Combined Procedure with Randomized Controls for the Parameters’ Confidence Region of Linear Plant under External Arbitrary Noise

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Introduction

- Any model is never been the real system perfect description.
- It is important to determine the bounds of uncertainties for the model in which it can still be used.
- The key question in the system identification is a development of methods and procedures which are applicable for wide range uncertainties.
A goal of control is to achieve some kind of generally stable state (if possible do not change over time). In this state, invariability provides a “very little information” and, therefore, it is impossible to identify or establish new links, values, and so on.

Feldbaum’s concept of dual control: control must be not only directing but also learning.
Control Via Feedback

Usually, we have

- substantial restrictions of resources,
- an insufficient number of data with the necessary diversity
to “extract” an information from the data in the real time environment.
Example

Let’s consider the simple problem of an unknown parameter $\theta_*$ estimating from the observations:

$$y_t = \theta_* \cdot u_t + v_t, \quad t = 1, 2, ..., N. \quad (1)$$

We can

- to choose the inputs (control actions) $u_t$;
- to measure the outputs $y_t$.

Figure: The model of observations.
The Source, Target and Detector of the Reflected Signal

Injector

Target

Detector

Jamming
Algorithm of $\theta_\star$ Estimation

1. Control $u_t$ selection and feeding it to the system input.
2. Receiving the response from the system $y_t$.
3. Estimation of the parameter $\theta_\star$ based on the data obtained $u_t, y_t$ (for example, calculation of an estimate $\hat{\theta}_t$ or set $\hat{\Theta}_t$ containing $\theta_\star$).
4. Repeat steps 1–3.

Figure: A model of an estimation algorithm.
Deterministic Algorithm

Definition

An algorithm is called a *deterministic algorithm* if each of its steps defined by the user is given by deterministic rules using the results of the previous steps, and obtained new data (output) is returned for using in subsequent steps of the algorithm.

Figure: A model of a deterministic algorithm.
Deterministic Approaches Often Failed!

- In theory and practice many difficulties arise when we try to make analytical investigation of “complex” systems.
- For many practical applications traditionally efficient deterministic methods fail to yield a result when the system dimension is high.
- In particular, this leads to the notion of $NP$-hard problems.
There are no Deterministic Algorithms Under Arbitrary External Noise!

Let be

$$\theta_\ast = 3, \quad \hat{\theta}_t = \frac{1}{t} \sum_{i=1}^{t} y_i$$

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<tr>
<th>$t$</th>
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$v_t = \text{rand}() - 0.5$

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<tr>
<td>$y_t$</td>
<td>2.9</td>
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<td>$\hat{\theta}_t$</td>
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<td>2.96</td>
<td>3.03</td>
<td>2.99</td>
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$v_t = \text{rand}() - 0.5 + m, \ m = 1$

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<td>3.96</td>
<td>4.03</td>
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Note, the bias $m$ is an unknown for the user!
Randomized Algorithms

Randomization is a powerful tool for solving a number of problems deemed unsolvable with deterministic methods

Definition

An algorithm is called a *randomized algorithm* when the execution of one or more steps, which are defined by the user, is based on a random rule (that is, among many deterministic rules one is chosen randomly according to a probability $P$).
“Enriched” Observations

Consider the following rule of a random input selection for the first step

\[ u_t = \begin{cases} 
+1, & \text{with probability } \frac{1}{2}, \\
-1, & \text{with probability } \frac{1}{2}.
\end{cases} \] (2)

At the second step from the known values \((u_t, y_t)\) we form a value

\[ \tilde{y}_t = u_t \cdot y_t. \]

For the “new” sequence of observations we have a similar to (1) model

\[ \tilde{y}_t = \tilde{u}_t \cdot \theta_* + \tilde{v}_t, \]

where \(\tilde{u}_t = u_t^2\) and \(\tilde{v}_t = u_t \cdot v_t\).
Diagram of Randomized Algorithm

random choice of $U_t$

computation of estimates

$y_t$
Two Kinds of Algorithms
Two Kinds of Algorithms

- Choice of $u_t$ → Computation of estimates
- Random choice of $u_t$ → Computation of estimates
Two Kinds of Algorithms

\[ Y_t \]

\( \cdots \)

\( U_t \)

random choice of \( U_t \)

computation of estimates
Preliminary result

\[ \theta_\star = 3, \quad \hat{\theta}_t = \frac{1}{t} \sum_{i=1}^{t} \tilde{y}_i = \frac{1}{t} \sum_{i=1}^{t} u_i y_i \]

Table:

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<tbody>
<tr>
<td>(u_t)</td>
<td>-1</td>
<td>1</td>
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<td>(v_t = \text{rand}() - 0.5 + m, \ m = 1)</td>
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<td>(y_t)</td>
<td>-2.1</td>
<td>3.8</td>
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<td>(\tilde{u}_t)</td>
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<td>(\tilde{y}_t)</td>
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<tr>
<td>(\hat{\theta}_t)</td>
<td>2.1</td>
<td>2.95</td>
<td>2.57</td>
<td>3.00</td>
<td>3.12</td>
<td>3.33</td>
<td>3.19</td>
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\(\forall t, \forall \varepsilon > 0 \quad \text{Prob}\{|\hat{\theta}_t - \theta_\star| \geq \varepsilon\} \leq \frac{1}{t} \frac{1}{\varepsilon^2} E\{v_t^2\} + o\left(\frac{1}{t}\right)\).

[Granichin, TAC, 2004]
Non-Asymptotic Result

For the finite number of observations \((N = 9)\) a new rigorous mathematical result of a guaranteed set of possible values of the unknown parameter \(\theta_*\) can be obtained for an arbitrary external noise \(\nu_t\) following by the method described by M. Campi [EJC, 2010]:

1. Let be \(M = 10\) and select randomly nine (= \(M - 1\)) different groups of four indexes \(T_1, \ldots, T_9\).
2. Compute the partial sums \(\bar{s}_i = \frac{1}{4} \sum_{j \in T_i} \bar{y}_j, \ i = 1, \ldots, 9\).
3. Build the confidence interval

\[
\hat{\Theta} = [\min_{i \in 1..9} \bar{s}_i; \max_{i \in 1..9} \bar{s}_i],
\]

which contains \(\theta_*\) with the probability \(p = 80\%\) (= \(1 - 2 \cdot 1/M\)).
The Confidence Interval

\[
\begin{pmatrix}
    s_1 \\
    s_2 \\
    s_3 \\
    s_4 \\
    s_5 \\
    s_6 \\
    s_7 \\
    s_8 \\
    s_9
\end{pmatrix} = \frac{1}{4}
\begin{pmatrix}
    0 & 1 & 1 & 1 & 1 & 0 & 0 \\
    1 & 0 & 1 & 1 & 0 & 1 & 0 \\
    0 & 1 & 1 & 0 & 1 & 1 & 0 \\
    1 & 1 & 0 & 0 & 0 & 1 & 1 \\
    1 & 0 & 0 & 1 & 1 & 0 & 1 \\
    0 & 1 & 1 & 0 & 1 & 0 & 1 \\
    1 & 0 & 0 & 1 & 1 & 0 & 1 \\
    1 & 1 & 1 & 0 & 0 & 1 & 0 \\
    1 & 0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
    -1 \cdot (-2.1) \\
    1 \cdot 3.8 \\
    -1 \cdot (-1.8) \\
    1 \cdot 4.3 \\
    1 \cdot 3.6 \\
    1 \cdot 4.4 \\
    -1 \cdot (-2.3)
\end{pmatrix} =
\begin{pmatrix}
    3.375 \\
    3.15 \\
    3.4 \\
    3.15 \\
    3.075 \\
    2.875 \\
    3.275 \\
    3.025 \\
    3.275
\end{pmatrix}
\]

- The unknown parameter \( \theta_x \) belongs to the interval \( \hat{\Theta} = [2.875, 3.4] \) with probability \( p = 80\% \).

The randomization in the process of the input data selection can get quite reasonable results.
Randomized and Bayesian Approaches

An alternative probabilistic approach is a Bayesian estimation when the noise $v_t$ probability is attributed a priori to a nature $Q$. But Bayesian and randomized approaches are quite different from the practical point of view.

In a Bayesian approach the probability $Q$ describes a probability of a value of $v_t$ in a comparison with other, i.e. the choice of $Q$ is a part of the problem model.

In contrast, the probability $P$ in a randomized approach is selected artificially. $P$ exists only in our algorithm, and therefore, there is no a traditional problem of “a bad model” as can happen with the $Q$ in a Bayesian approach.
Randomization . . .

1930 . . .
- Fisher (remove bias)

1950 . . . 1975
- Metropolis, Ulam (method Monte-Carlo)
- Rastrigin, Kirkpatrick, Holland (random search, simulation annealing, genetic algorithm)

1980 . . . 1999
- Granichin, Fomin, Chen, Guo (randomized control strategies)
- Polyak, Thzubakov, Luing, Guffi, Spall (fast algorithms)
- Granichin (arbitrary noise)
- Vadiyasagar (randomized learning theory)

2000 . . .
- Tempo, Campi, Calafiore, Dabbene, Polyak, Sherbakov etc. (probabilistic methods in a control synthesis, scenario approach)
- Candes, Donoho, Romberg, Tao (compressive sensing)
Best Features

- Significantly decreasing the number of operations
- Annihilating the systematic errors (the bias effect or an arbitrary noise)
- Accuracy usually not depend on the dimension of data
Adaptive Control

Let’s consider the dynamical system:

\[ A_\star(z^{-1})y_t = B_\star(z^{-1})u_t + v_t, \quad t = 1, 2, \ldots, N, \quad (3) \]

where \( A_\star(\lambda) = 1 + a_\star^{(1)} \lambda + \cdots + a_\star^{(n_a)} \lambda^{n_a} \), \( B_\star(\lambda) = b_\star^{(l)} \lambda^l + b_\star^{(l+1)} \lambda^{l+1} + \cdots + b_\star^{(n_b)} \lambda^{n_b} \), \( \tau_\star = \text{col}(a_\star^{(1)}, a_\star^{(2)}, \ldots, a_\star^{(n_a)}, b_\star^{(l)}, b_\star^{(l+1)}, \ldots, b_\star^{(n_b)}) \) is the vector of parameters some of which are unknown.
Control Strategy Randomization

Goal: \( \lim_{t \to \infty} |y_t| \to \min, \sup_t |y_t| + |u_t| < \infty \)

(Granichin, Fomin, ARC, 1986)

Let be \( s \leq n_a + n_b - l + 1 \) and \( N = s \cdot K \cdot N_\Delta \).

\[
\begin{align*}
u_{sn+i-l} &= \begin{cases} 
\beta_{n \div N_\Delta} \Delta_n + \bar{u}_{sn-1}, & i = 0, \\
\bar{u}_{sn+i-l}, & i \in [1..s - 1] \text{ or } i \in [-s + 1..-1], 
\end{cases}
\end{align*}
\]

\( n \in [1..KN_\Delta] \) and “own” controls \( \{\bar{u}_t\} \) are determined by the adjustable feedback law

\[
\bar{u}_t = \mathcal{U}_t(y_t, y_{t-1}, \ldots, \bar{u}_{t-1}, \ldots), \quad \bar{u}_t = 0, \quad t \geq 0.
\]
“Strip”-algorithm

The type and characteristics of a feedback depend on practical problems specifics:
1) $\tilde{u}_t = 0$, $t \in [1..N - l]$ 
2) stabilized regulator

$$C(z^{-1}, \tilde{\tau}_t)\tilde{u}_t = D(z^{-1}, \tilde{\tau}_t)y_t$$ (4)

with parameters $\tilde{\tau}_t = \hat{\tau}_{t-s}$ which are tuning by the “Strip”-algorithm

$$\hat{\tau}_t = \hat{\tau}_{t-1} - \frac{(\varphi_t^T\hat{\tau}_{t-1} - y_t)\mathbf{1}_{\{|\varphi_t^T\hat{\tau}_{t-1} - y_t| - 2C_v - \delta \|\varphi_t\| > 0\}}}{\|\varphi_t\|^2} \varphi_t, \quad (5)$$

where $\varphi_t = (-y_{t-1}, \ldots, -y_{t-n_a}, u_{t-l}, \ldots, u_{t-n_b})^T$, and 
$A(\lambda, \tau)C(\lambda, \tau) - B(\lambda, \tau)D(\lambda, \tau)$ is a stable polynomial.
Random Perturbations

$\Delta_0, \Delta_1, \ldots, \Delta_{KN\Delta-1}$ are measurable:

$$E\{\Delta_n\} = E\{\Delta^3_n\} = 0, \ E\{\Delta^2_n\} = \sigma^2_\Delta, \ E\{\Delta^4_n\} \leq M_4$$

For example, $\Delta_n = \begin{cases} +1, \text{with probability } \frac{1}{2}, \\ -1, \text{with probability } \frac{1}{2}. \end{cases}$

**A1.** The user can choose $\Delta_n$ and this choice does not affect to the external noise $v_{sn}, \ldots, v_{s(n+1)-1}$. 
Diagram of Adaptive Control
Reparametrization

We can rewrite the model (3) as a linear regression

\[ y_{sn} = \beta_{n/N} \Delta_n \theta_\star^{(1)} + \theta_\star^{(1)} \bar{u}_{sn-l} + \bar{v}_{sn}, \quad \theta_\star^{(1)} = b_\star^{(l)} \]

and

\[ y_{sn+i-1} = \beta_{n/N} \Delta_n \theta_\star^{(i)} + \sum_{j=0}^{i-1} \theta_\star^{(i-j)} \bar{u}_{sn-l+j} + \bar{v}_{sn+i-1}. \] (6)

\[ \theta_\star = \theta(\tau_\star), \quad \theta(\tau) = A^{-1}(\tau)B(\tau), \] (7)

\[ A = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots \\
0 & \ldots & 1 & \ldots & 1
\end{pmatrix}, \quad B = \begin{pmatrix}
b_\star^{(l)} \\
\vdots \\
b_\star^{(n_b)}
\end{pmatrix}. \]

(Granichin, Fomin, ARC, 1986)
How to choose $s$?

\[ A2. \quad s : \exists \tau(\theta) = \theta^{-1}(\tau) \]

\[ s = n_a + n_b \text{ if polynomials } A_\star(\lambda) \text{ and } B_\star(\lambda) \text{ are mutually prime} \]
Example
Consider the second-order plant

\[
y_t + a^{(1)}_* y_{t-1} + y_{t-2} = b^{(1)}_* u_{t-1} + 1.6 u_{t-2} + v_t,
\]

\(t = 1, 2, \ldots, N\), with unknown coefficients \(a^{(1)}_*\) and \(b^{(1)}_* \neq 0\). Denote

\[
\tau_* = \text{col}(a^{(1)}_*, b^{(1)}_*).
\]

Let \(s = 2\) and vector \(\theta_*\) of the “new” parameters be

\[
\theta_* = \begin{pmatrix} b^{(1)}_* \\ 1.6 - a^{(1)}_* b^{(1)}_* \end{pmatrix} \in \mathbb{R}^2.
\]

In this case, the inverse function \(\tau(\theta)\) is

\[
\tau(\theta) = \begin{pmatrix} \frac{1.6 - \theta^{(2)}}{\theta^{(1)}} \\ \theta^{(1)} \end{pmatrix}.
\]
Unknown Parameters Estimation

We combine two different algorithms:

- Stochastic approximation algorithm with perturbation in the input
  (Granichin, Fomin, ARC, 1986)
- Leave-out Sign-dominant Correlation Regions (LSCR) method
  (M. Campi and E. Weyer, TAC, 2010)
Stochastic Approximation Algorithm

\[
\begin{aligned}
&\left\{
\tilde{\tau}_t = \tau(\hat{\theta}_{n-1}), \ t \in [s(n - 1) + 1..sn], \ n \in [1..KN\Delta], \\
&\hat{\theta}_n^{(i)} = \hat{\theta}_{n-1}^{(i)} - \frac{1}{n}\Delta_n (u_{sn-l}\hat{\theta}_{n-1}^{(i)} - y_{sn+i-1}), \ i \in [1..s].
\end{aligned}
\]

\[
\tilde{\tau}_t = \begin{cases} 
\tilde{\tau}_t, & \text{if } |y_t| + |u_{t-1}| < \bar{R}, \\
\hat{\tau}_{t-s} & \text{otherwise.}
\end{cases}
\]

Theorem

If A1–A2 and \(2\sigma^2 > 1\), \(E\{v^2_t\} \leq \sigma^2_v\), \(\sum_{n=1}^{\infty} \frac{\beta^{-1}_n N\Delta}{n} = \infty\),
\(\sum_{n=1}^{\infty} \frac{\beta^{-2}_n N\Delta}{n^2} < \infty\) then \(\forall \rho > 0\)

\[
E\{|\hat{\theta}_n^{(i)} - \theta^{(i)}_\bullet|^2\} \leq \frac{\beta^{-1}_n N\Delta}{n} \frac{\rho C^2_{\bar{u}} + \sigma^2_{i,\bar{v}}}{2\sigma^2_\Delta - 1} + o \left(\frac{\beta^{-1}_n N\Delta}{n}\right), \ i \in [1..s],
\]

\[
C_{\bar{u}} = \sup_t \bar{u}_t, \ \sigma^2_{i,\bar{v}} = \sigma^2_v + (C^2_{\bar{u}} + \sigma^2_\Delta) \sum_{j=1}^{i-1} |\theta^{(j)}_\bullet|^2.
\]
LSCR method

1. ∀k ∈ [1..K] consider [k′s..k′s + sNΔ − 1] where k′ = (k − 1)NΔ.

2.  
\[ \hat{y}_{k's+sn+i-1}(\theta) = \beta_k \Delta_{k'+n} \theta^{(i)} + \sum_{j=0}^{i-1} \theta^{(i-j)} \bar{u}_{k's+sn+i-l-j}. \]

3.  
\[ \epsilon_t(\theta) = y_t - \hat{y}_t(\theta), \ t \in [k's..k's + sNΔ − 1]. \]

4.  
\[ f_{k's+sn+i-1}(\theta) = \Delta_{k'+n} \epsilon_{k's+sn+i-1}(\theta). \]

5. Choose M > 2s and construct M different binary stochastic strings (h_{j,1}, \ldots, h_{j,sNΔ}). We calculate

\[ g_{k,j}^{(i)}(\theta) = \sum_{n=0}^{NΔ−1} h_{j,ns+i} \cdot f_{k's+ns+i-1}(\theta), \ i \in [1..s]. \]

6. Choose q ∈ [1, M/2s] and construct \( \hat{\Theta}_k^{(i)} \): at least q of the \( g_{k,j}^{(i)}(\theta) \) functions are strictly higher than 0 and at least q functions are strictly lower than 0.
Confidence Set

We define the confidence set by the formula

\[ \hat{\Theta}_k = \bigcap_{i=1}^s \hat{\Theta}^{(i)}_k. \]  \hspace{1cm} (12)

Theorem

Let condition \textbf{A1} be satisfied. Consider \( i \in [1..s] \) and assume that \( \Pr(\theta^{\star} = 0) = 0 \). Then

\[ \Pr\{\theta^{\star} \in \hat{\Theta}^{(i)}_k\} = 1 - \frac{2q}{M} \]  \hspace{1cm} (13)

\[ \Pr\{\theta^{\star} \in \hat{\Theta}_k\} \geq 1 - \frac{2sq}{M} \]  \hspace{1cm} (14)

where \( M, q \) and \( \hat{\Theta}^{(i)}_k \) are from steps 5 and 6 of the above-described procedure.
Combined Algorithm

1. For \( k = 1, 2, \ldots, K \).
2. To generate the sequence \( \{\hat{\theta}_n\} \) by the algorithm (9).
3. To choose \( \varepsilon > 0 \) and to build the parallelepiped

\[
\bar{\Theta}_k = \prod_{i=1}^{s} \{ \theta^{(i)} : |\hat{\theta}^{(i)}_{kN\Delta} - \theta^{(i)}| \leq \varepsilon \}.
\]

4. To choose \( q \) and \( M \) and to compute the region \( \hat{\Theta}_k \) by the algorithm (12).
5. We define the confidence set as the intersection \( \tilde{\Theta}_k = \hat{\Theta}_k \cap \bar{\Theta}_k \).

**Theorem**

\[
Prob\{\theta_* \in \tilde{\Theta}_k\} \geq (1 - 2sq/M) \left( 1 - \frac{\beta_k}{kN\Delta} \frac{\rho C_{\bar{u}}^2 + \sigma_{s,\bar{v}}^2}{\varepsilon^2 (2\sigma_{\Delta}^2 - 1)} \right)^s + o \left( \frac{\beta_k}{kN\Delta} \right).
\]
Thank you for your attention!
Example

\[ y_t - 2y_{t-1} + y_{t-2} = b \star u_{t-1} + 1.6u_{t-2} + v_t, \ t = 1, \ldots, 15, \]

\[ y_0 = y_{-1} = u_{-1} = 0, \ b \star \text{ is an unknown coefficient, } v_t \text{ is an unknown external arbitrary noise.} \]

LSCR, \( s = 1 \)
Example

\[ y_t - 2y_{t-1} + y_{t-2} = b_* u_{t-1} + 1.6u_{t-2} + v_t, \quad t = 1, \ldots, 15, \]

\[ y_0 = y_{-1} = u_{-1} = 0, \quad b_* \text{ is an unknown coefficient, } v_t \text{ is an unknown external arbitrary noise.} \]

LSCR + adaptive stabilizing feedback
Example

\[ y_t - 2y_{t-1} + y_{t-2} = b_x u_{t-1} + 1.6u_{t-2} + v_t, \quad t = 1, \ldots, 15, \]

\[ y_0 = y_{-1} = u_{-1} = 0, \quad b_x \text{ is an unknown coefficient}, \quad v_t \text{ is an unknown external arbitrary noise}. \]

LSCR

\[ b_x = 1, \quad Ev_t = 0.5, \quad \sigma_v = 0.1 \]

\[ g_i(b) = \sum_{t=1}^{15} h_{i,t} \cdot \Delta_{t-1} \epsilon_t(b), \]

\[ h_{i,t} \in \{0, 1\}, \quad i = 1, \ldots, 19. \]

The confidence interval is [0.834; 1.090]
Example

\[ y_t + a_1^* y_{t-1} + y_{t-2} = b_1^* u_{t-1} + 1.6 u_{t-2} + v_t, \quad t = 1, 2, \ldots, 960, \]
\[ P_{\text{rob}}\{\theta_* \in \hat{\Theta}^k\} = 1 - \frac{2q}{M}, \]
\[ 95\% = (1 - 2 \cdot 2 \cdot 6/480) \cdot 100\%. \]
Randomized algorithm for the small UAV flight optimization

\[
\begin{align*}
   u_t &= \tilde{u}_{t-1} + \Delta_t, \\
   \hat{\theta}_{t+1} &= \hat{\theta}_t - \alpha \Delta_t \varepsilon_t, \\
   \tilde{u}_t &= \frac{b}{a} \sin \hat{\theta}_{t+1}.
\end{align*}
\]
Simulation results

<table>
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<th>External noise</th>
<th>RA</th>
<th>KF</th>
<th>SKF</th>
</tr>
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<tbody>
<tr>
<td>$v_t = 10 \cdot (\text{rand}() \cdot 4 - 2)$</td>
<td>41.36</td>
<td>38.15</td>
<td>42.65</td>
</tr>
<tr>
<td>$v_t = 0.1 \cdot \sin(t) + 19 \cdot \text{sign}(50 - t \mod 100)$</td>
<td>53.4</td>
<td>197.64</td>
<td>212.45</td>
</tr>
<tr>
<td>$v_t = 20$</td>
<td>45.15</td>
<td>276.35</td>
<td>169.48</td>
</tr>
</tbody>
</table>
Assumption: \( \{\Delta_n\}_{n=0}^{N-1} \) and \( \{v_t\}_{t=0}^{N} \) are independent. Hence, adaptive control schemes are not applicable. (We can not use the current estimates of parameters in a feedback).

Randomization adds to the control channel at each step. It disturbs the system permanently.

Algorithm dimension is so high (even for the simplest cases).
Proposition 1: Fix $k \in [1, \ldots, s]$. Let $H$ be a stochastic $M \times N_\Delta$ matrix with elements $h_{i,ns+k}$, $i = 0, 1, \ldots, M - 1$, $n = 0, \ldots, N_\Delta - 1$, from step 4 of the algorithms in Section VI, and let $\eta = \text{col}(\eta_1, \ldots, \eta_{N_\Delta})$ be a vector independent of $H$, consisting of mutually uncorrelated random variables symmetrically distributed around zero. Given an $i \in [0, M - 1]$, let $H_i$ be the $M \times N$ matrix, whose rows are equal to the $i$-th row of $H$. Then, $H\eta$ and $(H - H_i)\eta$ have the same $M$-dimensional distribution provided that the $i$-th element of $(H - H_i)\eta$ (which is 0) is repositioned as the first element of the vector. (M. Campi and E. Weyer, TAC, 2010)
**Sketch of the proof**

Denote $\eta_n := \Delta_{n-1}\epsilon_{(n-1)s+k-1}(\theta_*)$.

For the correlation between $\eta_i$ and $\eta_j$, $i > j$:

$$E[\eta_i\eta_j] = E[\Delta_{i-1}]E[\epsilon_{(i-1)s+k-1}(\theta_*)\Delta_{j-1}\epsilon_{(j-1)s+k-1}(\theta_*)] = 0$$

$$E[\Delta_{i-1}] = 0 \quad (\eta_1, \ldots, \eta_{N\Delta} \text{ are mutually uncorrelated}).$$

Take $g_i^{(k)}(\theta_*)$ in the $r$-th position.

$$g_i^{(k)}(\theta_*) - g_{\bar{i}}^{(k)}(\theta_*) = \sum_{n=0}^{N\Delta} (h_{i,ns+k} - h_{\bar{i},ns+k})\eta_n < 0$$

for $r - 1$ selection of $i \in [0, M - 1]$.

From Proposition 1: $Prob\{"r - 1 entries of (H - H_{\bar{i}})\eta are negative"\} = Prob\{"r - 1 entries of H\eta are negative"\}$, and it does not depend on $\bar{i}$. 