Some new properties of the applied-physics related Boubaker polynomials

Tinggang Zhao
Dept. of Math., Lanzhou City University, Lanzhou 730070, P. R. CHINA

B. K. Ben Mahmoud
ESSTT/ 63 Rue Sidi Jabeur 5100 Mahdia, TUNISIA

M. A. Toumi
Dép. des Math., Faculté des Sciences de Bizerte, 7021, Zarzouna, Bizerte, TUNISIA

O. P. Faromika
Department of Physics, Federal Univ. of Technology, Akure, Ondo State, NIGERIA

M. Dada
Department of Physics, Federal Univ. of Technology, Minna, Niger State, NIGERIA

O. B. Awojoyogbe
Department of Physics, Federal Univ. of Technology, Minna, Niger State, NIGERIA

J. Magnuson
9517 Hartford Circle, Eden Prairie, MN 55347, USA

F. Lin
Dep. of Elect. and Computer Engin., Wayne State University, Detroit, MI 48202, USA.

Abstract

Some new properties of the Boubaker polynomials are presented in this paper. Among others, it is shown that that all positive zeros of the Boubaker polynomial $B_n(x)$ are in $[0,2]$. Also, there are only two pure imaginary zeros of $B_n(x)$.


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1 Introduction

Recently, the use of polynomial expansions took a big part of the most known mathematical expansion schemes and yielded meaningful results to both numerical and analytical analysis [1-8]. In this context, the Boubaker polynomials were established as a guide for solving some applied physics problems [9-22] where appears, i.e. the following equation:

\[
\frac{\partial^2 u(x,t)}{\partial x^2} = k \frac{\partial f(x,t)}{\partial t}
\]  

defined in the domain D:

\[
D: \begin{cases} -H < x < 0 \\ t > 0 \end{cases}
\]

In this paper, we intend to give some new properties of the Boubaker polynomials. We will show among others that all positive zeros of the Boubaker polynomial \( B_n(x) \) are in \([0,2]\). Also, we try to demonstrate that there are only two pure imaginary zeros of \( B_n(x) \).

2 History of the Boubaker polynomials

2.1 The Boubaker polynomials

The first monomial definition of the Boubaker polynomials [9-12] appeared in a physical study that yielded an analytical solution to heat equation inside a physical model. This monomial definition is defined by [12-18]:

\[
B_n(X) = \sum_{p=0}^{\zeta(n)} \left[ \frac{(n-4p)}{(n-p)} \right] C_{n-p}^p \cdot (-1)^p \cdot X^{n-2p}
\]

where:

\[
\zeta(n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{2n + ((-1)^n - 1)}{4}
\]

(The symbol: \( \left\lfloor \right\rfloor \) designates the floor function)
The Boubaker polynomials, which are a polynomial sequence with integer coefficients, have the explicit monic expression as follow:

\[ B_n(X) = X^n - (n - 4)X^{n-2} + \sum_{p=2}^{(n)} \frac{(n - 4p)}{p!} \prod_{j=p+1}^{2p-1} (n - j) \cdot (-1)^p \cdot X^{n-2p} \]  

(4)

The recurrence relation of the Boubaker polynomials is

\[
\begin{align*}
B_0(X) &= 1 \\
B_1(X) &= X \\
B_2(X) &= X^2 + 2 \\
B_m(X) &= X.B_{m-1}(X) - B_{m-2}(X) \quad \text{for} \ m \geq 2
\end{align*}
\]  

(5)

2.2 The modified Boubaker polynomials (Boubaker-Turki polynomials)

The Boubaker-Turki polynomials or modified Boubaker polynomials [10,17], which are an enhanced form of the formerly defined polynomials, have been established as solutions of the second order differential equation:

\[
(X^2 - 1)(3nX^2 + n - 2)\left[\frac{d^2}{dX^2}B_n(X)\right] + P_n(X)\left[\frac{d}{dX}B_n(X)\right] + Q_n(X)B_n(X) = 0
\]  

(6)

where

\[
\begin{align*}
P_n(X) &= 3X(nX^2 + 3n - 2) \\
Q_n(X) &= -n(3n^2X^2 + n^2 - 6n + 8)
\end{align*}
\]  

The modified Boubaker polynomials have a recursive coefficient definition [17] expressed by equation:
\[
\begin{align*}
\tilde{B}_n(X) &= \sum_{j=0}^{\xi(n)} [b_{n,j} X^{n-2j}] \\
\tilde{b}_{n,0} &= 2^n; \quad \tilde{b}_{n,1} = -2^{n-2}(n-4); \\
\tilde{b}_{n,j+1} &= \frac{(n-2j)(n-2j-1)}{(j+1)(n-j-1)} \times \frac{(n-4j-4)}{(n-4j)} \times \tilde{b}_{n,j} \\
\tilde{b}_{n,\xi(n)} &= \begin{cases} 
(-1)^{n/2} \times 2 & \text{if } n \text{ even} \\
2(-1)^{(n+1)/2} \times 2(n-2) & \text{if } n \text{ odd}
\end{cases}
\end{align*}
\]  

(7)

Both Boubaker and Boubaker-Turki polynomials are the source of several registered integer sequences [12-14].

The ordinary generating function of the Boubaker-Turki polynomials:

\[
f_B(X,t) = \frac{1 + 3t^2}{1 + t(t - 2X)}
\]  

(8)

2.3 The 4q-Boubaker polynomials subsequence

The Boubaker polynomials \(B_n\) explicit monomial form evoked, while prospected, some singularities for \(m=4, 8, 12,\) etc. In fact for the general case: \(m=4q\) the 2\(q\) rank monomial term is removed from the explicit form so that the whole expression contains only 2\(q\) effective terms. Correspondent 4\(q\)-order Boubaker polynomials [11] are presented in equation (9) as a general form and equation (10) as first functions:

\[
B_{4q}(X) = 4 \sum_{p=0}^{2q} \frac{(q-p)}{(4q-p)} C_{4q-p}^{p} (-1)^{p} X^{2(2q-p)}
\]  

(9)
3 The upper bound of the zeros of the Boubaker polynomial

**Theorem 3.1** Let $x_k$ ($1 \leq k \leq n$) be zeros of the Boubaker polynomial $B_n$, then:

$$|x_k| < 2, \quad \text{for} \quad 1 \leq k \leq n$$  \hspace{1cm} (11)

**Proof.** Making use of the recurrence relation (5), we obtain the following relation:

$$[M] \times [B] = x \times [B] + [C]$$  \hspace{1cm} (12)

where

$$[M] = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
-2 & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix}$$  \hspace{1cm} (13)

and

$$[B] = \begin{pmatrix}
B_0(X) = 1; \\
B_4(X) = X^4 - 2; \\
B_8(X) = X^8 - 4X^6 + 8X^2 - 2; \\
B_{12}(X) = X^{12} - 8X^{10} + 18X^8 - 35X^4 + 24X^2 - 2; \\
B_{16}(X) = X^{16} - 12X^{14} + 52X^{12} - 88X^{10} + 168X^8 - 168X^4 - 48X^2 - 2; \\
B_{20}(X) = X^{20} - 16X^{18} + 102X^{16} - 320X^{14} + 455X^{12} - 858X^8 + 1056X^6 - 495X^4 + 80X^2 - 2; \\
\text{.........................}
\end{pmatrix}$$
\[ [B] = \begin{pmatrix} B_0(x) \\ B_1(x) \\ \vdots \\ B_{n-1}(x) \end{pmatrix}; \quad [C] = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ B_n(x) \end{pmatrix} \]

Hence, from (12), zeros of the Boubaker polynomial \( B_n \) are also eigenvalues of matrix \([M]\). In fact, the eigen polynomial of the matrix is precisely the Boubaker polynomial. We can use the Gerschgorin’s theorem [6] to estimate the eigenvalues of \([M]\). By the special structure of \([M]\), it is easy to see that all eigenvalues are in the circle with centre at 0 and radius 2. This means that the result (11) holds.

**Theorem 3.2** There holds the following expression:

\[ B_n(2) = 4n - 2; \quad \text{for } n > 0. \quad \text{(14)} \]

**Proof.** From (5), for \( m > 2 \), one can get:

\[ B_m(2) - B_{m-1}(2) = B_{m-1}(2) - B_{m-2}(2) = \cdots = B_2(2) - B_1(2) \quad \text{(15)} \]

Summing the equalities together, gives the desired result as required.

**4 Some properties of the Boubaker polynomial \( B_n \)**

Now let us introduce the m-distribution notion [1]. A nondecreasing bounded function \( \alpha \) defined in \( ]-\infty, \infty[ \), is called an m-distribution, if it takes infinitely many distinct values, and its moments, that is, the improper Stieltjes integral:

\[ \int_{-\infty}^{\infty} x^n d\alpha(x) = \lim_{\epsilon_1 \to -\infty, \epsilon_2 \to \infty} \left[ \int_{\epsilon_1}^{\epsilon_2} x^n d\alpha(x) \right] \quad \text{(16)} \]
Lemma 4.1 Let \( \{p_n\}_{n=0}^{\infty} \) be the sequence of the orthogonal polynomials associated with an \( m \)-distribution \( \alpha \). Then each \( p_n \) has exactly \( n \) simple real zeros lying in the interior of the smallest interval containing \( \text{supp}(\alpha) \).

Note that \( \pm \sqrt{2i} (i = \sqrt{-1}) \) are two zeros of \( B_2(x) \) and \( \pm i \) are two zeros of \( B_3(x) \).

Theorem 4.1 The Boubaker polynomial \( B_n(x) \) does not belong to orthogonal polynomial system associated with any \( m \)-distribution.

The Boubaker polynomials have a similar Christoffel-Darboux formula.

Theorem 4.2 The following equality holds:

\[
\sum_{k=0}^{n} B_k(x)B_k(y) = 3 + \frac{B_{n+1}(x)B_n(y) - B_n(x)B_{n+1}(y)}{x - y}
\]  

(17)

for all \( x \neq y \)

Proof. The recurrence relation (5) yields that:

\[
B_{k+1}(x)B_k(y) - B_k(x)B_{k+1}(y) = (x - y)B_k(x)B_k(y) - [B_{k-1}(x)B_k(y) - B_k(x)B_{k-1}(y)]
\]

for \( k = 2,3,... \), so we have:

\[
B_k(x)B_k(y) = \frac{\Delta_k - \Delta_{k-1}}{x - y} \quad \text{for} \quad k = 2,3,...
\]

(19)

in which: \( \Delta_k = B_{k+1}(x)B_k(y) - B_k(x)B_{k+1}(y) \). Summing (19) from 0 to \( n \) gives the desired formula.

If \( x \to y \) in (17), we obtain the following Corollary

Corollary 4.1 The following equality is satisfied
\[
\sum_{k=0}^{n} B_k^2(x) = 3 + B_{n+1}'(x)B_n(x) - B_n'(x)B_{n+1}(x)
\]  

(20)

5 Further study on zeros of \( B_n(x) \)

**Lemma 5.1** Each \( B_n(x) \) \((n \geq 1)\) has exactly \( n \) simple zeros.

**Theorem 5.1** The Boubaker polynomial \( B_n \) \((n \geq 1)\) has \( \left\lfloor \frac{n}{2} \right\rfloor - 1 \) zeros for \( 0 < x < 2 \)

**Proof.** Thanks to the relation

\[
B_n(2\cos t) = 4\cos(t)\sin(nt) - 2\cos(nt), \quad \text{for } n > 1
\]  

(21)

to find the zeros of \( B_n(x) = 0 \) for \( 0 < x < 2 \), we set \( x = 2\cos t \) with \( 0 < t < \frac{\pi}{2} \) and solve:

\[
\tan t = 2\tan(nt)
\]  

(22)

It follows easily that \( B_n(x) = 0 \) has \( \left\lfloor \frac{n}{2} \right\rfloor - 1 \) zeros for \( 0 < x < 2 \) and the proof is complete.

**Remark 5.1** Now we can count the zeros of \( B_n \) as follows:

1. When \( n \) is even, all of the zeros involve \( \frac{n}{2} - 1 \) positive real zeros and \( \frac{n}{2} - 1 \) negative real zeros which locate symmetrically in \([-2,2]\] and 2 conjugate pure imaginary zeros.
2. When \( n \) is odd, all of the zeros involve \( \left(\frac{n-1}{2}\right) - 1 \) positive real zeros and \( \left(\frac{n-1}{2}\right) - 1 \) negative real zeros which locate symmetrically in \([-2,2]\] and 2 conjugate pure imaginary zeros.

The Boubaker polynomial \( B_n \) \((n \geq 1)\) has \( \left\lfloor \frac{n}{2} \right\rfloor - 1 \) zeros for...
Corollary 5.1  The Boubaker polynomial $B_n$ can’t have non-simple (or double) zeros

In fact, if we suppose that there exists $n$ such that $B_n$ has at least a non-simple (or double) zero, denoted by $x_0$: In view of Corollary 4.1, we deduce:

$$\sum_{k=0}^{n} B_k^2(x_0) = 3 + B_{n+1}'(x_0)B_n(x_0) - B'_n(x_0)B_{n+1}(x_0).$$

Therefore

$$\sum_{k=0}^{n} B_k^2(x_0) = \sum_{k=0}^{n-1} B_k^2(x_0) + B_n^2(x_0) = \sum_{k=0}^{n-1} B_k^2(x_0) = 3$$

which is absurd and we are done.

**Theorem 5.2**  Let $\pm t_ni$ be the two pure imaginary zeros of the Boubaker polynomial $B_n(x)$ $(n \geq 2)$ then $t_n$ converges to $\frac{2\sqrt{3}}{3}$

**Proof.** We also have

$$B_n(2i\sinh t) = \begin{cases} i^n(4\tanh t \sinh(nt) - 2\cosh(nt)), & \text{if } n \geq 2 \text{ is even} \\ i^n(4\tanh t \cosh(nt) - 2\sinh(nt)), & \text{if } n \geq 1 \text{ is odd} \end{cases}$$

To find solutions of $B_n(x)=0$ for positive imaginary $x$ we set.

$$2 \tanh t = \coth(nt) \quad \text{if } n \text{ is even}$$

and

$$2 \tanh t = \tanh(nt) \quad \text{if } n \text{ is odd}$$

If $n \geq 2$ there is a unique solution $t_n > 0$ Since $\tanh(nt) \to 1$ as $n \to \infty$ for each fixed $t > 0$, we obtain that $2 \tanh(t_n) \to 1$. Therefore $2 \sinh(t_n) \to 2\sqrt{2}/3$ .

**Theorem 5.3**  There are only two pure imaginary zeros of the Boubaker polynomial $B_{4q}(x)$ of degree $4q$
Proof. Let $a_1 (a>0)$ be a pure imaginary zero of $B_{4q}(x)$.

$f(a) = B_{4q}(ai)$

It is an easy task to show that the polynomial $f(a)$ in $a$ of degree $4q$ has only one positive real zero. From (9), we have:

$$f(a) = \sum_{p=0}^{2q-1} \left[ \frac{4(q - p)}{4q - p} C_{4q-p}^p a^{4q-2p} \right] - 2$$

(28)

Let us check the sign changes of the coefficients in $f(a)$. Let us denote the ratios of coefficients in $f(a)$ by $\alpha_f[p+1,p]$. Then:

$$\alpha_f[p+1,p]=\frac{(q - p - 1)(4q - 2p)(4q - 2p - 1)}{(q - p)(p + 1)(4q - p - 1)}$$

(29)

Hence, the coefficient of $a^{4q-2p}$ is positive if $p<q$, otherwise is negative. By denoting the number of sign changes of coefficients in $f(a)$ by $V_f$ and number of positive real zeros of $f(a)$ by $N_f$, we can use the Descartes's rule of signs to obtain:

$$N_f = V_f - 2k$$

(30)

where $k$ is an integer.

It is clear that $k=0$ and then $N_f=1$, which completes the proof.

6. Conclusion

The upper bound of the zeros of the Boubaker polynomials has been studied. This is of interest not only because of its application to determine new properties of Boubaker polynomials but also because the used method can be applied to solve problems in Physics, Chemistry Biology and Medicine. By means of these polynomials, appropriate mathematical algorithms and computational methods can easily be developed to reveal specific information needed to solve real physiological and pathological
problems. For example, using the Boubaker polynomials expansion scheme described here, one can solve the Bloch NMR flow equations for different flow systems. With these possibilities, we can still find new and robust algorithms to solve very old problems. These possibilities will be explored in our next investigation.

References


