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Ordinary differential equations

Stability and boundedness of solutions of nonlinear differential equations of third-order with delay

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Abstract

In this paper we investigate stability and boundedness of solutions of some nonlinear differential equations of third order with delay. By constructing a Lyapunov functional, sufficient conditions for the stability and boundedness of solutions for equations considered are obtained.

Keywords: Stability, boundedness, Lyapunov functional, differential equations of third order with delay.
AMS (MOS) Subject Classification: 34K20.

1. Introduction

The results existing in the literature on the stability and boundedness of solutions of nonlinear differential equations of third order with bounded delay have been developed over the last several decades. After a literature survey about nonlinear equations of third order with bounded delay, one can conclude that there are not so many results on the stability and boundedness of solutions. Up to this moment, the investigations concerning stability and boundedness of solutions of nonlinear equations of third order with bounded delay have not been fully developed. Certainly, these results should be obtained to be able to benefit from the applications of the theory of stability and boundedness of solutions. At the same time, we should recognize that some significant theoretical results concerning the stability and boundedness of solutions of third order nonlinear differential equations with delay have been achieved, see for example the papers of Sadek [9], Tejumola and Tchegnani [10], Tunç ([11], [12]), Zhu [14] and the references citied in these papers. It should be noted that, in 1969, Palusinski et al. [8] applied an energy metric algorithm for the generation of a Lyapunov function for third order ordinary nonlinear differential equation of the form:

$$x''' + a_1 x'' + f_2(x')x' + a_3 x = 0 .$$

They found some conditions for the stability of zero solution of this equation as follows:

$$a_1 > 0 ,\ f_2(x') > a_3 > 0 .$$

In this paper we are concerned with the third order ordinary nonlinear delay differential equations of the type

$$x'''(t) + a_1 x''(t) + f_2(x'(t - r(t)) + a_3 x(t) = p(t, x(t), x'(t), x(t - r(t)), x'(t - r(t)), x''(t)) \qquad (1)$$

or its equivalent system

$$x'(t) = y(t) ,\ y'(t) = z(t) ,$$
$$z'(t) = -a_1 z(t) - f_2(y(t)) - a_3 x(t) + \int_{t-r(t)}^{t} f_2'(y(s))z(s)ds$$
$$+ p(t, x(t), y(t), x(t - r(t)), y(t - r(t)), z(t)) , \qquad (2)$$

where r is a bounded delay, $0 \le r(t) \le \gamma$, $r'(t) \le \beta$, $0 < \beta < 1$, β and γ are some positive constants, γ which will be determined later; a_1 and a_3 are some positive constants; the functions f_2 and p depend only on the arguments displayed explicitly and the primes in equation (1) denote differentiation with respect to *t*. It is principally assumed that $f_2(0) = 0$ and the functions f_2 and p are continuous for all values their respective arguments on $\Re$ and $\Re^+ \times \Re^5$, $\Re^+ = (0, \infty)$, respectively. This fact guarantees the existence of the solution of delay differential equation (1). Besides, it is supposed that the derivative $f_2'(y) \equiv \frac{df_2}{dy}$ exists and is continuous. In addition, it is also assumed that the functions $f_2(y(t - r(t)))$ and $p(t, x(t), y(t), x(t - r(t)), y(t - r(t)), z(t))$ satisfy a Lipschitz condition in $x(t)$, $y(t), x(t - r(t))$, $y(t - r(t))$ and $z(t)$; throughout the paper $x(t)$, $y(t)$ and $z(t)$ are, respectively, abbreviated as x , y and z . Then the solution is unique (See [2, pp.14].)

The motivation for the present work has been inspired basically by the paper of Palusinski et al. [8] and the papers mentioned above. Our aim here is to discuss the result verified by Palusinski et al. [8] on the stability of the solutions to the equation (1) for the stability and boundedness of solutions of this equation in the case $p = 0$ and $p \ne 0$, respectively.

2. Preliminaries

In order to reach our main result, we give some important basic information for the general non-autonomous delay differential system (see also Burton [1], Èl'sgol'ts [2], Èl'sgol'ts and Norkin [3], Hale [4], Kolmanovskii and Myshkis [5], Kolmanovskii and Nosov [6], Krasovskii [7] and Yoshizawa [13].

Now, we consider the general non-autonomous delay differential system

$$\dot{x} = f(t, x_t) ,\ x_t(\theta) = x(t + \theta) ,\ -r \le \theta \le 0 ,\ t \ge 0 , \qquad (3)$$

where $f:[0,\infty)\times C_H \to \Re^n$ is a continuous mapping, $f(t,0)=0$, and we suppose that f takes closed bounded sets into bounded sets of $\Re^n$. Here $(C, \|.\|)$ is the Banach space of continuous function $\phi:[-r,0]\to\Re^n$ with supremum norm, $r>0$, C_H is the open H-ball in C; $C_H := \{\phi\in(C[-r,0], \Re^n): \|\phi\|<H\}$. Standard existence theory, see Burton [1, pp.312], shows that if $\phi\in C_H$ and $t\ge 0$, then there is at least one continuous solution $x(t,t_0,\phi)$ such that on $[t_0, t_0+\alpha)$ satisfying equation (3) for $t>t_0$, $x_{t_0}(s,t,\phi)=\phi_{t_0}(s)$ and α is a positive constant. If there is a closed subset $B\subset C_H$ such that the solution remains in B, then $\alpha=\infty$. Further, the symbol $|.|$ will denote the norm in $\Re^n$ with $|x| = \max_{1\le i\le n}|x_i|$.

Definition 1: (See [1, pp.223].) A continuous function $W:[0,\infty)\to[0,\infty)$ with $W(0)=0$, $W(s)>0$ if $s>0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i, where i an integer.)

Definition 2: (See [1, pp. 260].) Let $V(t,\phi)$ be a continuous functional defined for $t\ge 0$, $\phi\in C_H$. The derivative of V along solutions of (3) will be denoted by $\dot V_{(3)}$ and is defined by the following relation

$$\dot V_{(3)}(t,\phi) = \limsup_{h\to 0}\frac{V(t+h, x_{t+h}(t_0,\phi)) - V(t, x_t(t_0,\phi))}{h},$$

where $x(t_0,\phi)$ is the solution of (3) with $x_{t_0}(t_0,\phi)=\phi$.

Definition 3: (See [13, pp.184].) A function $x(t_0,\phi)$ is said to be a solution of (3) with the initial condition $\phi\in C_H$ at $t=t_0$, $t_0\ge 0$, if there is a constant $A>0$ such that $x(t_0,\phi)$ is a function from $[t_0-h, t_0+A]$ into $\Re^n$ with the properties:

(i) $x_t(t_0,\phi)\in C_H$ for $t_0\le t<t_0+A$,

(ii) $x_{t_0}(t_0,\phi)=\phi$,

(iii) $x(t_0,\phi)$ satisfies (3) for $t_0\le t<t_0+A$.

Theorem 1: (See [13, pp.184].) If $f(t,\phi)$ in (3) is continuous in t, ϕ, for every $\phi\in C_{H_1}$, $H_1<H$, and t_0, $0\le t_0<c$, where c is a positive constant, then there exist a solution of (3) with initial value ϕ at $t=t_0$, and this solution has a continuous derivative for $t>t_0$.

For the general autonomous delay differential system $\dot x = f(x_t)$, which is a special case of (3), the following lemma is given.

Proposition: (See [7].) Suppose $f(0)=0$. Let V be a continuous functional defined on $C_H = C$ with $V(0)=0$, and let $u(s)$ be a function, non-negative and continuous for $0\le s<\infty$, $u(s)\to\infty$ as $u\to\infty$ such that for all $\phi\in C$

(i) $u(|\phi(0)|) \le V(\phi),\ V(\phi) \ge 0$,

(ii) $\dot{V}_{(3)}(\phi) < 0$ for $\phi \ne 0$.

Then all solutions of $\dot{x} = f(x_t)$ approach zero as $t \to \infty$ and the origin is globally asymptotically stable.

Note that $C_H = C$ when $H = \infty$; and that the set R of ϕ in C for which $\dot{V}_{(3)}(\phi) = 0$ has a largest invariant set $M = \{0\}$ by the condition $\dot{V}_{(3)}(\phi) < 0$ for $\phi \ne 0$.

3. Main results

First for the case $p(t, x(t), y(t), x(t-r(t)), y(t-r(t)), z(t)) = 0$ the following result is established.

Theorem 2: In addition to the basic assumptions imposed on the functions f_2 and p that appeared in equation (1), we assume that there are positive constants a_1, a_2, a_3, ε_0, L and μ such that the following conditions are satisfied

$$a_1 a_2 - a_3 > 0,\ f_2(0) = 0,\ \frac{f_2(y)}{y} - a_2 \ge \varepsilon_0,\ (y \ne 0), \text{ and } |f_2'(y)| \le L \text{ for all } y.$$

Then for sufficiently small γ the zero solution of (1) is globally asymptotically stable provided that

$$\gamma < \min\left\{ \frac{2\varepsilon_0}{L}, \frac{2(a_1 a_2 - a_3)}{a_2 L + 2\mu} \right\}.$$

Proof: The proof of this theorem depends on a scalar differentiable Lyapunov functional. $V = V(x_t, y_t, z_t)$. The idea of Lyapunov's method is to impose some conditions on the functional V and its time derivative $\frac{d}{dt}V(x_t, y_t, z_t)$ which both imply the stability of the zero solution of equation (1). We introduce the Lyapunov's functional $V = V(x_t, y_t, z_t)$:

$$V(x_t, y_t, z_t) = \frac{1}{2}a_3^2 x^2 + a_2 a_3 xy + \frac{1}{2}a_2 z^2 + a_3 yz + a_2 \int_0^y f_2(\xi)d\xi$$

$$+\frac{1}{2}a_1 a_3 y^2 + \mu \int_{-r(t)}^{0} \int_{t+s}^{t} z^2(\theta)d\theta ds, \tag{4}$$

where a_1, a_2, a_3 and μ are some positive constants and the constant μ which will be determined later in the proof. Now, the Lyapunov functional $V = V(x_t, y_t, z_t)$ defined in (4) can be rearranged in the form:

$$V(x_t, y_t, z_t) = \frac{1}{2}a_3^2\left(x + \frac{a_2}{a_3}y\right)^2 + \frac{1}{2}a_2\left(z + \frac{a_3}{a_2}y\right)^2$$

$$+\int_0^y\left[a_1a_3-a_2^2-\frac{a_3^2}{a_2}+a_2\frac{f_2(\xi)}{\xi}\right]\xi d\xi+\mu\int_{-r(t)}^{0}\int_{t+s}^{t}z^2(\theta)d\theta ds. \qquad (5)$$

In view of the assumption $\frac{f_2(y)}{y}\geq a_2+\varepsilon_0$, it is clear that

$$\int_0^y\left[a_1a_3-a_2^2-\frac{a_3^2}{a_2}+a_2\frac{f_2(\xi)}{\xi}\right]\xi d\xi\geq\int_0^y\left[a_1a_3-a_2^2-\frac{a_3^2}{a_2}+a_2\left(a_2+\varepsilon_0\right)\right]\xi d\xi$$

$$=\int_0^y\left[a_1a_3-\frac{a_3^2}{a_2}+a_2\varepsilon_0\right]\xi d\xi$$

$$=\left(\frac{a_1a_2a_3-a_3^2+a_2^2\varepsilon_0}{2a_2}\right)y^2>0.$$

Hence, it is evident, from the terms contained in (5), that there exist sufficiently small positive constants $D_i,(i=1,2,3)$, such that

$$V\geq D_1x^2+D_2y^2+D_3z^2+\mu\int_{-r(t)}^{0}\int_{t+s}^{t}z^2(\theta)d\theta ds$$

$$\geq D_4\left(x^2+y^2+z^2\right), \qquad (6)$$

since the integral $\mu\int_{-r(t)}^{0}\int_{t+s}^{t}z^2(\theta)d\theta ds$ is non-negative, where $D_4=\min\{D_1,D_2,D_3\}$.

Now, calculating the time derivative of the functional $V(x_t,y_t,z_t)$ with respect to t along a solution $(x(t),y(t),z(t))$ of the system (2), we have

$$\frac{d}{dt}V(x_t,y_t,z_t)=-\left(a_1a_2-a_3\right)z^2-a_3\left(\frac{f_2(y)}{y}-a_2\right)y^2$$

$$+a_2z\int_{t-r(t)}^{t}f_2'(y(s))z(s)ds+a_3y\int_{t-r(t)}^{t}f_2'(y(s))z(s)ds$$

$$+\mu r(t)z^2-\mu(1-r'(t))\int_{t-r(t)}^{t}z^2(s)ds$$

$$=-\left(a_1a_2-a_3-\mu r(t)\right)z^2-a_3\left(\frac{f_2(y)}{y}-a_2\right)y^2$$

$$+a_2z\int_{t-r(t)}^{t}f_2'(y(s))z(s)ds+a_3y\int_{t-r(t)}^{t}f_2'(y(s))z(s)ds$$

$$-\mu(1-r'(t))\int_{t-r(t)}^{t}z^2(s)ds. \qquad (7)$$

By noting the assumption $\left|f_2'(y)\right| \le L$ and the inequality $2|ab| \le a^2 + b^2$, we obtain the following relations:

$$a_2 z \int_{t-r(t)}^{t} f_2'(y(s))z(s)ds \le \frac{a_2 L}{2}\, r(t) z^2(t) + \frac{a_2 L}{2} \int_{t-r(t)}^{t} z^2(s)ds$$

and

$$a_3 y \int_{t-r(t)}^{t} f_2'(y(s))z(s)ds \le \frac{a_3 L}{2}\, r(t) y^2(t) + \frac{a_3 L}{2} \int_{t-r(t)}^{t} z^2(s)ds\,.$$

Hence, using the assumptions $\dfrac{f_2(y)}{y} - a_2 \ge \varepsilon_0$, $0 \le r(t) \le \gamma$, $r'(t) \le \beta$, $0 < \beta < 1$, and the above discussion, we get from (7) that

$$\frac{d}{dt}V(x_t, y_t, z_t) \le -a_3\left[\left(\frac{f_2(y)}{y} - a_2\right) - \frac{L}{2} r(t)\right] y^2$$

$$-\left[\left(a_1 a_2 - a_3\right) - \left(\frac{a_2 L + 2\mu}{2}\right) r(t)\right] z^2$$

$$+\left(a_2 + a_3\right)\frac{L}{2} \int_{t-r(t)}^{t} z^2(s)ds \text{ - } \mu(1 - r'(t)) \int_{t-r(t)}^{t} z^2(s)ds$$

$$\le -a_3\left[\varepsilon_0 - \frac{L}{2}\gamma\right] y^2 - \left[\left(a_1 a_2 - a_3\right) - \left(\frac{a_2 L + 2\mu}{2}\right)\gamma\right] z^2$$

$$+\left[\frac{(a_2 + a_3)L - \mu(1-\beta)}{2}\right] \int_{t-r(t)}^{t} z^2(s)ds\,. \qquad (8)$$

If we choose $\mu = \dfrac{(a_2 + a_3)L}{1-\beta}$, then we have from (8) that

$$\frac{d}{dt}V(x_t, y_t, z_t) \le -a_3\left[\varepsilon_0 - \frac{L}{2}\gamma\right] y^2 - \left[\left(a_1 a_2 - a_3\right) - \left(\frac{a_2 L + 2\mu}{2}\right)\gamma\right] z^2\,. \qquad (9)$$

Therefore, in view of (9), one can conclude for some positive constants α and ρ that

$$\frac{d}{dt}V(x_t, y_t, z_t) \le -\alpha y^2 - \rho z^2 \qquad (10)$$

provided

$$\gamma < \min\left\{\frac{2\varepsilon_0}{L}, \frac{2(a_1 a_2 - a_3)}{a_2 L + 2\mu}\right\}.$$

Finally, it is followed that $\dfrac{d}{dt}V(x_t, y_t, z_t) \equiv 0$ if and only if $y_t = z_t = 0$, $\dfrac{d}{dt}V(\phi) < 0$ for $\phi \ne 0$ and $V(\phi) \ge u(|\phi(0)|) \ge 0$. Thus, in view (6), (10) and the last discussion, it is seen that all the conditions of the above Proposition are satisfied. This shows that the trivial solution of equation (1) is globally asymptotically stable. Hence, the proof of Theorem 2 is complete.

Example 1: Consider the third order nonlinear delay differential equation

$$x'''(t) + 3x''(t) + 4x'(t - r(t)) + \sin x'(t - r(t)) + 2x(t) = 0 \tag{11}$$

Equation (11) is equivalent to the system

$$\begin{aligned} &x'(t) = y(t),\; y'(t) = z(t), \\ &z'(t) = -3z(t) - 4y(t) - \sin y(t) - 2x(t) + \int_{t-r(t)}^{t} \left(4 + \cos y(s)\right) z(s)\, ds\;, \end{aligned} \tag{12}$$

where we suppose that $0 \le r(t) \le \gamma$, $r'(t) \le \beta$, $0 < \beta < 1$, β and γ are positive constants, γ which will be determined later, $t \in [0, \infty)$. It is obvious that

$$3 \le 4 + \frac{\sin y}{y}$$

for all y, $(y \ne 0)$.
Our main tool is the Lyapunov functional

$$\begin{aligned} V(x_t, y_t, z_t) = 2\left(x + \frac{y}{2}\right)^2 + \frac{1}{2}(z + 2y)^2 + \int_0^y \left[5 + \frac{\sin \xi}{\xi}\right] \xi\, d\xi \\ + \mu \int_{-r(t)}^{0} \int_{t+s}^{t} z^2(\theta)\, d\theta\, ds, \end{aligned} \tag{13}$$

where μ is a positive constant which will be determined later.
It is clear that the functional $V(x_t, y_t, z_t)$ is positive definite. Hence, it is evident, from the terms contained in (13), that there exist sufficiently small positive constants D_i, $(i = 5, 6, 7)$, such that

$$\begin{aligned} V(x_t, y_t, z_t) &\ge D_5 x^2 + D_6 y^2 + D_7 z^2 + \mu \int_{-r(t)}^{0} \int_{t+s}^{t} z^2(\theta)\, d\theta\, ds \\ &\ge D_5 x^2 + D_6 y^2 + D_7 z^2 \\ &\ge D_8 \left(x^2 + y^2 + z^2\right), \end{aligned}$$

where $D_8 = \min\{D_5, D_6, D_7\}$.

Now, the time derivative of the functional $V(x_t, y_t, z_t)$ in (13) with respect to the system (12) can be calculated as follows:

$$\frac{d}{dt} V(x_t, y_t, z_t) = -\left(1 - \mu r(t)\right) z^2 - 2\left(3 + \frac{\sin y}{y}\right) y^2$$

$$+z\int_{t-r(t)}^{t}(4+\cos y(s))z(s)ds+2y\int_{t-r(t)}^{t}(4+\cos y(s))z(s)ds$$

$$-\mu(1-r'(t))\int_{t-r(t)}^{t}z^2(s)ds. \tag{14}$$

Making use of the facts $|4+\cos y|\le 5$, $\left|\dfrac{\sin y}{y}\right|\le 1$, $0\le r(t)\le\gamma$, $r'(t)\le\beta$, $0<\beta<1$ and the inequality $2|uv|\le u^2+v^2$, we obtain the following inequalities for all terms contained in the equality (14), respectively:

$$-(1-\mu r(t))z^2\le-(1-\mu\gamma)z^2,$$

$$-2\left(3+\frac{\sin y}{y}\right)y^2\le-4y^2,$$

$$z\int_{t-r(t)}^{t}(4+\cos y(s))z(s)ds\le\frac{5}{2}\,r(t)z^2(t)+\frac{5}{2}\int_{t-r(t)}^{t}z^2(s)ds$$

$$\le\frac{5\gamma}{2}\,z^2(t)+\frac{5}{2}\int_{t-r(t)}^{t}z^2(s)ds,$$

$$2y\int_{t-r(t)}^{t}(4+\cos y(s))z(s)ds\le 5\,r(t)y^2(t)+5\int_{t-r(t)}^{t}z^2(s)ds$$

$$\le 5\gamma\,y^2(t)+5\int_{t-r(t)}^{t}z^2(s)ds$$

and

$$-\mu(1-r'(t))\int_{t-r(t)}^{t}z^2(s)ds\le-\mu(1-\beta)\int_{t-r(t)}^{t}z^2(s)ds.$$

Gathering all of these inequalities into (14), we have

$$\frac{d}{dt}V(x_t,y_t,z_t)\le-2\left(2-\frac{5\gamma}{2}\right)y^2-\left(1-\left(\mu+\frac{5}{2}\right)\gamma\right)z^2-\left(\mu(1-\beta)-\frac{15}{2}\right)\int_{t-r(t)}^{t}z^2(s)ds.$$

Let us choose $\mu=\dfrac{15}{2(1-\beta)}$. Then, it easy to see that

$$\frac{d}{dt}V(x_t,y_t,z_t)\le-2\left(2-\frac{5\gamma}{2}\right)y^2-\left(1-\left(\mu+\frac{5}{2}\right)\gamma\right)z^2. \tag{15}$$

Now, in view of (15), one can conclude for some positive constants α and ρ that

$$\frac{d}{dt}V(x_t, y_t, z_t) \le -\alpha y^2 - \rho z^2. \qquad (16)$$

provided

$$\gamma < \min\left\{\frac{2}{2\mu+5}, \frac{4}{5}\right\}.$$

It is also easy to see that $\frac{d}{dt}V(x_t, y_t, z_t) \equiv 0$ if and only if $z_t = x_t = 0$, $\frac{d}{dt}V(\phi) < 0$ for $\phi \ne 0$ and $V(\phi) \ge u(|\phi(0)|) \ge 0$. Thus all the conditions of the above Proposition are satisfied. This shows that the trivial solution of equation (11) is globally asymptotically stable.

For the case $p(t, x(t), y(t), x(t-r(t)), y(t-r(t)), z(t)) \ne 0$ the following result is established.

Theorem 3: In addition to the basic assumptions imposed on the functions f_2 and p that appeared in equation (1) , we assume that there are positive constants a_1, a_2, a_3, ε_0, L, μ, H and H_1 such that the following conditions are satisfied for every x, y and z in

$$\Omega := \{(x, y, z) \in \Re^3 : |x| < H_1,\ |y| < H_1,\ |z| < H_1, H_1 < H\}:$$

(i) $a_1 a_2 - a_3 > 0$, $f_2(0) = 0$, $\frac{f_2(y)}{y} - a_2 \ge \varepsilon_0$, $(y \ne 0)$, and $|f_2'(y)| \le L$.

(ii) $|p(t, x(t), y(t), x(t-r(t)), y(t-r(t)), z(t))| \le q(t)$,

where $\max q(t) < \infty$ and $q \in L^1(0,\infty)$, $L^1(0,\infty)$ is space of integrable Lebesgue functions.

Then, there exists a finite positive constant K such that the solution $x(t)$ of equation (1) defined by the initial functions

$$x(t) = \phi(t),\ x'(t) = \phi'(t),\ x''(t) = \phi''(t)$$

satisfies the inequalities

$$|x(t)| \le K,\ |x'(t)| \le K,\ |x''(t)| \le K$$

for all $t \ge t_0$, where $\phi \in C^2([t_0 - r, t_0], \Re)$, provided that

$$\gamma < \min\left\{\frac{2\varepsilon_0}{L}, \frac{2(a_1 a_2 - a_3)}{a_2 L + 2\mu}\right\}.$$

Proof: As in the Theorem 2, the proof of this theorem also depends on the scalar differentiable Lyapunov functional $V = V(x_t, y_t, z_t)$, which is defined in (4). Now, since $p(t, x(t), y(t), x(t-r(t)), y(t-r(t)), z(t)) \ne 0$, in view of (4), (2) and (10), it can be easily followed that the time derivative of the functional $V(x_t, y_t, z_t)$ satisfies the following inequality:

$$\frac{d}{dt}V(x_t,y_t,z_t) \le -\alpha y^2 - \rho z^2 + |a_3 y + a_2 z|.|p(t,x(t),y(t),x(t-r(t)),y(t-r(t)),y(t))|$$

$$\le -\alpha y^2 - \rho z^2 + |a_3 y + a_2 z| q(t).$$

Hence, it follows that

$$\frac{d}{dt}V(x_t,y_t,z_t) \le -\alpha y^2 - \rho z^2 + D_9(|y|+|z|)q(t)$$

$$\le D_9(|y|+|z|)q(t)$$

for a constant $D_9 > 0$, where $D_9 = \max\{a_2, a_3\}$.
Making use of the inequalities $|y| < 1 + y^2$ and $|z| < 1 + z^2$, it is clear that

$$\frac{d}{dt}V(x_t,y_t,z_t) \le D_9\left(2 + y^2 + z^2\right)q(t).$$

By (6), we have

$$\left(y^2 + z^2\right) \le D_4^{-1} V(x_t,y_t,z_t).$$

Hence

$$\frac{d}{dt}V(x_t,y_t,z_t) \le D_9\left(2 + D_4^{-1}V(x_t,y_t,z_t)\right)q(t).$$

Now, integrating the last inequality from 0 to t, using the assumption $q \in L^1(0,\infty)$ and Gronwall-Reid-Bellman inequality, we obtain

$$V(x_t,y_t,z_t) \le V(x_0,y_0,z_0) + 2D_9 A + D_9 D_4^{-1}\int_0^t \left(V(x_s,y_s,z_s)\right)q(s)ds$$

$$\le \left(V(x_0,y_0,z_0) + 2D_9 A\right)\exp\left(D_9 D_4^{-1}\int_0^t q(s)ds\right)$$

$$\le \left(V(x_0,y_0,z_0) + 2D_9 A\right)\exp\left(D_9 D_4^{-1} A\right) = K_1 < \infty, \tag{17}$$

where $K_1 > 0$ is a constant, $K_1 = \left(V(x_0,y_0,z_0) + 2D_9 A\right)\exp\left(D_9 D_4^{-1} A\right)$ and $A = \int_0^\infty q(s)ds$.

Now, the inequalities (6) and (17) together yield that

$$x^2(t) + y^2(t) + z^2(t) \le D_4^{-1}V(x_t,y_t,z_t) \le K,$$

where $K = K_1 D_4^{-1}$. Thus, we conclude that

$$|x(t)| \le K, \ |y(t)| \le K, \ |z(t)| \le K$$

for all $t \ge t_0$. That is,

$$|x(t)| \le K, \ |x'(t)| \le K, \ |x''(t)| \le K$$

for all $t \geq t_0$.
The proof of the theorem is now complete.

Example 2: Consider the third order nonlinear delay differential equation

$$x'''(t) + 3x''(t) + 4x'(t - r(t)) + \sin x'(t - r(t)) + 2x(t)$$

$$= \frac{2}{1 + t^2 + x^2(t) + x'^2(t) + x^2(t - r(t)) + x'^2(t - r(t)) + x''^2(t)} . \qquad (18)$$

Clearly, equation (18) is equivalent to the system

$$x'(t) = y(t), \; y'(t) = z(t),$$

$$z'(t) = -3z(t) - 4y(t) - \sin y(t) - 2x(t) + \int_{t-r(t)}^{t} \left(4 + \cos y(t)\right) z(s) ds$$

$$+ \frac{2}{1 + t^2 + x^2(t) + y^2(t) + x^2(t - r(t)) + y^2(t - r(t)) + z^2(t)} , \qquad (19)$$

Observe that

$$\frac{2}{1 + t^2 + x^2(t) + y^2(t) + x^2(t - r(t)) + y^2(t - r(t)) + z^2(t)} \leq \frac{2}{1 + t^2} = q(t)$$

for all $t \in \Re^+$, $x(t)$, $y(t)$, $x(t - r(t))$, $y(t - r(t))$, $z(t)$, and

$$\int_0^\infty q(s)ds = \int_0^\infty \frac{2}{1 + s^2} ds = \pi < \infty \text{, that is, } \; q \in L^1(0, \infty) .$$

To show the boundedness of the solutions we use as a main tool the Lyapunov functional in (13). Now, in view of (16), the time derivative of the functional $V(x_t, y_t, z_t)$ with respect to the system (19) can be revised as follows:

$$\frac{d}{dt} V(x_t, y_t, z_t) = -\alpha y^2 - \rho z^2 + \frac{4y + 2z}{1 + t^2 + x^2(t) + y^2(t) + x^2(t - r(t)) + y^2(t - r(t)) + z^2(t)} . \qquad (20)$$

Making use of the fact

$$\frac{1}{1 + t^2 + x^2(t) + y^2(t) + x^2(t - r(t)) + y^2(t - r(t)) + z^2(t)} \leq \frac{1}{1 + t^2}$$

we get

$$\frac{d}{dt} V(x_t, y_t, z_t) \leq -\alpha y^2 - \rho z^2 + \frac{2|2y + z|}{1 + t^2} .$$

Hence, it is obvious that

$$\frac{d}{dt}V(x_t, y_t, z_t) \le \frac{2|2y+z|}{1+t^2} \le \frac{4(|y|+|z|)}{1+t^2}$$

$$\le \frac{4(2+y^2+z^2)}{1+t^2} \le \frac{8}{1+t^2} + \frac{4(y^2+z^2)}{1+t^2}$$

$$\le \frac{8}{1+t^2} + \frac{4D_8^{-1}}{1+t^2}V(x_t, y_t, z_t). \qquad (21)$$

Now, integrating (21) from 0 to t, using the fact $\frac{1}{1+t^2} \in L^1(0,\infty)$ and Gronwall-Reid-Bellman inequality, it can be easily concluded the boundedness of all solutions of equation (18).

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