Existence of Three Bounded Positive Solutions of Quasi-linear functional differential equations [1]

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Abstract. By applying Leggett-Williams fixed-point theorem in cones in Banach spaces, we obtain existence results of at least three bounded positive solutions for a functional differential equation with p-Laplacian.

Keywords and phrases. Second order functional differential equation on half line; bounded positive solution.

AMS Subject Classification. 34B10, 34B15, 35B10

1 Introduction

Recently an increasing interest has been observed in investigating the oscillations of solutions of second order differential equations on half line.

For example, Kamenev [2], Philos [3] and Yang [4] dealt with the oscillatory behavior of second order differential equations. Indeed, they extensively investigated the oscillation of the following second order differential equations without or with damping

$$(r(t)x'(t))' + p(t)x(t) = 0.$$

Several authors including Li and Yeh [5], Hsu and Yeh [6], Wong and Agarwal [7] have investigated the oscillation criteria of second order half-linear differential equations of the form

$$(r(t)|x'(t)|^{p-2}x'(t))' + p(t)|x(t)|^{p-2}x(t) = 0.$$

Takasi and Yoshida [8] considered the oscillation of the following second order quasilinear differential equation

$$(r(t)|x'(t)|^{p-2}x'(t))' + p(t)|x(t)|^{q-2}x(t) = 0.$$

In recent papers [10-12], by comparing with the oscillatory behavior of a certain associated linear second-order differential equation, the authors studied the oscillatory properties of solutions of the following nonlinear differential equation

$$(r(t)\Phi(x'(t)))' + c(t)\Phi(x(t)) = 0,$$

where $\Phi(x) = |x|^{p-2}x$ with $p > 1$.

However, the existence of multiple positive solutions of above mentioned equations have received much less attention.

In this paper, we consider the following functional differential equation with p-Laplacian on half line

$$\begin{cases} [p(t)\phi(x'(t))]' + q(t)f(t, x(t), x(\mu(t)) = 0, & t \in (0, +\infty), \\ x(t) = \psi(t), & t \in [-r, 0], \end{cases} \tag{1}$$

where

- $\psi : [-r, 0] \to R$ is continuous with $\psi(0) = 0$;
- $\mu : [0, +\infty) \to [-r, +\infty)$ with $\mu(t) \leq t$;

- $f : [0, +\infty) \times [0, +\infty) \times R \to [0, +\infty)$ is an S-Carathéodory function and $f(t, 0, 0) \neq 0$ on each subinterval of $[0, +\infty)$;

- $p : [0, +\infty) \to (0, +\infty)$ with $p \in C^1[0, +\infty)$ and there exists the limit $\lim_{t \to +\infty} p(t)$ and $\int_0^{+\infty} \phi^{-1}\left(\frac{1}{p(s)}\right) ds = +\infty$;

- $q : [0, +\infty) \to [0, +\infty)$ with $q \in C^0[0, +\infty)$ and $0 < \int_0^{+\infty} q(s)ds < +\infty$;

- $\phi : R \to R$ is called quasi-linear operator satisfying that $\phi'(x) > 0$ for all $t \in R, t \neq 0$, its inverse function is denoted by $\phi^{-1}(x)$.

The purpose of this paper is to establish existence results for at least three bounded positive solutions of equation (1) by applying fixed point theorem in cones in Banach spaces. Based on the results in this paper, the existence of three bounded positive solutions of the equation

$$[p(t)\phi(x'(t))]' + f(t, x(t)) = 0$$

subjected to the initial condition

$$x(t) = \psi(t), \quad t \in [-r, 0]$$

is established.

We call a function $x : [-r, +\infty) \to R$ is a positive solution of equation (1) if x satisfies equation

$$[p(t)\phi(x'(t))]' + q(t)f(t, x(t), x(\mu(t))) = 0, \quad t \in (0, +\infty)$$

and the initial conditions

$$x(t) = \psi(t), \quad t \in [-r, 0]$$

and $x(t) > 0$ for all $t > 0$.

The remainder of this paper is organized as follows: the main result and an example to illustrate the main result are presented in Section 2. The proof of the main result is given in Section 3.

2 Main Results and Examples

In this section, we present the main result of the paper, an example is given to illustrate the main theorem. whereas the known results in the current literature do not cover that example.

Definition 2.1. $f : [0, +\infty) \times R^2 \to R$ is called an S-Carathéodory function if

(i) for each $u, v \in R$, $t \to f(t, u, v)$ is measurable on $[0, +\infty)$;

(ii) for a.e. $t \in [0, +\infty)$, $(u, v) \to f(t, u, v)$ is continuous on $[0, +\infty) \times R$;

(iii) for each $r > 0$, there exists $\phi_r \in L^1[0, +\infty)$ satisfying $\phi_r(t) > 0, t \in (0, +\infty)$ and $\int_0^{+\infty} \phi_r(s)ds < +\infty$ such that $|u|, |v| \leq r$ implies $|f(t, (1+t)u, \sigma(t)v)| \leq \phi_r(t)$, a.e. $t \in [0, +\infty)$, where $\sigma(t) = 1 + \mu(t)$ if $\mu(t) \geq 0$ and $\sigma(t) = \psi(t)$ if $\mu(t) \leq 0$.

Suppose $k > 1$. Denote

$$M = \phi^{-1}\left(\frac{1}{p(0)}\int_0^{+\infty} q(u)du\right),$$

$$L = \frac{k}{1+k}\int_0^{1/k} \phi^{-1}\left(\frac{1}{p(s)}\int_{1/k}^{k} q(u)du\right)ds.$$

Theorem 2.1. Suppose that f is an nonnegative S-Carathéodory function, $k > 1$ a constant and there exist constants e_1, e_2 and c such that

$$0 < e_1 < e_2/k < (1+k)e_2 < c$$

and

(A1) $f(t, (1+t)u, (1+\mu(t))v) < \phi\left(\frac{c}{M}\right)$ for $t \in [0, +\infty)$, $u, v \in [0, c]$;

(A2) $f(t, (1+t)u, \psi(\mu(t))) < \phi\left(\frac{c}{M}\right)$ for $t \in [0, +\infty)$ with $\mu(t) \leq 0$, $u \in [0, c]$;

(A3) $f(t, (1+t)x, (1+\mu(t))v) < \phi\left(\frac{e_1}{M}\right)$ for $t \in [0, +\infty)$ and $x, v \in [0, e_1]$;

(A4) $f(t, (1+t)x, \psi(\mu(t))) < \phi\left(\frac{e_1}{M}\right)$ for $t \in [0, +\infty)$ with $\mu(t) \leq 0$ and $x \in [0, e_1]$;

(A5) $f(t, (1+t)x, (1+\mu(t))v) > \phi\left(\frac{e_2}{L}\right)$ for $t \in [1/k, k]$ and $x, v \in \left[\frac{e_2}{k}, (1+k)e_2\right]$;

(A6) $f(t, (1+t)x, \psi(\mu(t))) > \phi\left(\frac{e_2}{L}\right)$ for $t \in [1/k, k]$ with $\mu(t) \leq 0$ and $x \in \left[\frac{e_2}{k}, (1+k)e_2\right]$.

Then equation (1) has at least three positive solutions y_1, y_2 and y_3 such that

$$y_i(t) = \begin{cases} x_i(t), \ t \geq 0, \\ \psi(t), \ t \in [-r, 0], \end{cases} \quad i = 1, 2, 3$$

with x_1, x_2 and x_3 satisfying

$$\sup_{t\in[0,+\infty)} \frac{x_1(t)}{1+t} < e_1, \quad \min_{t\in[1/k,k]} x_2(t) > \frac{1+k}{k}e_2$$

and

$$\sup_{t\in[0,+\infty)} \frac{x_3(t)}{1+t} > e_1, \quad \min_{t\in[1/k,k]} x_3(t) < \frac{1+k}{k}e_2.$$

Now, we present an example, which can not be covered by known results, to illustrate the main results.

Example 2.1. Consider the following BVP

$$\begin{cases} \left[[x'(t)]^3 \right]' + e^{-t}f(t, x(t), x(t-2)) = 0, & t \in (0, +\infty), \\ x(t) = -t^2, \ t \in [-2, 0], \end{cases} \tag{2}$$

where f is defined by

$$f_0(x) = \begin{cases} 500, & x \in [0, 10], \\ 500 + (x - 10)\frac{5\times10^{23}+\frac{101^3\times5\times10^{11}}{e^{-0.01}-e^{-100}}-500}{100-10}, & x \in [10, 100], \\ 5 \times 10^{23} + \frac{101^3\times5\times10^{11}}{e^{-0.01}-e^{-100}}, & x \in [100, 10^8], \\ \left(5 \times 10^{23} + \frac{101^3\times5\times10^{11}}{e^{-0.01}-e^{-100}}\right)e^{x-10^8}, & x \geq 10^8 \end{cases}$$

and

$$f(t, x, y) = \frac{t}{10^{29}} + f_0\left(\frac{x}{1+t}\right) + \frac{1}{10^{29} + y^2}.$$

Corresponding to equation (1), one sees that $\phi(x) = x^3$, $p(t) = 1$, $q(t) = e^{-t}$, $\mu(t) = t - 2$ and $\psi(t) = -t^2$. Then $\phi^{-1}(x) = x^{\frac{1}{3}}$.

Choose $k = 100$. One finds that

$$M = \phi^{-1}\left(\frac{1}{p(0)}\int_0^{+\infty} q(u)du\right) = 1,$$

$$L = \frac{k}{1+k}\int_0^{1/k}\phi^{-1}\left(\frac{1}{p(s)}\int_{1/k}^k q(u)du\right)ds = \frac{\sqrt[3]{e^{-0.01}-e^{-100}}}{101}.$$

Choose $e_1 = 10, e_2 = 10000, c = 10^8$. One sees that

- $f(t, (1+t)u, (t-1)v) < \phi\left(\frac{c}{M}\right) = 10^{24}$ for $t \geq 2$, $u, v \in [0, 10^8]$;
- $f(t, (1+t)u, -(t-2)^2) < \phi\left(\frac{c}{M}\right) = 10^{24}$ for $0 \leq t \leq 2$, $u \in [0, 10^8]$;

- $f(t, (1+t)x, (t-1)v) < \phi\left(\frac{e_1}{M}\right) = 1000$ for $t \geq 2$ and $x, v \in [0, 10]$;
- $f(t, (1+t)x, -(t-2)^2) < \phi\left(\frac{e_1}{M}\right) = 1000$ for $0 \leq t \leq 2$ and $x \in [0, 10]$;
- $f(t, (1+t)x, (t-1)v) > \phi\left(\frac{e_2}{L}\right) = \frac{101^3 \times 10^{12}}{e^{-0.01} - e^{-100}}$ for $t \in [2, k]$ and $x, v \in [100, 1010000]$;
- $f(t, (1+t)x, -(t-2)^2) > \phi\left(\frac{e_2}{L}\right) = \frac{101^3 \times 10^{12}}{e^{-0.01} - e^{-100}}$ for $t \in [0.01, 2]$ and $x \in [100, 1010000]$.

It is easy to see that $(A1) - (A6)$ hold. Theorem 2.1 implies that equation (2) has at least three positive solutions y_1, y_2 and y_3 such that

$$y_i(t) = \begin{cases} x_i(t), \ t \geq 0, \\ -t^2, \ t \in [-2, 0], \end{cases} \quad i = 1, 2, 3$$

with x_1, x_2 and x_3 satisfying

$$\sup_{t \in [0, +\infty)} \frac{x_1(t)}{1+t} < 10, \quad \min_{t \in [0.01, 100]} x_2(t) > 10100$$

and

$$\sup_{t \in [0, +\infty)} \frac{x_3(t)}{1+t} > 10, \quad \min_{[0.01, 100]} x_3(t) < 10100.$$

Remark. It is easy to see that Example 2.1 can not be covered by the theorems in [1-26].

3   Proofs of Theorem 2.1

In this section, we first present some background definitions in Banach spaces and then theorem 2.1 is proved.

As usual, let X be a semi-ordered real Banach space. The nonempty convex closed subset P of X is called a cone in X if $ax \in P$ and $x+y \in P$ for all $x, y \in P$ and $a \geq 0$, and $x \in X$ and $-x \in X$ imply $x = 0$. A map $\psi : P \to [0, +\infty)$ is a nonnegative continuous concave (or convex) functional map provided ψ is nonnegative, continuous and satisfies

$$\psi(tx + (1-t)y) \geq \ (\text{ or } \leq \ ) \ t\psi(x) + (1-t)\psi(y) \text{ for all } x, y \in P, t \in [0, 1].$$

An operator $T; X \to X$ is completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Let X be a Banach space, P be a cone of X, $\psi : P \to P$ be a nonnegative convex continuous functional. Denote the sets by

$$P_c = \{x \in P : ||x|| < c\}, \ \ \overline{P}_c = \{x \in P : ||x|| \leq c\}$$

and

$$P(\psi; b, d) = \{x \in P : \psi(x) \geq b, ||x|| \leq d\}.$$

Theorem 3.1[1]. Suppose that X is a Banach space and P is a cone of X. Let $T : \overline{P}_c \to \overline{P}_c$ be a completely continuous operator and let ψ be a nonnegative continuous concave functional on P such that $\psi(y) \leq ||y||$ for all $y \in \overline{P}_c$. Suppose that there exist $0 < a < b < d \leq c$ such that

(C1) $\{y \in P(\psi; b, d) | \psi(y) > b\} \neq \emptyset$ and $\psi(Ty) > b$ for $y \in P(\psi; b, d)$;

(C2) $||Ty|| < a$ for $||y|| \leq a$;

(C3) $\psi(Ty) > b$ for $y \in P(\psi; b, c)$ with $||Ty|| > d$.

Then T has at least three fixed points y_1, y_2 and y_3 such that $||y_1|| < a$, $\psi(y_2) > b$ and $||y_3|| > a$ with $\psi(y_3) < b$.

Choose

$$X = \left\{ x \in C^0[0, +\infty) : \text{ there exists the limt } \lim_{t \to +\infty} x(t) \right\}.$$

We call $x \ll y$ for $x, y \in X$ if $x(t) \leq y(t)$ for all $t \in [0, +\infty)$. Since $x \in X$ implies that there exists the limit $\lim_{t \to +\infty} x(t)$. Then $\lim_{t \to +\infty} \frac{x(t)}{1+t} = 0$. Define the norm

$$||x|| = \sup_{t \in [0, +\infty)} \frac{|x(t)|}{1+t} \text{ for } x \in X.$$

It is easy to see that X is a semi-ordered real Banach space. Choose $k > 1$. Define the cone in X by

$$P = \left\{ \begin{array}{l} x \in X : x(t) \geq 0 \text{ for all } t \in [0, +\infty) \\ \text{ and is increasing, concave on } [0, +\infty), \\ x\left(\frac{1}{k}\right) \geq \frac{1}{k} \sup_{t \in [0, +\infty)} \frac{x(t)}{1+t}. \end{array} \right\}.$$

Define the functionals on $P \to R$ by

$$\psi(y) = \frac{k}{1+k} \min_{t \in [1/k, k]} |y(t)|, \ y \in P.$$

It is easy to see that ψ be a nonnegative continuous concave functional on P such that $\psi(y) \le ||y||$ for all $y \in P$.

Lemma 3.1. Suppose that $x \in X$ is a solution of equation (1). Then

$$x(t) = \begin{cases} \int_0^t \phi^{-1}\left(\frac{1}{p(s)} \int_s^{+\infty} q(u)f(u, x(u), x(\mu(u)))du\right) ds, & t \in [0, +\infty) \\ \psi(t), & t \in [-r, 0]. \end{cases}$$

Proof. Since $x \in X$, we get that $\lim_{t \to +\infty} x'(t) = 0$ and x is bounded. Since $f : [0, +\infty) \times [0, +\infty) \times R \to [0, +\infty)$ is an S-Carathéodory function, we have $\int_0^{+\infty} f(u, x(u), x(\mu(u)))du < +\infty$. Integrating (1) from t to $+\infty$, We get

$$p(t)\phi(x'(t)) = \int_t^{+\infty} q(u)f(u, x(u), x(\mu(u)))du, \quad t \in [0, +\infty).$$

Then

$$x(t) = \int_0^t \phi^{-1}\left(\frac{1}{p(s)} \int_s^{+\infty} q(u)f(u, x(u))du\right) ds, \quad t \in [0, +\infty).$$

Then

$$x(t) = \begin{cases} \int_0^t \phi^{-1}\left(\frac{1}{p(s)} \int_s^{+\infty} q(u)f(u, x(u), x(\mu(u)))du\right) ds, & t \in [0, +\infty) \\ \psi(t), & t \in [-r, 0]. \end{cases}$$

The proof is completed.

Lemma 3.2. Suppose that $x \in X$ is a solution of equation (1). Then $x'(t) \ge 0$ for all $t \in [0, +\infty)$ and $x(t)$ is positive and concave on $(0, +\infty)$.

Proof. First, we prove that x' is positive on $[0, +\infty)$. Since $x \in X$ is a solution of (1), we get from Lemma 3.1 that $x'(+\infty) = 0$. Since f is nonnegative, we gat that $[p(t)\phi(x'(t))]' \le 0$ for all $t \in [0, +\infty)$. Then

$$p(+\infty)\phi(x'(+\infty)) - p(t)\phi(x'(t)) \le 0, \quad t \in [0, +\infty).$$

It follows that $p(t)\phi(x'(t)) \ge 0$. Then $x'(t) \ge 0$ for all $t \in [0, +\infty)$.

Second, we prove that $x'(t)$ is decreasing on $[0, +\infty)$. For $t \in [0, +\infty)$, let

$$\tau(t) = \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\right) ds.$$

It is easy to see that $\tau \in C([0, +\infty), [0, +\infty))$ and

$$\frac{d\tau}{dt} = \phi^{-1}\left(\frac{1}{p(t)}\right) > 0.$$

Thus

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{dx}{d\tau}\phi^{-1}\left(\frac{1}{p(t)}\right).$$

It follows that

$$p(t)\phi\left(\frac{dx}{dt}\right) = \phi\left(\frac{dx}{d\tau}\right).$$

Hence

$$\left[p(t)\phi\left(\frac{dx}{dt}\right)\right]' = \phi'\left(\frac{dx}{d\tau}\right)\frac{d^2x}{d\tau^2}\frac{d\tau}{dt}.$$

So

$$\frac{d^2x}{d\tau^2} = \frac{\left[p(t)\phi\left(\frac{dx}{dt}\right)\right]'}{\phi'\left(\frac{dx}{d\tau}\right)\frac{d\tau}{dt}}.$$

Since $[p(t)\phi(x'(t))]' \leq 0$, $\phi'(x) > 0 (x > 0)$ and $\frac{d\tau}{dt} \geq 0$, we get that $\frac{d^2x}{d\tau^2} \leq 0$. Hence $x'(\tau)$ is decreasing on $[0, +\infty)$. So x is concave on $[0, +\infty)$.

Finally, we prove that x is positive on $[0, +\infty)$. From above discussion, we get that $x'(t) \geq 0$ and $x'(t)$ is decreasing on $[0, +\infty)$. Then $x(0) = 0$ implies that $x(t) > 0$ on $(0, +\infty)$. We get that $x(t)$ is positive on $(0, +\infty)$. The proof is completed.

Define the nonlinear operator $T : X \to X$ by

$$(Tx)(t) = \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\int_s^{+\infty} q(u)f(u, x(u), x(\mu(u)))du\right)ds, \quad t \in [0, +\infty).$$

Lemma 3.3. Suppose that $k > 1$. It is easy to show that

(i) Tx satisfies

$$[p(t)\phi((Tx)'(t))]' + q(t)f(t, x(t), x(\mu(t))) = 0, \quad t \in [0, +\infty)$$

and

$$(Tx)(0) = 0;$$

(ii) $Ty \in P$ for each $y \in P$;

(iii) If x is a solution of the operator equation $x = Tx$ in P, then

$$x_1(t) = \begin{cases} x(t), \ t \in [0, +\infty), \\ \psi(t), \ \ t \in [-r, 0] \end{cases}$$

is a positive solution of equation (1).

Proof. The proofs of (i) and (iii) are simple and are omitted.

For $x \in P$, by the definition of T, we know that Tx is defined on $[0, +\infty)$. It follows from Lemma 3.1 and 3.2 that Tx is increasing, concave and positive on $[0, +\infty)$. It is easy to see that the function $(Tx)(t)/(1 + t)$ achieves its maximum at $[0, +\infty)$, then

$$\sup_{t \in [0, +\infty)} (Tx)(t)/(1 + t) = (Tx)(\sigma)/(1 + \sigma).$$

Then

$$(Tx)\left(\frac{1}{k}\right) = x\left(\frac{k - 1 + k\sigma}{k + k\sigma}\frac{1}{k - 1 + k\sigma} + \frac{1}{k + k\sigma}\sigma\right) \geq \frac{1}{k}(Tx)(\sigma)/(1 + \sigma).$$

Then $Tx \in P$. The proof of (ii) is completed.

Lemma 3.4[26]. Let $V = \{x \in X : ||x|| < l\}(l > 0)$. If $\left\{\frac{x(t)}{1+t} : x \in V\right\}$ is equicontinuous on any compact intervals of $[0, +\infty)$ and equiconvergent at infinity, where

$$V_1 =: \left\{\frac{x(t)}{1 + t} : x \in V\right\}$$

is called equiconvergent at infinity if and only if for all $\epsilon > 0$, there exists $T = T(\epsilon) > 0$ such that for all $x \in V_1$, it holds

$$\left|\frac{x(t_1)}{1 + t_1} - \frac{x(t_2)}{1 + t_2}\right| < \epsilon, \ t_1, t_2 > T.$$

Then V is relatively compact on X.

Lemma 3.5. $T : P \to P$ is completely continuous;

Proof. It is easy to verify that $T : P \to P$ is well defined. Now we prove that T is continuous and compact respectively.

Let $x_n \to x_0$ as $n \to +\infty$ in P, then there exists r_0 such that $\sup_{n \in N \cup \{0\}} ||x_n|| < r_0$ and $|\psi(t)| \leq r_0$ for all $t \in [-r, 0]$. Set

$$B_{r_0}(t) = \sup_{u, |v| \in [0, r_0]} q(t) f(t, (1 + t)u, \sigma(t)v),$$

where

$$\sigma(t) = \begin{cases} (1 + \mu(t)), & \mu(t) \geq 0, \\ 1, & \mu(t) < 0, \end{cases}$$

and we have

$$\int_0^{+\infty} |q(s)f(s, x_n(s), x_n(\mu(s))) - q(s)f(s, x_0(s), x_0(\mu(s)))|ds \leq 2\int_0^{+\infty} B_{r_0}(s)ds.$$

Therefore by the Lebesgue dominated convergence theorem, one arrives at

$$|p(t)\phi((Tx_n)'(t)) - p(t)\phi((Tx_0)'(t))|$$
$$= \int_t^{+\infty} |q(s)f(s, x_n(s), x_n(\mu(s))) - q(s)f(s, x_0(s), x_0(\mu(s)))|ds$$
$$\leq \int_0^{+\infty} q(s)|f(s, x_n(s), x_n(\mu(s))) - f(s, x_0(s), , x_0(\mu(s)))|ds$$
$$\to 0 \text{ as } n \to +\infty.$$

Since $p(+\infty) < +\infty$, we get

$$|\phi((Tx_n)'(t)) - \phi((Tx_0)'(t))| \to 0 \text{ as } n \to +\infty.$$

Furthermore $(Tx_n)(0) = (Tx_0)(0) = 0$ imply that

$$0 \leq \frac{[(Tx_n) - (Tx_0)](t)}{1 + t}$$
$$= \frac{1}{1 + t}\left(\int_0^t [(Tx_n) - (Tx_0)]'(s)ds + [(Tx_n) - (Tx_0)](0)\right)$$
$$\leq \frac{1}{1 + t}\int_0^t |(Tx_n)'(s) - (Tx_0)'(s)|ds$$
$$\leq \max_{t \in [0, +\infty)} |[(Tx_n) - (Tx_0)]'(t)|.$$

It follows that

$$||(Tx_n) - (Tx_0)|| \leq \max_{t \in [0, +\infty)} ||(Tx_n)' - (Tx_0)]'|| \to 0$$

as $n \to +\infty$. So, T is continuous.

T is compact provided that it maps bounded sets into pre-compact sets. Let Ω be any bounded subset of P. Then there exists $r > 0$ such that $||x|| \leq r$ for all $x \in \Omega$ and $|\psi(t)| \leq r$. Denote

$$B_r(t) = \sup_{u,v \in [0,r]} q(t)f(t, (1 + t)u, \sigma(t)v).$$

Obviously, Lemma 3.1 implies that

$$
\begin{aligned}
0 \le (Tx)'(t) &= \frac{1}{p(t)}\phi^{-1}\left(\int_t^{+\infty} q(s)f(s,x(s),x(\mu(s)))ds\right) \\
&\le \frac{1}{p(t)}\phi^{-1}\left(\int_t^{+\infty} q(s)f(s,x(s),x(\mu(s)))ds\right) \\
&\le \frac{1}{p(t)}\phi^{-1}\left(\int_0^{+\infty} q(s)f(s,x(s),x(\mu(s)))ds\right) \\
&\le \frac{1}{\min_{t\in[0,+\infty)} p(t)}\phi^{-1}\left(\int_0^{+\infty} B_r(s)ds\right).
\end{aligned}
$$

Therefore,

$$
||Tx|| \le \sup_{t\in[0,+\infty)}(Tx)'(t) \le \frac{1}{\min_{t\in[0,+\infty)} p(t)}\phi^{-1}\left(\int_0^{+\infty} B_r(s)ds\right), \ x \in \Omega.
$$

So $T\Omega$ is bounded.

Moreover, for any $T \in (0,+\infty)$ and $t_1, t_2 \in [0,T]$, one has from Lemma 2.1 that

$$
\begin{aligned}
&\left|\frac{(Tx)(t_1)}{1+t_1} - \frac{(Tx)(t_2)}{1+t_2}\right| \\
\le\ &\left|\frac{(Tx)(t_2)-(Tx)(t_1)}{1+t_1}\right| + \left|\frac{1}{1+t_1}-\frac{1}{1+t_2}\right||(Tx)(t_2)| \\
\le\ &\frac{1}{1+t_1}\left|\int_{t_1}^{t_2}(Tx)'(s)ds\right| + \left|\frac{1}{1+t_1}-\frac{1}{1+t_2}\right|\times \\
&\left|\int_0^{t_2}\phi^{-1}\left(\frac{1}{p(s)}\int_s^{+\infty} q(u)f(u,x(u),x(\mu(u)))du\right)ds\right| \\
\le\ &\frac{1}{1+t_1}\left|\int_{t_1}^{t_2}\phi^{-1}\left(\frac{1}{p(s)}\int_s^{+\infty} q(u)f(u,x(u),x(\mu(u)))du\right)ds\right| \\
&+\left|\frac{1}{1+t_1}-\frac{1}{1+t_2}\right|\left|\int_0^{t_2}\phi^{-1}\left(\frac{1}{p(s)}\int_s^{+\infty} q(u)f(u,x(u),x(\mu(u)))du\right)ds\right| \\
\le\ &\frac{1}{1+t_1}\left|\int_{t_1}^{t_2}\phi^{-1}\left(\frac{1}{p(s)}\right)ds\right|\phi^{-1}\left(\int_0^{+\infty} B_r(u)du\right) \\
&+\left|\frac{1}{1+t_1}-\frac{1}{1+t_2}\right|\int_0^T\phi^{-1}\left(\frac{1}{p(s)}\right)ds\phi^{-1}\left(\int_0^{+\infty} B_r(u)du\right) \\
\to\ &0 \text{ uniformly as } t_1 \to t_2
\end{aligned}
$$

for all $x \in \Omega$. So T is equicontinuous on any compact interval of $[0,+\infty)$.

Finally, for any $x \in \Omega$, one has

$$\lim_{t \to +\infty} (Tx)'(t) = 0.$$

We get that

$$\lim_{t \to +\infty} \frac{(Tx)(t)}{1+t} = \lim_{t \to +\infty} (Tx)'(t) = 0 \text{ for all } x \in \Omega.$$

So T is equiconvergent at infinity. By using Lemma 3.4, we obtain that T is pre-compact, that is, T is a compact operator. Above all $T : P \to P$ is completely continuous. The proof is completed.

Proof of Theorem 2.1. By the definition of ψ, it is easy to see that ψ is a nonnegative convex continuous functional on the cone P. $\psi(y) \le ||y||$ for all $y \in P$. For $x \in P$, it follows from Lemma 3.3 that $TP \subseteq P$. From Lemma 3.5, $T : P \to P$ is completely continuous.

Choose

$$c = e, \quad d = (1+k)e_2, \quad b = e_2, \quad a = e_1.$$

We divide the remainder of the proof into four steps.

Step 1. Prove that $T(\overline{P_e}) \subset \overline{P_e}$.

For $x \in \overline{P_e}$, one has $||x|| \le e$. Then

$$0 \le \frac{x(t)}{1+t} \le e, \ t \in [0, +\infty).$$

It follows from (A1) and (A2) that

$$f(t, x(t), x(\mu(t))) = f\left(t, (1+t)\frac{x(t)}{1+t}, (1+\mu(t))\frac{x(\mu(t))}{1+\mu(t)}\right) \le$$

$$\phi\left(\frac{c}{M}\right), t \in [0, +\infty), \mu(t) \ge 0$$

and

$$f(t, x(t), x(\mu(t))) = f\left(t, (1+t)\frac{x(t)}{1+t}, \psi(\mu(t))\right) \le$$

$$\phi\left(\frac{c}{M}\right), t \in [0, +\infty), \mu(t) \le 0.$$

Then $Tx \in P$ implies that

$$
\begin{aligned}
||Tx|| &= \sup_{t \in [0,+\infty)} \frac{(Tx)(t)}{1+t} = \sup_{t \in [0,+\infty)} \frac{\int_0^t (Tx)'(s)ds + (Tx)(0)}{1+t} \\
&\leq \sup_{t \in [0,+\infty)} \frac{t \sup_{t \in [0,+\infty)} |(Tx)'(t)|}{1+t} \leq \sup_{t \in [0,+\infty)} |(Tx)'(t)| \\
&= \sup_{s \in [0,+\infty)} \phi^{-1} \left(\frac{1}{p(s)} \int_s^{+\infty} q(u)f(u,x(u),x(\mu(u)))du \right) \\
&\leq \phi^{-1} \left(\frac{1}{p(0)} \int_0^{+\infty} q(u)f(u,x(u),x(\mu(u)))du \right) \\
&\leq \phi^{-1} \left(\frac{1}{p(0)} \int_{u \in [0,+\infty) \text{ with } \mu(u) \geq 0} q(u) \times \right. \\
&\qquad \left. f\left(u, (1+u)\frac{x(u)}{1+u}, (1+\mu(u))\frac{x(\mu(u))}{1+\mu(u)}\right) du \right) \\
&\quad + \phi^{-1} \left(\frac{1}{p(0)} \int_{u \in [0,+\infty) \text{ with} \mu(u)<0} q(u) \times \right. \\
&\qquad \left. f\left(u, (1+u)\frac{x(u)}{1+u}, \psi(\mu(u))\right) du \right) \\
&\leq \phi^{-1} \left(\frac{1}{p(0)} \int_0^{+\infty} q(u)\phi\left(\frac{c}{M}\right) du \right) \\
&= e.
\end{aligned}
$$

Then $Tx \in \overline{P_e}$, Hence $T(\overline{P_e}) \subset P_e$. This completes the proof of Step 1.

Step 2. Prove that

$$
\{y \in P(\psi; b, d) | \psi(y) > b\} = \{y \in P(\psi; e_2, (1+k)e_2) | \psi(y) > e_2\} \neq \emptyset
$$

$$
\text{and } \psi(Ty) > b = e_2
$$

for $y \in P(\psi; b, d) = P(\psi; e_2, (1+k)e_2)$.

It is easy to see that $\{x \in P(\psi, e_2, (1+k)e_2), \psi(x) > e_2\} \neq \emptyset$.

For $x \in P(\psi, e_2, (1+k)e_2)$, then $\psi(x) \geq e_2$ and $||x|| \leq (1+k)e_2$. Then

$$
\frac{k}{1+k} \min_{t \in [1/k,k]} x(t) \geq e_2, \quad \sup_{t \in [0,+\infty)} \frac{x(t)}{1+t} \leq (1+k)e_2.
$$

Hence

$$\frac{x(t)}{1+t} \le (1+k)e_2, \ t \in [0,+\infty)$$

and

$$\frac{x(t)}{1+t} \ge \frac{1+k}{k}e_2\frac{1}{1+t} \ge \frac{1+k}{k}e_2\frac{1}{1+k} = \frac{e_2}{k}, \ t \in [1/k, k].$$

It follows that

$$\frac{e_2}{k} \le \frac{x(t)}{1+t} \le (1+k)e_2, \ t \in [1/k, k].$$

Hence (A5) and (A6) imply that

$$f(t, x(t), x(\mu(t))) \ge \phi\left(\frac{e_2}{L}\right), t \in [0,+\infty), \mu(t) \ge 0$$

and

$$f(t, x(t), \psi(\mu(t))) \ge \phi\left(\frac{e_2}{L}\right), t \in [0,+\infty), \mu(t) \le 0.$$

Then

$$
\begin{aligned}
\psi(Tx) &= \frac{k}{1+k}\min_{t\in[1/k,k]}(Tx)(t) = \frac{k}{1+k}(Tx)\left(\frac{1}{k}\right)\\
&= \frac{k}{1+k}\left(\int_0^{1/k}\phi^{-1}\left(\frac{1}{p(s)}\int_s^{+\infty}q(u)f(u,x(u),x(\mu(u)))du\right)ds\right)\\
&\ge \frac{k}{1+k}\int_0^{1/k}\phi^{-1}\left(\frac{1}{p(s)}\int_{u\in[1/k,k],\mu(u)\ge 0}q(u)\times\right.\\
&\qquad \left. f\left(u,(1+u)\frac{x(u)}{1+u},(1+\mu(u))\frac{x(\mu(u))}{1+\mu(u)}\right)du\right)ds\\
&\quad +\frac{k}{1+k}\int_0^{1/k}\phi^{-1}\left(\frac{1}{p(s)}\int_{u\in[1/k,k],\mu(u)\le 0}q(u)\times\right.\\
&\qquad \left. f\left(u,(1+u)\frac{x(u)}{1+u},\psi(\mu(u))\right)du\right)ds\\
&\ge \frac{k}{1+k}\int_0^{1/k}\phi^{-1}\left(\frac{1}{p(s)}\int_{1/k}^k q(u)\phi\left(\frac{e_2}{L}\right)du\right)ds\\
&= e_2.
\end{aligned}
$$

This completes the proof of Step 2.

Step 3. Prove that $||Ty|| < a = e_1$ for $y \in P$ with $||y|| \le a$.

For $x \in \overline{P_{e_1}}$, we have

$$\sup_{t \in [0,+\infty)} \frac{x(t)}{1+t} \le e_1 = a.$$

It follows from $(A3), (A4)$ and $Tx \in P$ that

$$f(t, (1+t)x, (1+\mu(t))v) \le \frac{e_1}{M}, \ t \in [0,+\infty), \mu(t) \ge 0, \ 0 \le x, v \le e_1$$

and

$$f(t, (1+t)x, \psi(\mu(t))) \le \frac{e_1}{M}, \ t \in [0,+\infty), \mu(t) \le 0, 0 \le x \le e_1.$$

Then

$$
\begin{aligned}
||Tx|| &= \sup_{t \in [0,+\infty)} \frac{(Tx)(t)}{1+t} \\
&= \sup_{t \in [0,+\infty)} |(Tx)'(t)| \\
&= \sup_{t \in [0,+\infty)} \phi^{-1}\left(\frac{1}{p(s)} \int_s^{+\infty} q(u)f(u, x(u), x(\mu(u))du\right) \\
&\le \phi^{-1}\left(\frac{1}{p(0)} \int_{u \in [0,+\infty), \mu(u) \ge 0} q(u) \times \right. \\
&\qquad \left. f\left(u, (1+u)\frac{x(u)}{1+u}, (1+\mu(u))\frac{x(\mu(u))}{1+\mu(u)}\right) du\right) \\
&\quad + \phi^{-1}\left(\frac{1}{p(0)} \int_{u \in [0,+\infty), \mu(u) \le 0} q(u) \times \right. \\
&\qquad \left. f\left(u, (1+u)\frac{x(u)}{1+u}, (1+\mu(u))\frac{x(\mu(u))}{1+\mu(u)}\right) du\right) \\
&< \frac{e_1}{M} \phi^{-1}\left(\frac{1}{p(0)} \int_0^{+\infty} q(u)du\right) \\
&= e_1.
\end{aligned}
$$

Then $||Ty|| < e_1$ for $||y|| \le e_1$. This completes that proof of Step 3.

Step 4. Prove that $\psi(Ty) > b$ for $y \in P(\psi; b, c)$ with $||Ty|| > d$.

For $x \in P(\psi; b, c) = P(\psi, e_2, e)$ and $||Tx|| > d = (1+k)e_2$, then

$$\sup_{t \in [0,+\infty)} \frac{(Tx)(t)}{1+t} \ge (1+k)e_2 \text{ and } ||x|| = \sup_{t \in [0,+\infty)} \frac{x(t)}{1+t} \le e.$$

Hence we have from $Tx \in P$ that

$$
\begin{aligned}
\psi(Tx) &= \frac{k}{1+k} \min_{t \in [1/k,k]} (Tx)(t) \\
&= \frac{k}{1+k} (Tx)\left(\frac{1}{k}\right) \\
&\geq \frac{k}{1+k}\frac{1}{k} \sup_{t \in [0,+\infty)} \frac{(Tx)(t)}{1+t} \\
&= \frac{1}{1+k}(1+k)e_2 = e_2 = b.
\end{aligned}
$$

This completes the proof of Step 4.

From above steps, $(C1)$, $(C2)$ and $(C3)$ of Theorem 3.1 are satisfied. Then, by Theorem 3.1, T has three fixed points x_1, x_2 and $x_3 \in \overline{P_e}$ such that

$$||x_1|| < e_1, \ \psi(x_2) > e_2, \ ||x_3|| \geq e_1, \ \psi(x_3) \leq e_2, \ ||x_i|| \leq e \text{ for } i = 1, 2, 3,$$

i.e., x_1, x_2 and x_3 satisfy

$$\sup_{t \in [0,+\infty)} \frac{x_1(t)}{1+t} < e_1, \quad \min_{t \in [1/k,k]} x_2(t) > \frac{1+k}{k}e_2$$

and

$$\sup_{t \in [0,+\infty)} \frac{x_3(t)}{1+t} > e_1, \quad \min_{t \in [1/k,k]} x_3(t) < \frac{1+k}{k}e_2.$$

Hence equation (1) has at least three positive solutions

$$
y_i(t) = \begin{cases} x_i(t), \ t \geq 0, \\ \psi(t), \ t \in [-r,0], \end{cases} \quad i = 1, 2, 3.
$$

The proof is completed.

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