppose that there is no more than one entry of each variable x_1, \dots, x_p into the expression. If some variable has two or

more entries, we introduce additional ones so that the above presupposition would be fulfilled. Let us prove the lemma by induction on the number of variables.

For $p = 1$, the statement of the lemma is obvious. If $p = 2$, there are three possibilities

$$x_1 \vee x_2, \quad x_1 \wedge x_2 \quad \text{and} \quad x_1 + x_2,$$

and it is clear that the statement is also true.

Assume that the statement of the lemma is true up to some value $p - 1$. Consider an expression φ of p variables. Clearly, there is an operation in the expression that should be performed after the other ones. Denote this operation by the asterisk $*$. In this case, we have $\varphi = \varphi_1 * \varphi_2$, where φ_1 and φ_2 are expressions such that each of them cannot include all the variables x_1, \dots, x_p . By the assumption, the statement of the lemma holds for both φ_1 and φ_2 . Now, we have three possibilities for the operation $*$.

1. \vee . This is obvious.
2. \wedge . It is sufficient to apply Axiom 1.
3. $+$. To obtain the representation in this case, one has to apply successively Axioms 1, 2 and 3.

Consequently, the statement of the lemma is true for $\varphi = \varphi_1 * \varphi_2$. □

4 Estimates of Gradient

To optimize the network performance measure $F(\theta) = E[f(\theta, \omega)]$, one often needs information about the gradients $\partial F(\theta)/\partial \theta$. In the absence of analytical formulae for the gradient, Monte Carlo experiments may be performed to estimate its values. There are three general methods of estimating $\partial F(\theta)/\partial \theta$ based on data obtained by simulation [1, 3, 7]. In the first two methods, the gradient is approximated by the finite differences and then estimated by using the Monte Carlo approach. To illustrate these two methods, assume θ to be a scalar and consider the following estimates:

The crude Monte Carlo (CMC) estimate:

$$G_{CMC} = \frac{1}{N\Delta\theta} \sum_{i=1}^N (f(\theta + \Delta\theta, \omega_i) - f(\theta, \omega_{N+i}));$$

the common random number (CRN) estimate:

$$G_{CRN} = \frac{1}{N\Delta\theta} \sum_{i=1}^N (f(\theta + \Delta\theta, \omega_i) - f(\theta, \omega_i)),$$

where ω_i , $i = 1, \dots, 2N$ are independent realizations of the random vector ω . The second estimate differs from the first in one respect: in the CRN estimate the random variables ω_i are the same for both $\theta + \Delta\theta$ and θ , whereas in the CMC estimate they are different. Note that each of them requires $2 \times N$ simulation runs (N at the

original value θ and N at $\theta + \Delta\theta$). Clearly, in the case of the vector $\theta \in R^n$, one must perform $(n + 1) \times N$ simulation experiments to get each estimate. In [1] has shown that the finite difference estimates have the mean square error (MSE) which is of order $O(N^{-1/3})$ for G_{CMC} and $O(N^{-1/2})$ for G_{CRN} .

We may somewhat improve the MSE properties of the estimate by using more sophisticated finite difference formulae. However, the estimates become very expensive in terms of computation time because they require a large number of additional simulation experiments. For example, the following symmetric difference estimate

$$G_{CRN}^{SD} = \frac{1}{2N\Delta\theta} \sum_{i=1}^N (f(\theta + \Delta\theta, \omega_i) - f(\theta - \Delta\theta, \omega_i))$$

requires $2 \times n \times N$ simulation runs, when $\theta \in R^n$).

An estimate of the third method can be written in the form

$$G = \frac{1}{N} \sum_{i=1}^N \frac{\partial}{\partial\theta} f(\theta, \omega_i), \quad (7)$$

provided that the gradient of the sample performance function (sample gradient) exists. It should be noted that although we may obtain values of the sample performance function by simulation, it can be rather difficult to evaluate its gradient.

Recently, a new technique called infinitesimal perturbation analysis (IPA) has been developed [2] as an efficient method of obtaining gradient information. The IPA method yields the exact values of the sample gradient $\partial f(\theta, \omega)/\partial\theta$ by performing one simulation run. The method is based on the analysis of the dynamics of the network and closely connected with the simulation technique. Therefore, one can easily combine an IPA procedure for calculating the sample gradient with a suitable algorithm of network simulation. Such a procedure provides all the partial derivatives of the sample gradient simultaneously during one simulation run. Furthermore, it needs an additional computation cost which is usually very small compared with that required for the simulation run alone.

The key question concerning the IPA method is whether it produces an unbiased estimate of the performance measure gradient. It can easily be shown that if $\partial f(\theta, \omega)/\partial\theta$ is an unbiased estimator of $\partial F(\theta)/\partial\theta$ then estimate (7) has MSE which is of order $O(N^{-1})$. In short, in the case of unbiasedness, this is a very efficient estimate which provides considerable savings in computation.

In the next section, using the algebraic representation of Section 3, we will examine properties of the network performance functions so as to derive the conditions for estimate (7) to be unbiased.

5 A Theoretical Background of Unbiased Estimation

It is easy to understand that a sufficient condition for the estimate (7) of the gradient $\partial E[f(\theta, \omega)]/\partial\theta$ at some $\theta \in \Theta$ to be unbiased is

$$\frac{\partial}{\partial\theta} E[f(\theta, \omega)] = E \left[\frac{\partial}{\partial\theta} f(\theta, \omega) \right]. \quad (8)$$

Cao showed in [7] that (8) holds in the case of $f(\theta, \omega)$ being uniformly differentiable at θ w.p. 1. Note that such a differentiability property is not easy to verify and hard to interpret for practical systems. A useful way to prove the interchange in (8) is to apply the Lebesgue dominated convergence theorem [8]. We use this theorem in the following form.

Theorem 4 *Let (Ω, \mathcal{F}, P) be a probability space, $\Theta \subset R^n$ and $f : \Theta \times \Omega \rightarrow R$ be a \mathcal{F} -measurable function for any $\theta \in \Theta$ and such that the following conditions hold:*

- (i) *for every $\theta \in \Theta$, there exists $\partial f(\theta, \omega)/\partial\theta$ at θ w.p. 1,*
- (ii) *for all $\theta_1, \theta_2 \in \Theta$, there is a random variable $\lambda(\omega)$ defined on the same probability space, with $E\lambda < \infty$ and such that*

$$|f(\theta_1, \omega) - f(\theta_2, \omega)| \leq \lambda(\omega) \|\theta_1 - \theta_2\| \quad \text{w.p. 1.} \quad (9)$$

Then equation (8) holds on Θ .

As an important consequence, we may state that the function $F(\theta) = E[f(\theta, \omega)]$ is a Lipschitz one with a constant $L = E\lambda$ and continuously differentiable on Θ , provided f satisfies the theorem conditions.

Definition 1 *A function $f(\theta, \omega)$ defined on the probability space (Ω, \mathcal{F}, P) at every $\theta \in \Theta$ belongs to the set $\mathcal{D}_{\Theta, \Omega}$ (or simply \mathcal{D}) if and only if it satisfies the conditions of Theorem 4.*

Example 1 Random variables which arise from a simulation study of networks, can be treated as members of a family of random variables [1]. There are few families one usually applies, namely the Exponential family, the Gaussian family, etc. Various random variables of a family may be obtained from the standard variable by using a suitable transformation. An ordinary way to transform random variables is based on changing the location and scale parameters.

Let $\xi(\omega)$ be the standard random variable of a family. Define

$$f(\theta, \omega) = \theta_1 \xi(\omega) + \theta_2,$$

where $\theta = (\theta_1, \theta_2)^\top \in \Theta \subset R^2$. Let us check whether it holds that $f \in \mathcal{D}$. Obviously, the partial derivatives of f with respect to θ_1 and θ_2 exist for almost all ω and equal

$$\frac{\partial}{\partial \theta_1} f(\theta, \omega) = \xi(\omega) \quad \text{and} \quad \frac{\partial}{\partial \theta_2} f(\theta, \omega) = 1.$$

In addition, it is easy to verify that f satisfies the condition (ii) of Theorem 4 with $\lambda = |\xi| + 1$. If $E|\xi| < \infty$, as is usually the case, then the conditions of Theorem 4 are fulfilled for f and we have $f \in \mathcal{D}$.

The next technical lemmatae give the sufficient conditions for the arithmetic operations and the operation max and min not to break the main properties of the functions from \mathcal{D} .

Lemma 5 *Let $f, g \in \mathcal{D}$ and let λ_1 and λ_2 be the random variables that provide the condition (ii) of Theorem 4 for f and g , respectively. Let μ_1, μ_2 and ν be positive random variables. Then the following are satisfied*

- (i) $f + g \in \mathcal{D}$;
- (ii) If α is a bounded random variable, then $\alpha f \in \mathcal{D}$;
- (iii) If $|f| \leq \mu_1$ and $|g| \leq \mu_2$ hold w.p. 1 for any $\theta \in \Theta$ and $E[\lambda_1 \mu_2 + \lambda_2 \mu_1] < \infty$, then $fg \in \mathcal{D}$;
- (iv) If $|f| \leq \mu_1$ and $|g| \geq \nu$ hold w.p. 1 for any $\theta \in \Theta$ and $E[\mu_1 \lambda_2 / \nu^2 + \lambda_1 / \nu] < \infty$, then $f/g \in \mathcal{D}$.

Proof. Clearly, $f + g$, αf , fg and f/g are measurable functions of ω and differentiable ones on Θ w.p. 1. Since for all of these functions the proofs of inequality (9) are quite similar, we verify it for only one of them. For instance, we examine $h = fg$.

For all $\theta_1, \theta_2 \in \Theta$ we have

$$\begin{aligned} & |h(\theta_1, \omega) - h(\theta_2, \omega)| \\ &= |f(\theta_1, \omega)g(\theta_1, \omega) - f(\theta_2, \omega)g(\theta_2, \omega)| \\ &= |f(\theta_1, \omega)g(\theta_1, \omega) - f(\theta_2, \omega)g(\theta_1, \omega) + f(\theta_2, \omega)g(\theta_1, \omega) - f(\theta_2, \omega)g(\theta_2, \omega)| \\ &\leq |g(\theta_1, \omega)||f(\theta_1, \omega) - f(\theta_2, \omega)| + |f(\theta_2, \omega)||g(\theta_1, \omega) - g(\theta_2, \omega)| \\ &\leq (\lambda_1(\omega)\mu_2(\omega) + \lambda_2(\omega)\mu_1(\omega))\|\theta_1 - \theta_2\| \quad \text{w.p. 1.} \end{aligned}$$

In short,

$$|h(\theta_1, \omega) - h(\theta_2, \omega)| \leq \lambda(\omega)\|\theta_1 - \theta_2\| \quad \text{w.p. 1,}$$

where

$$\lambda = \lambda_1 \mu_2 + \lambda_2 \mu_1, \quad E\lambda = E[\lambda_1 \mu_2 + \lambda_2 \mu_1] < \infty.$$

By Theorem 4, we conclude $fg \in \mathcal{D}$. □

Notice, from Lemma 5 (i) and (ii) it follows that being closed for the operations of addition and multiplication by bounded random variables, \mathcal{D} is a linear space of functions with these two operations.

Lemma 6 *Let $f, g \in \mathcal{D}$. Suppose that for any $\theta_0 \in \Theta$, there exists a neighbourhood $U_\omega(\theta_0)$ of θ_0 w.p. 1 such that one and only one of the following conditions*

- (i) $f(\theta, \omega) = g(\theta, \omega)$,
- (ii) $f(\theta, \omega) < g(\theta, \omega)$,
- (iii) $f(\theta, \omega) > g(\theta, \omega)$

is satisfied for all $\theta \in U_\omega(\theta_0)$.

Then $f \vee g \in \mathcal{D}$ and $f \wedge g \in \mathcal{D}$.

Proof. Consider $h(\theta, \omega) = f(\theta, \omega) \vee g(\theta, \omega)$. It is clear that h is measurable with respect to ω . In order to prove differentiability of h w.p. 1 on Θ , we examine an arbitrary $\theta \in \Theta$. There are only two possibility for h not to be differentiable. Firstly, it is possible that the derivative of h at θ does not exist if at least one of the derivatives

$$\left. \frac{\partial f(\theta, \omega)}{\partial \theta} \right|_{\theta=\theta_0} \quad \text{and} \quad \left. \frac{\partial g(\theta, \omega)}{\partial \theta} \right|_{\theta=\theta_0}$$

does not. In addition, h may not be differentiable at θ if the maximum of the functions f and g changes over from f to g at this point or vice versa. The last case is equivalent to that there exists $\omega \in \Omega$ such that all the neighborhoods $U_\omega(\theta_0) \subset \Theta$ contain both points at which

$$f(\theta, \omega) = g(\theta, \omega) \quad \text{and} \quad f(\theta, \omega) \neq g(\theta, \omega).$$

By the assumption of the lemma, both of these cases may occur only with zero probability. Therefore, there exists $\partial h(\theta, \omega) / \partial \theta|_{\theta=\theta_0}$ at all $\theta \in \Theta$ w.p. 1.

For the function h , the proof will be completed if we show that h satisfies condition (ii) of Theorem 4. Since $f, g \in \mathcal{D}$, there are random variables λ_1 and λ_2 with $E\lambda_1 < \infty$ and $E\lambda_2 < \infty$ such that the inequalities

$$\begin{aligned} |f(\theta_1, \omega) - f(\theta_2, \omega)| &\leq \lambda_1(\omega) \|\theta_1 - \theta_2\| \quad \text{w.p. 1} \\ |g(\theta_1, \omega) - g(\theta_2, \omega)| &\leq \lambda_2(\omega) \|\theta_1 - \theta_2\| \quad \text{w.p. 1} \end{aligned}$$

hold for all $\theta_1, \theta_2 \in \Theta$. Let ω be an arbitrary element of Ω at which both these inequalities hold. Divide Θ into two subsets:

$$\begin{aligned} \mathbf{X}_\omega &= \{\theta \in \Theta | f(\theta, \omega) \geq g(\theta, \omega)\}, \\ \mathbf{Y}_\omega &= \{\theta \in \Theta | f(\theta, \omega) < g(\theta, \omega)\}. \end{aligned}$$

Obviously, it holds

$$|h(\theta_1, \omega) - h(\theta_2, \omega)| \leq \lambda_1(\omega) \|\theta_1 - \theta_2\|$$

for all $\theta_1, \theta_2 \in \mathbf{X}_\omega$ and

$$|h(\theta_1, \omega) - h(\theta_2, \omega)| \leq \lambda_2(\omega) \|\theta_1 - \theta_2\|$$

for all $\theta_1, \theta_2 \in \mathbf{Y}_\omega$. Assume $\theta_1 \in \mathbf{X}_\omega, \theta_2 \in \mathbf{Y}_\omega$. If $h(\theta_1, \omega) \geq h(\theta_2, \omega)$, we deduce

$$\begin{aligned} & |h(\theta_1, \omega) - h(\theta_2, \omega)| \\ &= |f(\theta_1, \omega) - g(\theta_2, \omega)| \\ &< |f(\theta_1, \omega) - f(\theta_2, \omega)| \leq \lambda_1(\omega) \|\theta_1 - \theta_2\|. \end{aligned}$$

Similarly, if $h(\theta_1, \omega) < h(\theta_2, \omega)$, we have

$$|h(\theta_1, \omega) - h(\theta_2, \omega)| \leq \lambda_2(\omega) \|\theta_1 - \theta_2\|.$$

It follows that

$$|h(\theta_1, \omega) - h(\theta_2, \omega)| \leq \lambda(\omega) \|\theta_1 - \theta_2\|, \quad \lambda(\omega) = \lambda_1(\omega) \vee \lambda_2(\omega),$$

for all $\theta_1, \theta_2 \in \Theta$. Since this inequality holds for almost all $\omega \in \Omega$, we conclude that

$$|h(\theta_1, \omega) - h(\theta_2, \omega)| \leq \lambda(\omega) \|\theta_1 - \theta_2\| \quad \text{w.p. 1,}$$

and $E\lambda = E[\lambda_1 \vee \lambda_2] \leq E\lambda_1 + E\lambda_2 < \infty$.

In other words, h satisfies the conditions of Theorem 4. Consequently, $f \vee g \in \mathcal{D}$. The proof of the statement $f \wedge g \in \mathcal{D}$, is analogous. \square

It should be noted that the condition of Lemma 6 is not necessary, as the next example shows.

Example 2 Let $\Theta = [-1, 1]$, (Ω, \mathcal{F}, P) be a probability space, where $\Omega = [0, 1]$, \mathcal{F} is the σ -field of Borel sets of Ω , and P is the Lebesgue measure on Ω . Consider the following functions:

$$f(\theta, \omega) = -\theta^3 + \omega, \quad g(\theta, \omega) = \theta^2 + \omega$$

and

$$h(\theta, \omega) = f(\theta, \omega) \vee g(\theta, \omega) = \begin{cases} -\theta^3 + \omega, & \text{if } -1 \leq \theta \leq 0, \\ \theta^2 + \omega, & \text{if } 0 < \theta \leq 1. \end{cases}$$

One can easily verify that for any neighbourhood of $\theta = 0$, there exist both points with $f > g$ and $f < g$ w.p. 1. The conditions of Lemma 6 are therefore violated. Nevertheless, h is differentiable at 0 for all $\omega \in \Omega$. Moreover, it holds that $h \in \mathcal{D}$.

Corollary 7 *Let $f, g \in \mathcal{D}$. If for every $\theta \in \Theta$ it holds that $f \neq g$ w.p. 1, then $f \vee g \in \mathcal{D}$ and $f \wedge g \in \mathcal{D}$.*

Proof. Clearly, the condition of the corollary implies that either $f - g > 0$ or $f - g < 0$ holds at every $\theta \in \Theta$ w.p. 1. Since $f, g \in \mathcal{D}$, these two functions are continuous functions of θ w.p. 1 as well as $f - g$. Because of continuity, $f - g > 0$ ($f - g < 0$) holds w.p. 1 not only at θ , but also at every points of a neighbourhood of θ . It remains to apply Lemma 6. \square

Using Corollary 7, we give the following general conditions for \mathcal{D} to provide closeness with respect to the operations \vee and \wedge .

Lemma 8 *Let $f, g \in \mathcal{D}$. If for any $\theta \in \Theta$ it holds that the random variables $f(\theta, \omega)$ and $g(\theta, \omega)$*

- (i) *are independent,*
- (ii) *at least one of them is continuous,*

then $f \vee g \in \mathcal{D}$ and $f \wedge g \in \mathcal{D}$.

To prove the lemma it is sufficient to see that its conditions lead to that of Corollary 7.

The next two examples show that both conditions of Lemma 8 are essential.

Example 3 Let (Ω, \mathcal{F}, P) and Θ be defined as in Example 2. Also define

$$f(\theta, \omega) = -\theta + \omega \quad \text{and} \quad g(\theta, \omega) = \theta + \omega.$$

Let us consider the function

$$h(\theta, \omega) = f(\theta, \omega) \vee g(\theta, \omega) = \begin{cases} -\theta + \omega, & \text{if } -1 \leq \theta \leq 0, \\ \theta + \omega, & \text{if } 0 < \theta \leq 1. \end{cases}$$

It is clear that $f, g \in \mathcal{D}$ and for every $\theta \in \Theta$, the random variables $f(\theta, \omega)$ and $g(\theta, \omega)$ are continuous. Although inequality (9) holds with $\lambda = 1$ for h , this function is not differentiable at $\theta = 0$ for all $\omega \in \Omega$. Therefore, $h \notin \mathcal{D}$.

Example 4 Let $\Theta = [0, 1]$, $\Omega_1 = \Omega_2 = [0, 1]$ and P be the Lebesgue measure on $\Omega = \Omega_1 \times \Omega_2$. Denote $\omega = (\omega_1, \omega_2)^\top$ and consider the following functions:

$$\begin{aligned} f(\theta, \omega) &= \begin{cases} \frac{1}{2}\theta, & \text{if } \omega_1 \leq \frac{1}{2}, \\ 1, & \text{if } \omega_1 > \frac{1}{2}, \end{cases} \\ g(\theta, \omega) &= \begin{cases} \theta^2, & \text{if } \omega_2 \leq \frac{1}{2}, \\ 1, & \text{if } \omega_2 > \frac{1}{2}, \end{cases} \end{aligned}$$

and

$$h(\theta, \omega) = f(\theta, \omega) \vee g(\theta, \omega) = \begin{cases} \max\{\frac{1}{2}\theta, \theta^2\}, & \text{if } \omega_1 \leq \frac{1}{2} \text{ and } \omega_2 \leq \frac{1}{2} \\ 1, & \text{otherwise.} \end{cases}$$

One can see that $f, g \in \mathcal{D}$ and for every $\theta \in \Theta$, the random variables $f(\theta, \omega)$ and $g(\theta, \omega)$ are independent. In addition, condition (ii) of Theorem 4 holds for h with $\lambda = 2$. Nevertheless, $h = \max\{\frac{1}{2}\theta, \theta^2\}$ with probability $\frac{1}{4}$, that is not a differentiable function at $\theta = \frac{1}{2}$. In that case, $h \notin \mathcal{D}$.

Lemma 9 *Let \mathcal{M} be a set of functions from \mathcal{D} such that for any $f, g \in \mathcal{M}$, the conditions of Lemma 6 are fulfilled. Then \mathcal{M} is closed for the operations \max and \min .*

Proof. Let $f, g \in \mathcal{M}$ and let us define $h = f \vee g$. Note that $h \in \mathcal{D}$ by Lemma 6. We have to prove the conditions of Lemma 6 are satisfied for h and any $u \in \mathcal{M}$.

If u is either f or g , say $u \equiv f$, we may write

$$h - u = f \vee g - f = \begin{cases} g - f, & \text{if } f < g, \\ 0, & \text{if } f \geq g. \end{cases}$$

Since $f, g \in \mathcal{M}$, for any point of Θ , there is a neighbourhood on which only one of the conditions $f - g < 0$, $f - g = 0$, or $f - g > 0$ holds w.p. 1. From the above identity this also holds for $h - u$ on the neighbourhood. Consequently, in this case the conditions of Lemma 6 are fulfilled.

Now we assume $u \in \mathcal{M} \setminus \{f, g\}$. We have

$$h - u = f \vee g - u = \begin{cases} g - f, & \text{if } f < g, \\ f - u, & \text{if } f \geq g. \end{cases}$$

Let us examine any $\theta \in \Theta$. Suppose that $f < g$ w.p. 1 at θ . Since f, g , and u belong to \mathcal{M} , there are neighborhoods $\mathbf{U}_\omega(\theta)$ and $\mathbf{V}_\omega(\theta)$ where the conditions of Lemma 6 are fulfilled for each pairs of functions (f, g) and (g, u) , respectively. It follows from the above expression that the neighborhood $\mathbf{U}_\omega \cap \mathbf{V}_\omega(\theta)$ is that Lemma 6 requires for h and u . If it holds that $f \geq g$ or $f = g$ at θ , the reasoning is the same.

In short, we have shown that the conditions of Lemma 6 are satisfied for h and any $u \in \mathcal{M}$ and, therefore, $h = f \vee g \in \mathcal{M}$. In the case of minimum, the proof is analogous. \square

Corollary 10 *If $f_j \in \mathcal{M}$ for every $j = 1, \dots, N$, then it holds*

$$\bigvee_{i \in I} \bigwedge_{j \in J_i} f_j \in \mathcal{M},$$

where I is a finite set of indices and $J_i \subset \{1, \dots, N\}$ for every $i \in I$.

This is an immediate consequence of the previous lemma.

The next example is of importance to the main result of the section.

Example 5 Let $f_j \in \mathcal{D}$ for all $j = 1, \dots, N$. Suppose that at every $\theta \in \Theta$, all the random variables $f_j(\theta, \omega)$ are continuous and independent. Define \mathcal{L} to be a set of linear combinations $\sum_{i \in I} a_i f_i$ with integer coefficients a_i , $i \in I \subset \{1, \dots, N\}$. Obviously, \mathcal{L} is stable for addition. For all functions $u = \sum_{i \in I} a_i f_i$ and $v = \sum_{j \in J} b_j f_j$, we have $u - v = \sum_{k \in K} c_k f_k$. It is clear that for every $\theta \in \Theta$, $u - v$ is a continuous random variable because of the properties of f (except for the case of all $c_k = 0$ which is obvious). Therefore, it holds that $u - v \neq 0$ w.p. 1 at every $\theta \in \Theta$. Similarly as in Corollary 7, one can deduce that u and v satisfy the conditions of Lemma 6. From this we conclude that \mathcal{L} may be treated as an example of \mathcal{M} .

One can easily see that the condition of continuity is essential to this reasoning. To illustrate the important role of independence, consider the following functions

$$f(\theta, \omega) = -2\theta + 2\omega, \quad g(\theta, \omega) = \theta - \omega, \quad \text{and} \quad u(\theta, \omega) = \theta + \omega,$$

under the same assumption as in Example 3. It is easy to verify that the conditions of Lemma 6 are fulfilled for any two functions of them. Nevertheless, the functions u and $v = f + g$ do not satisfy the conditions, as Example 3 has shown.

Now, we may formulate the main result of the section. We first introduce some definitions. Let \mathcal{A} be the algebra of all functions $f : \Theta \times \Omega \rightarrow R$ being defined on the probability space (Ω, \mathcal{F}, P) at every $\theta \in \Theta$ with the operations \vee , \wedge , and $+$. In other words, this is a closed system of the functions for these operations.

Definition 2 Let \mathbf{T} be a finite subset of functions of \mathcal{A} . We define $[\mathbf{T}]_{\mathcal{A}}$ to be the set generated by \mathbf{T} in \mathcal{A} , that is the set of all functions being obtained from ones of \mathbf{T} by means of the operations \vee , \wedge , and $+$.

Theorem 11 Let $\mathbf{T} \in \mathcal{D}$. Suppose that for all $\tau \in \mathbf{T}$, $\tau(\theta, \omega)$ are continuous and independent random variables at any $\theta \in \Theta$.

Then it holds $[\mathbf{T}]_{\mathcal{A}} \subset \mathcal{D}$.

Proof. It results from Lemma 3 that every $f \in [\mathbf{T}]_{\mathcal{A}}$ can be represented as

$$f = \bigvee_{i \in I} \bigwedge_{j \in J_i} \sum_{\tau \in \mathbf{T}} a_{ij}^{\tau} \tau,$$

where all a_{ij}^{τ} are integers. It has been shown in Example 5 that the functions of the family $\{\sum_{\tau \in \mathbf{T}} a_k^{\tau} \tau\}_{k=1,2,\dots}$ satisfy the conditions of Lemma 6. Applying Corollary 10, we conclude that the statement of the theorem is true. \square

It is important to note that the conditions of Theorem 11 are rather general and usually fulfilled in the network simulation. In particular, in contrast with the traditional approaches (cf., for example, existing results on the unbiasedness of IPA estimates in [2, 3]), we may not restrict ourselves to the exponential distribution.

In short, to satisfy the theorem only the following are required for the functions of the set \mathbf{T} :

- (i) for any $\theta \in \Theta$, all $\tau \in \mathbf{T}$ are continuous and independent random variables;
- (ii) each $\tau \in \mathbf{T}$ as a function of θ is differentiable w.p. 1 and Lipschitz one with an integrable random variable as a Lipschitz constant.

In the next section we will show how these results can be applied to some problems to verify the unbiasedness of gradient estimates.

6 Applications

Now we discuss the applications of the previous results to optimizing the networks. In particular, we describe algorithms of obtaining sample gradients, based on the algebraic representation of the networks. In this section we keep using the notations (Ω, \mathcal{F}, P) and Θ for the underlying probability space and the parameter space, respectively.

We begin with the stochastic activity network. Let the duration of the j th activity be represented by the function $\tau_j(\theta, \omega)$. Denote the set of all such functions of the network by \mathbf{T} . As we have seen, a sample completion time of the network $t(\theta, \omega)$ may be expressed by functions of \mathbf{T} by using only the operations \max and $+$. This implies $t \in [\mathbf{T}]_{\mathcal{A}}$.

Suppose that $\mathbf{T} \in \mathcal{D}$, and all $\tau \in \mathbf{T}$ are continuous and independent random variables at every $\theta \in \Theta$. For the mean completion time $T(\theta) = E[t(\theta, \omega)]$, it follows from Theorem 11 that $(1/N) \sum_{i=1}^N \partial t(\theta, \omega_i) / \partial \theta$, where $\omega_i \in \Omega$, is an unbiased estimate of the gradient $\partial T(\theta) / \partial \theta$.

As an example, suppose $\tau(\theta, \omega) = -\theta \ln(1 - \omega)$, where $\theta \in R$ and the random variable ω is uniformly distributed on $[0, 1]$. It is well known [1] that $-\ln(1 - \omega)$ has an exponential distribution with mean 1. Similarly as in Example 1, we have $\tau \in \mathcal{D}$. In addition, durations of the activities are normally considered as independent in the probabilistic sense. Our results are therefore applicable in this case.

Now suppose that there is a simulation procedure for the activity network with L nodes to provide a simulation experiment for any fixed $\theta \in \Theta$ and a realization of ω . One can easily combine it with the following algorithm.

Algorithm 1

- Step (i). At the initial time, fix values of θ and ω ; set $g_j = 0$ for $j = 1, \dots, L$, and set $c = 0$.
- Step (ii). Upon the completion of any activity i , add the value of $\partial \tau_i(\theta, \omega) / \partial \theta$ to g_i and add 1 to c ; if $c = L$, then save g_i as the value of $\partial t(\theta, \omega) / \partial \theta$ and stop; otherwise go to Step (iii).
- Step (iii). Determine the set $\mathbf{N}_D(i)$. For every $j \in \mathbf{N}_D(i)$, if all activities of the set $\mathbf{N}_F(j)$ have been completed, then set $g_j = g_i$.

To verify the correctness of Algorithm 1, it suffices to see that it is simply based on recursive equation (1).

For a reliability network, one can apply Theorem 11 in a similar way. As in Section 3, denote the sample lifetime of a system by $t(\theta, \omega)$. It is not difficult to construct the next algorithm that calculates the sample gradient $\partial t(\theta, \omega) / \partial \theta$.

Algorithm 2

- Step (i). At the initial time, fix values of θ and ω .

Step (ii). Upon the failure of element i , exclude all nodes representing the elements that are now not able to keep working from the set \mathbf{N} as well as the corresponding arcs from the set \mathbf{A} .

Step (iii). If for the reduced set \mathbf{N} it holds $\mathbf{N} \cap \mathbf{N}_E = \emptyset$, then save $\partial\tau_i(\theta, \omega)/\partial\theta$ as the value of $\partial t(\theta, \omega)/\partial\theta$ and stop; otherwise go to Step (ii).

Finally, we consider the queueing network which is a rather complicated model. Applying Theorem 11 to a network with a deterministic routing mechanism, we may conclude that

- (i) if $\mathbf{T} \in \mathcal{D}$, and for all $\tau \in \mathbf{T}$, $\tau(\theta, \omega)$ are continuous and independent random variables for every $\theta \in \Theta$, then the estimate (7) is unbiased for both the expected average total time T and the expected average waiting time W ;
- (ii) if in addition to previous assumptions, for all $\tau_1, \tau_2 \in T$, condition (iv) of Lemma 5 is fulfilled, then the estimate (7) is unbiased for the expected average utilization U , the expected average number of customers C and the expected average queue length Q .

In the case of the stochastic routing mechanism with a random routing table, the above conclusions still hold true. This follows from representation (4) and Lemma 5 (ii) because of the boundedness of the indicator random variable.

In order to construct a useful algorithm of calculating a sample performance function gradient, we first consider the identities at (3) in the form

$$\delta_{ij}(\theta, \omega) = \max\{\alpha_{ij}(\theta, \omega), \delta_{ij-1}(\theta, \omega)\} + \tau_{ij}(\theta, \omega).$$

Differentiating the function $\delta_{ij}(\theta, \omega)$ for a fixed $\omega \in \Omega$ at θ , we have

$$\frac{\partial}{\partial\theta}\delta_{ij}(\theta, \omega) = \begin{cases} \frac{\partial}{\partial\theta}\delta_{ij-1}(\theta, \omega) + \frac{\partial}{\partial\theta}\tau_{ij}(\theta, \omega), & \text{if } \alpha_{ij}(\theta, \omega) < \delta_{ij-1}(\theta, \omega), \\ \frac{\partial}{\partial\theta}\alpha_{ij}(\theta, \omega) + \frac{\partial}{\partial\theta}\tau_{ij}(\theta, \omega), & \text{if } \alpha_{ij}(\theta, \omega) > \delta_{ij-1}(\theta, \omega). \end{cases}$$

The inequalities in the conditions of the right-hand side mean the following. The inequality $\alpha_{ij} > \delta_{ij-1}$ implies that at the arrival of the j th customer into node i , the service of the previous one, the $(j-1)$ th, has been completed. Therefore, at that time the server is free. The meaning of the contrary inequality is that the server is busy at the arrival of the j th customer.

In the first case, the value of $\partial\delta_{ij}(\theta, \omega)/\partial\theta$ is defined by $\partial\alpha_{ij}(\theta, \omega)/\partial\theta$. Note that the function $\alpha_{ij}(\theta, \omega) \equiv \delta_{km}(\theta, \omega)$, where $\delta_{km}(\theta, \omega)$ represents some m th completion time at a node k . It is the service completion of the customer that is the j th to arrive into node i . In other words, in that case one have to add the value of $\partial\delta_{km}(\theta, \omega)/\partial\theta$ to $\partial\tau_{ij}(\theta, \omega)/\partial\theta$ to obtain $\partial\delta_{ij}(\theta, \omega)/\partial\theta$.

If it holds $\alpha_{ij} < \delta_{ij-1}$, then the value of $\partial\delta_{ij}(\theta, \omega)/\partial\theta$ is calculated by addition $\partial\delta_{ij-1}(\theta, \omega)/\partial\theta$ and $\partial\tau_{ij}(\theta, \omega)/\partial\theta$.

It results from Section 5 that if for any $\theta \in \Theta$, all $\tau \in \mathbf{T}$ are continuous and independent random variables, then $\alpha_{ij}(\theta, \omega) \neq \delta_{ij-1}(\theta, \omega)$ w.p. 1. Therefore, the above expression defines the sample gradient w.p. 1.

Now, we consider an algorithm that provide the value of $\partial\delta_{KM}(\theta, \omega)/\partial\theta$ for fixed $\theta \in \Theta \subset R$, $\omega \in \Omega$, $K \in \{1, \dots, L\}$, and $M \in \{1, 2, \dots\}$. We suppose that there is a simulation procedure into which the algorithm may be incorporated.

Algorithm 3

- Step (i). At the initial time, fix values of θ and ω ; set $g_j = 0$ for $j = 1, \dots, L$.
- Step (ii). Upon the j th completion at node i , add the value of $\partial\tau_{ij}(\theta, \omega)/\partial\theta$ to g_i ; if both $i = K$, and $j = M$, then save g_K as the value of $\partial\delta_{KM}(\theta, \omega)/\partial\theta$ and stop; otherwise go to Step (iii).
- Step (iii). Determine the next node $r = \sigma_{ij}(\omega)$ to be visited by the customer; if the server of node r is free, then set $g_r = g_i$.

Note that in the case of a vector of parameters, $\Theta \subset R^n$, the algorithm is analogous. It only needs to change g_i for the vector $\mathbf{g}_i = (g_{i1}, \dots, g_{in})^\top$ and to treat the arithmetic operations as the vector ones.

It is easy to see that the algorithm of evaluating $\delta_{ij}(\theta, \omega)/\partial\theta$ plays the key role in calculating gradients of sample performance functions of the network. It is included as the main part in other algorithms. To illustrate this, we consider an algorithm for the sample function $u(\theta, \omega) = \sum_{j=1}^M \tau_{Kj}(\theta, \omega)/\delta_{KM}(\theta, \omega)$, the average utilization per unit time.

Algorithm 4

- Step (i). At the initial time, fix values of θ and ω ; set $t, d = 0$, and $g_j = 0$ for $j = 1, \dots, L$.
- Step (ii). Upon the j th completion at node i , add the value of $\partial\tau_{ij}(\theta, \omega)/\partial\theta$ to g_i ; if $i = K$, then add the value of $\partial\tau_{ij}(\theta, \omega)/\partial\theta$ to d , and add the value of $\tau_{Kj}(\theta, \omega)$ to t ; if both $i = K$, and $j = M$, then set $h = \delta_{KM}(\theta, \omega)$ and stop; otherwise go to Step (iii).
- Step (iii). Determine the next node $r = \sigma_{ij}(\omega)$ to be visited by the customer; if the server at node r is free, then set $g_r = g_i$.

Upon the completion of the algorithm we get $(dh - tg_K)/h^2$ as the value of $\partial u(\theta, \omega)/\partial\theta$.

It is easy to see that Algorithms 3 and 4 are quite similar to those of IPA method in [3].

In conclusion, note that the algorithms are rather simple. In fact, they only require calculating gradients of given functions and performing some trivial operations to produce values of the sample gradients. Using these values, one can easily estimate the gradients of network performance measures so as to apply efficient optimization procedures.

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References

- [1] Ermakov, S.M.: *Die Monte-Carlo-Methode und verwandte Fragen*, VEB Deutscher Verlag der Wissenschaften, 1975.
- [2] Ho, Y.C.: Performance evaluation and perturbation analysis of discrete event dynamic systems, *IEEE Trans. Automat. Contr.*, **AC-32**(7) (1987), 563–572.
- [3] Suri, R.: Perturbation analysis: The state of the art and research issues explained via the GI/G/1 queue, *Proc. IEEE*, **77** (1) (1989), 114–137.
- [4] Glynn, P.W.: Optimization of stochastic systems, in *Proc. of 1986 Winter Simulation Conf.*, 1986, 52–58.
- [5] Krivulin, N.K.: Optimization of complex systems for simulation modeling, *Vestnik Leningrad Univ. Math.*, **23**(2) (1990), 64–67.
- [6] Krivulin, N.K.: Optimization of discrete event dynamic systems by using simulation, PhD Dissertation. St. Petersburg University, 1990. (in Russian).
- [7] Cao, X.R.: Convergence of parameter sensitivity estimates in a stochastic experiment, *IEEE Trans. Automat. Contr.*, **AC-30**(9) (1985), 845–853.
- [8] Loève, M.: *Probability Theory*, Van Nostrand, New York, 1960.