AUTOMATION
AND
REMOTE CONTROL

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PROCEDURE OF STOCHASTIC APPROXIMATION
WITH DISTURBANCES AT THE INPUT

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A method and an algorithm for identification of unknown parameters of a certain function when observing its values at a series of points on the background of dependent noise is proposed. For substantiation of the method, practically no assumptions on the degree of correlation of the noise in observations are made. Consistency of the identification algorithm is assured by means of an additional feeding to the input of a measurable test perturbation fading in time which is assumed to be uncorrelated with the noise in observations.

1. INTRODUCTION

To solve problems of parametric identification of objects or processes observed on a noisy background recur algorithms [1-3] are used. The problem of convergence of various recursive algorithms to the vector of the true parameter has been sufficiently thoroughly studied; however, to justify convergence in the case of dependent errors in observations almost in all the papers quite "rigid" assumptions on the degree of correlation of the errors are imposed [4-5] and its statistical properties assumed to be largely known. Actually, there may be no such information about the errors available.

In the cases when it is difficult to assert anything substantial about properties of uncontrolled noise, it was proposed in [6] for an "enrichment" of the sequence of observations — to input in the feedback channel of the control system a perturbation with known statistical properties subject to an efficient control. In [2, 7, 8] to identify parameters of a discrete dynamic object, an algorithm of stochastic approximation was proposed with a perturbation at the input. In [9] it is shown that in a number of cases the algorithm proposed in [8] is suitable for estimation of the mean value of the vector unknown parameters of a static object with drifting parameters. In [10] an algorithm of stochastic approximation with disturbance at the input is proposed aimed at determination of a scalar stationary value of the scalar function observed on background of dependent perturbations.

In [11] the problem of the rate of convergence of a method for estimating the minimum of a function of several variables observed on the background of a centered noise is investigated. The method is similar to one proposed in this paper.

2. THE BASIC RESULT

Consider the problem of determining a vector of unknown parameters \( \theta^* = (\theta_1^*, \ldots, \theta_n^*)^T \) of a function \( B(\theta') : \mathbb{R} \to \mathbb{R}^n \) under the condition that its values are observed on noisy background so that at the instants of time 1, 2, ... at points \( \theta_1', \theta_2', \ldots \) the quantities \( Y_1 = B(\theta_1') + \xi_1, Y_2 = B(\theta_2') + \xi_2, \ldots \) in this case \( \xi_1, \xi_2, \ldots \) can be measured. The design of observations (the sequence \( \{\xi_t\}_{t=1}^{\infty} \)) can either be deterministic or randomized.

We state sufficient conditions for consistency of the proposed algorithm for identification of the vector \( \theta^* \).

1. Let the vector of unknown parameter \( \theta^* \) belong to a convex compact set \( \Theta \) such that the function \( B(\theta') \) is defined and twice continuous-differentiable on the set \( \Theta = \{\theta' : \exists \theta \in \Theta : ||\theta - \theta'|| < \rho\} \).
2. For any \( \theta \in \Theta \) the inequality

\[
\nabla B(\theta) (\theta_k - \theta) \leq -L||\theta_k - \theta||^2
\]

is fulfilled for some constant \( L > 0 \).
3. The second moments of the noise in observations $\xi_t$ are bounded:

$$M_2\xi_t^4 < C, \quad t=1, 2, \ldots$$

4. Let an observed sequence $\{w_t\}_{t=1}^{\infty}$ of bounded independent identically distributed random vectors $w_t \in \mathbb{R}^n$ with the properties

$$Mw_t=0, \quad Mw_t w_t^T = \sigma_w I, \quad \sigma_w > 0, \quad \|w_t\| < C_n.$$  

be assigned, uncorrelated with the noise in observations $\{\xi_t\}_{t=1}^{\infty}$:

$$Mw_t \xi_k = 0, \quad t=1, 2, \ldots; \quad k < t.$$  

Let $P_\theta$ be a projector on the set $\Theta$. To construct a sequence of estimators $\{\theta_t\}_{t=1}^{\infty}$ of the vector of the unknown parameters $\theta^*$, consider the following identification algorithm:

$$\theta_t = \theta_0 + \gamma_t w_t,$$

$$Y_t = B(\theta_t), \quad \gamma_t = \tilde{\delta}_t w_t,$$

where $\gamma_t$ and $\tilde{\delta}_t$ are sequences of positive numbers with the properties

$$\sum_{t=1}^{\infty} \gamma_t = \infty, \quad \sum_{t=1}^{\infty} \tilde{\delta}_t = \infty.$$  

THEOREM 1. If the conditions 1-4 are fulfilled, the identification algorithm (4) provides consistent estimators of the vector of the unknown parameters $\theta^*$, i.e., $\theta_t \to \theta^*$ as $t \to \infty$ with probability 1.

It should be mentioned that in the identification algorithm (4) the design of observations is randomized and in the sequence of the observation points $\theta_0 + \delta_1 w_1, \theta_1 + \delta_2 w_2, \ldots$ the second summand serves as a test perturbation (a test signal) which fades in time in view of (5).

The proof of the theorem is presented in the Appendix. Condition 2 is the most important condition of the theorem. Condition 3 could be substantially relaxed, as well as condition (5) imposed on the numerical sequences. Condition 3 is the hardest to verify. Actual convergence of algorithm (4) is assured by choosing, uncorrelated with $\xi_t$, a measurable test perturbation $\delta_t w_t$ with given statistical properties. As the numerical sequences $\{\gamma_t\}_{t=1}^{\infty}$ and $\{\tilde{\delta}_t\}_{t=1}^{\infty}$ one could choose $\gamma_t = \sqrt{\epsilon} t^{-0.25}, \quad t=1, 2, \ldots$

From the aspect of computational complexity, to obtain estimators of the vector of unknown parameters, the number of operations in algorithm (4) at each cycle is rather small and does not increase with the growth of $t$. This is not the case when, for example, the extended LS method [3, 5] is applied.

Algorithm (4) in the scalar case (for $n = 1$) coincides with the corresponding procedure described in [10].

3. EXAMPLE

Consider the problem of detecting a known signal when observing over a noisy background. Let the quantities

$$y_t = \theta x_t + \nu_t, \quad t=1, 2, \ldots$$

be observed.

Here $x_t$ is a known signal; $\nu_t$ is a sequence of noises in observations; $\theta^*$ is the estimated quantity — an attribute of the type "yes—no": $\theta^* = 1$ indicates the presence of a signal, $\theta^* = 0$ its absence.

We shall assume that the working signal $\{x_t\}_{t=1}^{\infty}$ is a sequence of independent identically distributed random variables satisfying the conditions

$$Mx_t=0, \quad Mx_t x_t^T = \sigma_x > 0, \quad |x_t| < C, \quad t=1, 2, \ldots.$$
and for the sequence of errors in observations, the conditions
\[ Mv_i^2 \leq C_v Y_t, \quad Mx,v_i=0, \quad t=1, 2, \ldots; \quad k \leq t \]
are fulfilled.

Consider the identification algorithm for parameter \( \theta_* \):
\[
\begin{align*}
\theta_i &= 0, \quad \xi_i = 0, \\
\theta_t &= \frac{1}{2} x_t, \\
Y_t &= 0.5 \left( \theta_t - \theta_* \right)^2 - \xi_t^2.
\end{align*}
\]

We shall show that it is an algorithm of type (4) and that the conditions of Theorem 1 are fulfilled for it.

For the function \( B(\theta') = 0.5(\theta' - \theta_*)^2 \), conditions 1 and 2 of the theorem are evidently fulfilled with \( \Theta = [0, 1] \), \( \rho = C \), \( L = -1 \).

In the model (4) we select the sequence
\[
0,1,0,2,0,3,\ldots,0,4,0,5,0,6,0,7,0,8,0,9,0,10,
\]
as the design of observations.

It is easy to verify that in this case
\[
Y_t = H(\theta_t) + \xi_t,
\]
where \( \xi_t = \theta_t - \theta_* - 0.5(\theta_* - \theta_t)^2 - \xi_t^2 \). In view of (6) and the boundedness of \( \theta_t: 0 < \theta_t < 1 \), it is easy to obtain \( M\xi_t^2 \leq 3.75 + 3C_v \). Fulfillment of condition (3) is assured also due to (6) and the independence of the sequence \( \{x_t\}_{t=1}^{\infty} \).

Thus, for the algorithm (7) all the conditions of Theorem 1 are fulfilled and hence the sequence of estimators \( \{\theta_t\}_{t=0}^{\infty} \) converges with probability 1 to the estimated quantity \( \theta_* \).

4. IDENTIFICATION OF PARAMETERS OF A LINEAR DYNAMIC OBJECT

Consider yet another example of utilization of algorithm (4) for identification of parameters of a discrete control object (CO) defined by a linear scalar equation with an additive noise
\[
y_s + a_s y_{s-1} + \ldots + a_m y_{s-m} + b_s u_{s-1} + \ldots + b_m u_{s-m} + v_s,
\]
where \( y_s, u_s, v_s \) are correspondingly the output, control, and perturbation variables. Let \( Mv_s^2 < C_v \) (if \( v_s \) is not random, it is sufficient to require its boundedness). We shall assume that the vector of the true parameters of the CO
\[
\tau_* = (a_s, b_s, \ldots, a_s, b_s)
\]
is unknown, but a compact set \( T \) containing it is given. Assume that for any \( \tau \in T \) the corresponding CO is totally controllable [3].

To estimate vector \( \tau_* \) one cannot, of course, apply the algorithm (4) directly. To form the estimation algorithm, a special mapping \( \psi \) of the set \( T \) onto a convex set \( \Theta \) is required, corresponding to each vector from \( T \) another vector \( \theta \in \Theta \) by the rule determined by the recursive relations (see [8]):
\[
\begin{align*}
\theta^{(1)} &= b^{(1)}, \\
\theta^{(2)} &= b^{(2)} - a^{(1)} \theta^{(1)}, \\
\theta^{(3)} &= b^{(3)} - a^{(1)} \theta^{(2)} - a^{(2)} \theta^{(1)}, \\
&\vdots \\
\theta^{(m)} &= b^{(m)} - a^{(1)} \theta^{(m-1)} - \ldots - a^{(m-1)} \theta^{(1)}.
\end{align*}
\]
It is shown in [8] that if for any $r \in T$ the corresponding CO is completely controllable, the mapping $\psi$ is one-to-one and there exists an inverse mapping $\varphi: \Theta \to T$, which is continuous. If only part of the parameters of the CO is unknown, the corresponding set $\Theta$ can be determined to be of a smaller dimensionality than $n = m + \ell$ (see [8]).

Consider a sequence of controls $u_s$, $s = 0, 1, \ldots$, formed by the law of a linear feedback with fine-adjusted coefficients

$$u_s + \alpha_s u_{s-1} + \ldots + \alpha_s^{m-1} u_{s-m} = \beta_s y_{s-n+1} + \ldots + \beta_s^{n-2} y_{s-n+2},$$

in which the lag in the measurements utilized in formation of $u_s$ equals $n - 1$. For identification of parameters of CO in the controlling channel in addition to $u_s$ we shall add a test perturbation formed with the aid of $\{w_t\}_{t=1}^{\infty}$ — a sequence of independent, identically distributed $n$ random vectors from $\mathbb{R}$:

$$u_{t(n+1)} = v_{t(n+1)} - \Psi_t w_t,$$

Here

$$R_{t} = 1 + \sum_{i=1}^{n} |w_{t-m+i}| + \sum_{i=1}^{n} |w_{t-m+i}| + \sum_{i=1}^{n} |w_{t-m+i}|.$$

$\{\delta_t\}_{t=1}^{\infty}$ is a sequence of positive numbers.

Let $\theta_s = \psi_r$, for the sequence $\{w_t\}_{t=1}^{\infty}$ the conditions (2) be fulfilled, the test perturbation be uncorrelated with the noise $v_s$, $s \leq t(n+1)$ and for each $t$ the coefficients of the feedback $\alpha_s^{(1)}, \ldots, \alpha_s^{(n+m-2)}, \beta_s^{(1)}, \ldots, \beta_s^{(\ell)}, s < t(n+1)$ be uncorrelated with $w_t$. Then to construct the sequence of estimators $\{\hat{\theta}_t\}$ of the vector $\theta_*$ we apply the algorithm

$$\hat{\theta}_t = \Psi_t(\hat{\theta}_{t-1} + \gamma_t w_t y_t),$$

in which $\{\gamma_t\}_{t=1}^{\infty}$ and $\{\delta_t\}_{t=1}^{\infty}$ are numerical sequences satisfying the conditions (5) with a corresponding choice of the constant $\rho$.

We shall show that the algorithm (11) is of the type (4) and that the conditions of Theorem 1 are satisfied for it.

Eliminating the variables $Y_{s-1}, \ldots, Y_{s-n+1}$, Eq. (8) can be rewritten in the form

$$y_s = \sum_{i=1}^{n} \theta^{(i)}_s u_{s-i} + f(y_{s-n}, u_{s-n-1}^m) + e_s,$$

where $f(y_{s-n}, u_{s-n-1}^m)$ is a linear function of $\ell + m - 1$ arguments; $v_s$ is determined in terms of $v_s$, $v_{s+1}, \ldots, v_{s+n+1}$ and the vector of the unknown parameters $r$. From the last equation for $s = t(n+1)$, taking (10) into account, we arrive at

$$y_{t(n+1)} = R_{t-1} \delta \theta_* w_t + \sum_{i=1}^{n} \theta^{(i)}_s u_{t(n+1)-i} + f(y_{t(n-1)}, u_{t(n-1)}^m) + e_{t(n+1)}.$$

Denoting

$$z_t = \sum_{i=1}^{n} \theta^{(i)}_s u_{t(n+1)-i} + f(y_{t(n-1)}, u_{t(n-1)}^m) + e_{t(n+1)},$$

we obtain the formula

$$y_{t(n+1)} = R_{t-1} \delta \theta_* w_t + z_t.$$
Consider the function \( B(\theta') \) defined on the set \( \Theta: B(\theta') = 1/2(\theta' - \theta)'(\theta' - \theta) \). For this function conditions 1 and 2 of the theorem are fulfilled. If we choose the sequence \( \theta_t' = \theta_{t-1}' + \delta_t w_t \) as the design of observations in model (4), then taking into account the last obtained formula, it is easy to deduce that \( Y_t = B(\theta_{t-1}') + \xi_t \). Here \( \xi_t = \theta_t w_{t-1} - 0.5\|\theta_t\|^2 - \xi_t/R_{t-1} \) and in view of the assumptions imposed it satisfies conditions 3 and 3. As it is seen, all the conditions of the theorem are fulfilled and hence the sequence of estimators \( \theta_t \) converges with probability 1 to the vector of unknown parameters \( \theta* \).

Estimators of the vector \( \tau* \) can be calculated from the estimators of \( \theta* \) by means of the formula \( \tau_t = \varphi(\theta_t) \).

For the problems of adaptive control besides the determination of unknown coefficients of Eq. (8), the problem of stabilization of the motion, i.e., fulfillment of the condition

\[
\sup_s (|y_s| + |u_s|) < \infty
\]

is meaningful.

Under the control strategy chosen above, condition (12) can be assured if the coefficients of the controller (9) are selected at the instants of time \( t = nt, nt + 1, \ldots, n(t + 1) - 1 \) to be equal to the coefficients of the corresponding polynomials \( a_{nt-1}(\lambda), b_{nt-1}(\lambda) \), determined by the equation

\[
a_{nt-1}(\lambda) \alpha_{nt-1}(\lambda) - \lambda^{m-1} b_{nt-1}(\lambda) \beta_{nt-1}(\lambda) = 1 .
\]

and the condition that the degree of the polynomial \( \alpha_{nt-1}(\lambda) \), is less than \( n + m - 1 \). Here \( a_{nt-1}(\lambda) \) and \( b_{nt-1}(\lambda) \) are polynomials whose coefficients \( a_{nt-1}^{(1)}, \ldots, a_{nt-1}^{(1)}, b_{nt-1}^{(1)}, \ldots, b_{nt-1}^{(m)} \), are determined by the vector of current estimators \( \tau_{t-1} \). The proof of this nontrivial assertion is based on the fact that the estimators of the identification algorithm are convergent and could be carried out by the analogy with the proof of a similar assertion in [8].

Note that if condition (13) is fulfilled, quantities \( R_{t-1}, t = 1, 2, \ldots \) are bounded and hence as \( t \) increases, the test perturbation in the control channel attenuates more and more.

**THEOREM 2.** Let the vector of true parameters of CO \( \tau* \) belong to the compact set \( T \). For any \( \tau \in T \), the corresponding CO is totally controllable. The set \( \psi(T) \) is convex. The control \( u_t, s = 0, 1, \ldots \) is formed by the rules (9), (10). The coefficients of the controller (9) are determined by the relations (13). The sequences of positive numbers \( \{\gamma_t\}_{t=1}^{\infty} \) and \( \{\delta_t\}_{t=1}^{\infty} \) satisfy (5). Then the identification algorithm

\[
\tau_t = \varphi(P_{\tau, \tau_0}(\psi(\tau_{t-1}) + \gamma_t w_{t}, Y_t))
\]

and

\[
Y_t = -\frac{1}{2} \|\psi(\tau_{t-1}) + \delta_t w_t\|^2 - y_{nt+1}/R_{nt-1}
\]

provides for any \( \tau_0 \in T \) consistent estimators of the vector \( \tau* \) and the selected control strategy is stabilizing, i.e., (12) is fulfilled.

In view of the above, to prove Theorem 2 it is sufficient to observe that the coefficients of the controller (9) determined by the relations (13) are not correlated with the corresponding test perturbation \( w_t \).

If the noise sequence \( \{v_s\}_{s=1}^{\infty} \) in equation (8) is bounded: \( |v_s| < C_v, s = 1, 2, \ldots \), one could show (see [8]) that the control strategy (9), (10), (13) assures the fulfillment of the limiting inequality

\[
\lim_{t \to \infty} |y_t| < C_v \sum_{i=0}^{n+m-1} |a_{nt-1}^{(i)}|.
\]

Thus, the closed system is "time-tuned" to the vector of unknown parameters of an object. On the r.h.s. of the inequality we have a quantity which is equal to the value of the corresponding minimax quality functional of controlling CO with known coefficients under a control by means of a linear stationary controller when coefficients are determined by the relations (13) for \( \tau_{t-1} = \tau* \). In [8] an example of simulation on a computer of an identification process of a nonminimally-phase CO of the second order by means of test perturbations is presented.
APPENDIX

Proof Theorem 1. Consider the sequence of residuals \( A_0 \), \( t = 1, 2, \ldots \). In view of the identification algorithm (4), we have

\[
\|A_0\| \leq \|A_0\| + 2\tau \|B(\theta_{t-1} + \delta)w_t\| + 2\tau \|B(\theta_{t-1} + \delta)w_t\| + \frac{\gamma_t}{\tau} + \frac{\gamma_t}{\tau}.
\]

(A.1)

Utilizing the Taylor expansion of functions, we obtain

\[
B(\theta_{t-1} + \delta)w_t = B(\theta_{t-1}) + \nabla B(\theta_{t-1})^T \delta w_t + o(\|\delta\|^2).
\]

(A.2)

Substituting (A.2) into (A.1) and taking into account the boundedness of \( ||w_t|| \) and the compactness of \( \Theta \), we arrive at

\[
\|A_0\| \leq \|A_0\| + 2\tau \|B(\theta_{t-1})^T \delta w_t + \frac{\gamma_t}{\tau} + \frac{\gamma_t}{\tau} \|B(\theta_{t-1} + \delta)w_t\| + \frac{\gamma_t}{\tau} + \frac{\gamma_t}{\tau}.
\]

(A.3)

Denote by \( \mathcal{F}_t \) the \( \sigma \)-algebra generated by the random variables \( \theta_t, \ldots, \theta_t \). Averaging in the inequality (A.3) under the condition of the \( \sigma \)-algebra \( \mathcal{F}_t \), in view of the property of the function \( B(\theta) \) and (2), we obtain

\[
M(\|A_0\| \mid \mathcal{F}_{t-1}) \leq (1 - a_t) \|A_0\| + \beta_t.
\]

Since in view of the noncorrelatedness of the random variable \( \xi_t \) and \( w_t \), the third summand in (A.3) vanishes after averaging. Here we use the notation: \( a_t = 2\gamma_t \delta_t w_t^2 \), \( B_t = \max\{C_1, C_2, C_3\} \gamma_t (\gamma_t + \delta_t^2) \). In view of condition (5), the series consisting of the values \( a_t \) is divergent, while the one of \( \beta_t \) is convergent. The sequence \( \{\|A_0\|^2\} \) is almost supermartingale in a certain sense. In view of a corollary to Doob's theorem on the convergence of semimartingales (see [2]), we obtain from the last inequality that \( \lim \|A_0\|^2 = 0 \) as \( t \to \infty \) with probability 1 and hence the sequence \( \theta_t \) converges to \( \theta_t \) as \( t \to \infty \) with probability 1. The proof of Theorem 1 is completed.

LITERATURE CITED