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An Analytical Solution for the Unsteady Free Convection Flow near an Inclined Plate in a Rotating System

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Abstract

The unsteady free-convection flow of a viscous, heat conducting fluid near an infinite, inclined and rotating plate (or surface) is investigated, with the rotation vector titled from the vertical. Exact solution of this problem is obtained with the help of Laplace transform technique, when the Prandtl number is arbitrary and the plate rotates in its own plane with constant angular velocity. The asymptotic or steady-state solution which corresponds to the present problem for large time is expressed analytically. The results thus obtained are also discussed.

1. Introduction

Free-convection flows near the inclined or vertical plates or surfaces find applications in many engineering and environmental studies such as cooling of nuclear reactors, solar energy collectors, cooling of equipments and crystal growth, among others. It is also known that these convection flows are greatly influenced by rotation [1]. Especially, the Coriolis force, which is a result of a motion of rotating frame, is responsible for the production of rotation in the geofluid (cf. the rotation in the environment [2]).

Chandrasekhar [3] has made significant contributions to the theory of hydrodynamic flow phenomena in numerous situations. He pointed out the significant role of the Coriolis force on problems of thermal instability and on stability of viscous flow. Then, studies carried out by several investigators [4, 5, 6, 7] disclosed that Coriolis force is significant as compared to viscous and inertia occurring in the basic equations of the problems involving unsteady free-convection flow past an infinite or semi-infinity vertical plate in rotating fluid about horizontal axis.

On the other hand, Hathaway and Somerville [8] have studied the three-dimensional free-convection flow in an inclined rotating layer, with the rotation vector titled from the vertical. Using numerical simulation, they deduced that the titling of the rotation vector was significant change in the flow structure. A combined numerical and experimental study was carried out by Ker and Lin [9] in order to investigate the steady convective flow structure and flow stability in an inclined cubic air cavity subject to

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differential heating and to a constant rotation. Similar problem of natural convection flow of air in an inclined open cavity without rotation has been studied numerically by Nateghi and Armfield [10]. The problem of the unsteady free-convection flow of a viscous, heat conducting fluid near an infinity vertical plate in a rotating system has been solved exactly by Chandran et al. [11] for the special case of Prandtl number *(P)* equal to unity.

In the present work I extend the work of Chandran et al. [11], by regarding the free-convection flow in an inclined rotating plate, with the rotation vector titled from the vertical and without any restrictions on Prandtl number of the heat conducting fluid. So, the unsteady free-convection flow due to an inclined rotating plate is generalized and exact solution of this problem is obtained with the help of Laplace transform technique. The asymptotic nature of this unsteady flow field is investigated, given real expressions for this ultimate steady-state solution of hydrodynamic boundary flow. Finally, the results thus obtained are discussed in last section.

2. Analysis of the Problem

Let us consider three-dimensional free-convection flow of a viscous incompressible fluid near an infinite, inclined plate (or surface). On this plate an arbitrary point has been chosen as the origin O of a Cartesian coordinate system, with the axes Ox′ and Oy′ fixed on the plate and the Oz′ normal to it into the fluid. This plate is inclined from the vertically direction so that the axis Oz′ inclines from the vertically upward direction by an angle ϕ [9].

Initially, the fluid and the plate are at the same temperature T'_∞ and stationary condition. Subsequently $(t' > 0)$, this plate is assumed to be in state of rigid rotation in its own plane about the axis Oz′ with a constant angular velocity Ω' (>0), and the plate temperature is raised to $T'_w(\neq T'_\infty)$. Now, this fluid flow near the inclined plate is driven by the Carioles force and the thermal buoyancy.

On the physical grounds of the present problem all the quantities are assumed to be functions of the space coordinate z' and time t'; so that the vector of the velocity of the fluid is given by $(u',\ \upsilon',\ 0)$. Then, it can be shown that the resulting flow subject to rotation and under the Boussinesq approximation is governed by the following equations of energy and motion

$$P\frac{\partial \mathrm{T}'}{\partial t'} = \frac{\kappa}{\mathrm{c_P}\rho}\frac{\partial^2 \mathrm{T}'}{\partial \mathrm{z}'^2} \quad , \tag{1a}$$

$$\frac{\partial u'}{\partial t'} - 2\Omega'\upsilon' = \frac{\partial^2 u'}{\partial \mathrm{z}'^2} - \beta'\mathrm{g}(\mathrm{T}' - \mathrm{T}'_\infty)\sin\phi\ \cos\Omega' t' \quad , \tag{1b}$$

$$\frac{\partial \upsilon'}{\partial t'} + 2\Omega' u' = \frac{\partial^2 \upsilon'}{\partial \mathrm{z}'^2} - \beta'\mathrm{g}(\mathrm{T}' - \mathrm{T}'_\infty)\sin\phi\ \sin\Omega' t' \quad , \tag{1c}$$

where ρ denotes the fluid density, ν the kinematic viscosity, T' the temperature, g the acceleration due to gravity, β' the coefficient of volume expansion, κ the thermal conductivity, and c_P the specific heat at constant pressure [9]. The angle $\phi(\neq 0)$ of inclination of the rotating axis Oz′ from the direction of the vertical is assumed constant for free-convection flow. The terms $-2\Omega'\upsilon'$ and $2\Omega' u'$ are the components of the Coriolis acceleration [2]

Assuming that no slipping occurs between the plate and the fluid, the initial and the boundary conditions corresponding to the present problem are

$$\mathrm{u}' = 0 \ ,\quad \upsilon' = 0 \ ,\quad T' = T'_\infty \ ,\qquad \text{for} \quad z' \geq 0 \quad \text{and} \quad t' \leq 0 \ , \tag{2a}$$

$$\mathrm{u}' = 0 \ ,\quad \upsilon' = 0 \ ,\quad T' = T'_w \ ,\qquad \text{at} \quad z' = 0 \quad \text{for} \quad t' \rangle 0 \ , \tag{2b}$$

$$\mathrm{u}' \to 0 \ ,\quad \upsilon' \to 0 \ ,\quad T' \to T'_\infty \ ,\qquad \text{as} \quad z' \to \infty \quad \text{for} \quad t' \rangle 0 \ . \tag{2c}$$

The above equations can be reduced to non-dimensional forms by the introduction of the following dimensionless quantities:

$$z = z'/L \ , \ t = \nu t'/L^2 , \quad (u,\ \upsilon) = (u',\ \upsilon')L/\nu \ , \quad \upsilon_0 = \upsilon_0'/U_0 \ , \quad \grave{U} = \grave{U}' L^2/\nu \ , \tag{3a}$$

$$\theta = (T' - T_\infty')/(T_w' - T_\infty') \quad , \quad P = \nu c_P / \kappa \quad , \tag{3b}$$

$$L = \left[\frac{\nu^2}{g\beta' \sin\phi \, (T_w' - T_\infty')} \right]^{1/3} \quad , \tag{3c}$$

where θ is the non-dimensional temperature, P the Prandtl number and L a defined characteristic length. Then, we get the non-dimensional forms of the equations of our problem

$$P \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial y^2} \quad , \tag{4a}$$

$$\frac{\partial u}{\partial t} - 2\grave{U}\upsilon = \frac{\partial^2 u}{\partial y^2} + \theta \cos \grave{U}t \quad , \tag{4b}$$

$$\frac{\partial \upsilon}{\partial t} + 2\grave{U}u = \frac{\partial^2 \upsilon}{\partial y^2} - \theta \sin \grave{U}t \quad . \tag{4c}$$

where the appropriate initial and boundary conditions are in non-dimensional form [11]:

$$\mathrm{u} = 0 \ , \quad \upsilon = 0 \ , \quad \theta = 0 \ , \qquad \text{for} \quad z \ge 0 \quad \text{and} \quad t \le 0 \quad , \tag{4d}$$

$$\mathrm{u} = 0 \ , \quad \upsilon = 0 \ , \quad \theta = 1 \ , \qquad \text{at} \quad z = 0 \quad \text{for} \quad t > 0 \quad , \tag{4e}$$

$$\mathrm{u} \to 0 \ , \quad \upsilon \to 0 \ , \quad \theta \to 0 \ , \qquad \text{as} \quad z \to \infty \quad \text{for} \quad t > 0 \quad . \tag{4f}$$

The equations (4b) and (4c) can be combined into a single equation of motion

$$\frac{\partial^2 q}{\partial z^2} - \frac{\partial q}{\partial t} - 2i\grave{U}\ q = -\theta \ e^{-i\grave{U}\,t} \quad . \tag{5a}$$

where we use the non-dimensional complex velocity

$$q \equiv u + i\upsilon \quad , \qquad i = \sqrt{-1} \quad , \tag{5b}$$

The system of equations (4a) and (5a) are the governing equations of our problem, which we shall solved exactly in the following section, with the initial and boundary conditions

$$q = 0 \ , \quad \theta = 0 \ , \qquad \text{for} \quad z \ge 0 \quad \text{and} \quad t \le 0 \quad , \tag{6a}$$

$$q = 0 \ , \quad \theta = 1 \ , \qquad \text{at} \quad z = 0 \quad \text{for} \quad t > 0 \quad , \tag{6b}$$

$$q \to 0 \ , \quad \theta \to 0 \ , \qquad \text{as} \quad z \to \infty \quad \text{for} \quad t > 0 \quad . \tag{6c}$$

3. Solution of the Problem

In order to obtain the exact solution of the present problem we shall use the Laplace transform technique.

Applying the Laplace transform (with respect to time t) to the system of equations (4a) and (5a) and the boundary conditions (6), we get

$$\frac{\partial^2\bar{\theta}(z,s)}{\partial z^2}-sP\bar{\theta}(z,s)=0 \quad , \tag{7a}$$

$$\frac{\partial^2\bar{q}(z,s)}{\partial z^2}-(s+2i\grave{U})\ \bar{q}(z,s)=-\bar{\theta}(z,s+i\grave{U}) \quad , \tag{7b}$$

with boundary conditions

$$\bar{q}(0,s)=0 \ , \quad \bar{\theta}(0,s)=1/s \ , \tag{7c}$$

$$\bar{q}\to 0 \ , \quad \bar{\theta}\to 0 \quad \text{as} \quad z\to\infty \ , \tag{7d}$$

where a bar over a quantity denotes its Laplace transform with s as the transform variable.

The system of Eqs. (7a,b) is a ordinary differential equations system. So, it is found that the solution of this system in the transform domain is

$$\bar{\theta}(z,s)=\frac{1}{s}e^{-z(Ps)^{1/2}} \quad , \tag{8a}$$

$$\bar{q}(z,s)=\frac{i}{\grave{U}}\left[\frac{1}{s+i\grave{U}}-\frac{P-1}{(P-1)s+i\grave{U}(P-2)}\right]\left\{e^{-z(s+2i\grave{U})^{1/2}}-e^{-z\left[P(s+i\grave{U})\right]^{1/2}}\right\} \quad , \tag{8b}$$

Then, the exact solution of the system of equations (4a) and (5a) can be obtained by taking the inverse transforms of Eqs. (8). So, the solution of the problem for the temperature $\theta(z,t)$ and velocity $q(z,t)$ for $t>0$,

$$\theta(z,t)=erfc\left[\tfrac{1}{2}z(P/t)^{1/2}\right] \quad , \tag{9a}$$

$$q(z,t)=\frac{i}{\grave{U}}\left[q_0(z,t)+q_1(z,t)+q_2(z,t)\right] \quad , \tag{9b}$$

where

$$q_0(z,t)=-erfc\left[\tfrac{1}{2}z(P/t)^{1/2}\right]\ \exp\left(-i\grave{U}\ t\right) \quad , \tag{10a}$$

$$q_1(z,t)=\tfrac{1}{2}E\left[(i\grave{U})^{1/2},\ t,\ \tfrac{1}{2}z\right]\ \exp\left(-i\grave{U}\ t\right) \quad , \tag{10b}$$

$$q_2(z,t)=\tfrac{1}{2}\left[E\left(\lambda^{1/2},\ t,\ \tfrac{1}{2}zP^{1/2}\right)-E\left(\lambda^{1/2}P^{1/2},\ t,\ \tfrac{1}{2}z\right)\right]\ \exp\left[-\lambda\ (P-2)t\right] \quad , \tag{10c}$$

with the abbreviations

$$\lambda\equiv i\grave{U}/(P-1) \tag{11a}$$

$$E(\alpha,t,\beta)\equiv e^{-2\alpha\beta}erfc\left(\frac{\beta}{t^{1/2}}-\alpha\ t^{1/2}\right)+e^{2\alpha\beta}erfc\left(\frac{\beta}{t^{1/2}}+\alpha\ t^{1/2}\right) \quad . \tag{11b}$$

The general solution (9) was exemplified above without any restrictions on Prandtl number P of fluids, and we can prove that these results satisfy the equations of the present problem.

Indeed, Eqs. (9) are exact solutions of the system of differential equations (4a) and (5a). First of all, it can be easily verified that the initial and boundary conditions by $\theta(z,t)$ and $q(z,t)$. The verification of $\theta(z,t)$ given by (9a) as solution of (4a) is straightforward, and is not done here.

We shall, however, show that Eq. (9b) represents the exact solution of Eq. (5a). In order to do so, we must find out the partial derivatives of $q(z,t)$ given from (9b). So, we obtain

$$\frac{\partial^2 q}{\partial z^2}=\frac{i}{Ù}\left[\frac{\partial^2 q_0}{\partial z^2}+\frac{\partial^2 q_1}{\partial z^2}+\frac{\partial^2 q_2}{\partial z^2}\right]=-q_1-\frac{P}{P-1}q_2 \quad , \tag{12a}$$

$$\frac{\partial q}{\partial t}=\frac{i}{Ù}\left[\frac{\partial q_0}{\partial t}+\frac{\partial q_1}{\partial t}+\frac{\partial q_2}{\partial t}\right]=q_1+q_0+\frac{P-2}{P-1}q_2 \quad . \tag{12b}$$

Then, substituting expressions (12) and (9b) into the left-hand side of Eq. (5a) can be reduced in the form

$$\frac{\partial^2 q}{\partial y^2}-\frac{\partial q}{\partial t}-2i\Omega q = q_0 = -erfc\left[\tfrac{1}{2}z(P/t)^{1/2}\right]\exp(-i\Omega t)=-\theta\exp(-i\Omega t) \quad . \tag{13}$$

In latter equations, we use the relation (10a) and the solution (9a). So, it can be seen that Eq. (5a) is identically satisfied.

Finally, knowing the velocity field from Eq. (9b), we can now calculate the axial and transverse components of skin friction of the flow at the plate, which is important for practical applications. In non-dimensional form these are given in the complex form

$$\begin{aligned}\tau_x+i\tau_y=-\left(\frac{\partial q}{\partial z}\right)_{z=0} &= -\frac{i\lambda^{1/2}P^{1/2}}{\Omega}\left[erfc\,(\lambda t)^{1/2}-erfc\,(\lambda Pt)^{1/2}\right]\exp\left[-\lambda\,(P-2)t\right]\\ &-\frac{1}{(i\Omega)^{1/2}}erf\,(i\Omega t)^{1/2}\exp(-i\Omega t)\ , \qquad \text{for} \quad P\neq 1 \ ;\end{aligned} \tag{14a}$$

$$\tau_x+i\tau_y==-\frac{i\,\exp(-i\Omega t)}{\Omega\pi^{1/2}t^{1/2}}\left[\exp(-i\Omega t)-1+(i\pi\Omega t)^{1/2}erf\,(i\Omega t)^{1/2}\right] \ , \quad \text{for} \quad P=1 \ . \tag{14b}$$

4. The Asymptotic Solution for Large Time

The result (9b) for velocity includes the asymptotic or steady-state solution, which corresponds to the present problem for large time.

This result for steady-state flow can be deduced by taking into account the asymptotic representation of *erfc(z)* for a complex argument z in the form [4]

$$erfc(z)\cong\frac{1}{z\pi^{1/2}}e^{-z^2} \qquad \text{as} \quad |z|\to\infty \quad . \tag{15}$$

Indeed, one can readily obtain the asymptotic solution for the primary and secondary velocity components u and υ from (9b) for large time t in the form

$$\begin{aligned}u(z,t)\cong&-\frac{1}{\Omega}\sin\Omega t\ erfc\left[\tfrac{1}{2}z(P/t)^{1/2}\right]\\ &+(A_1\cos 2\Omega t+B_1\sin 2\Omega t)\exp\left(-z^2/4t\right)+(A_2\cos\Omega t+B_2\sin\Omega t)\exp\left(-z^2P/4t\right)\ ,\end{aligned} \tag{16a}$$

$$\begin{aligned}\upsilon(z,t)\cong&-\frac{1}{\Omega}\cos\Omega t\ erfc\left[\tfrac{1}{2}z(P/t)^{1/2}\right]\\ &+(B_1\cos 2\Omega t-A_1\sin 2\Omega t)\exp\left(-z^2/4t\right)+(B_2\cos\Omega t-A_2\sin\Omega t)\exp\left(-z^2P/4t\right)\ ,\end{aligned} \tag{16b}$$

where A_j and B_j , $j=1, 2$ are defined by

$$A_1 \equiv -\frac{8zt^{5/2}}{\pi^{1/2}}\left[\frac{1}{z^4+16\Omega^2 t^4}-\frac{P(P-1)}{z^4(P-1)^2+16\Omega^2 P^2 t^4}\right] \quad , \tag{17a}$$

$$B_1 \equiv \frac{2z^3 t^{1/2}}{\pi^{1/2}\Omega}\left[\frac{1}{z^4+16\Omega^2 t^4}-\frac{(P-1)^2}{z^4(P-1)^2+16\Omega^2 P^2 t^4}\right] \quad , \tag{17b}$$

$$A_2 \equiv -\frac{8zt^{5/2}P^{1/2}(P-1)}{\pi^{1/2}\left[z^4(P-1)^2+16\Omega^2 P^2 t^4\right]} \quad , \quad B_2 \equiv \frac{2z^3 t^{1/2}(P-1)^2}{\pi^{1/2}\Omega\left[z^4(P-1)^2+16\Omega^2 P^2 t^4\right]} \quad . \tag{17c,d}$$

Also, the axial and transverse components of skin friction in the steady state of this case are given by

$$\tau_x = \left(\frac{\partial u}{\partial z}\right)_{z=0} \cong \frac{1}{\Omega}\left(\frac{P}{t}\right)^{1/2}\sin \Omega t - \frac{1}{2\pi^{1/2}t^{3/2}\Omega^2 P}\left(\cos 2\Omega t - \frac{P-1}{P^{1/2}\Omega}\cos \Omega t\right) \quad , \tag{18a}$$

$$\tau_y = \left(\frac{\partial \upsilon}{\partial z}\right)_{z=0} \cong \frac{1}{\Omega}\left(\frac{P}{t}\right)^{1/2}\cos \Omega t + \frac{1}{2\pi^{1/2}t^{3/2}\Omega^2 P}\left(\sin 2\Omega t + \frac{P-1}{P^{1/2}\Omega}\sin \Omega t\right) \quad . \tag{18b}$$

5. Discussion

The general problem of the unsteady convective flow of a viscous, heat conducting fluid near a rotating, infinite and inclined plate has been solved exactly without any restrictions on Prandtl number (P).

The new general solution for the velocity field is exemplified in complex form (cf. Eq. (9b)). In order to obtain the primary and secondary velocity components u and υ, we use the real and imaginary parts of complex velocity q of the expression (9b), namely

$$u(z,t) = \mathrm{Re}(q) \quad , \qquad \upsilon(z,t) = \mathrm{Im}(q) \quad . \tag{19}$$

For numerical values of the velocity components u and υ directly from (19), one can calculate the complementary error functions (*erfc(z)*) of the complex variables into Equations (9b) – (11b) [12].

Another method of computation of the velocity components u and υ is to separate analytically these *erfc(z)* functions [11]. One way is to use the forms of the method of Stand [13] from Appendix. So, it can be shown from (19) that the primary and secondary velocity components u and υ are given by

$$u(z,t) = u_0(z,t) + u_1(z,t) + u_2(z,t) + u_3(z,t) \quad , \tag{20a}$$

$$\upsilon(z,t) = \upsilon_0(z,t) + \upsilon_1(z,t) + \upsilon_2(z,t) + \upsilon_3(z,t) \quad , \tag{20b}$$

where

$$u_0(z,t) \equiv -\frac{\sin \Omega t}{\Omega}\, erfz\left[\tfrac{1}{2}z(P/t)^{1/2}\right] \ , \tag{21a}$$

$$\upsilon_0(z,t) \equiv -\frac{\cos \Omega t}{\Omega}\, erfz\left[\tfrac{1}{2}z(P/t)^{1/2}\right] \ . \tag{21b}$$

The remains components of velocity of the expressions (20) depend every time from the values of the time. After extensive algebraic derivations, it can be shown that these components of velocity are expressed as following

$$u_1(z,t)=\begin{cases}\dfrac{\sin 2\Omega t}{2\Omega}\Phi_1(z,t)-\dfrac{\cos 2\Omega t}{2\Omega}\Psi_1(z,t)\ , & \text{if}\quad t\langle z/(2\Omega)^{1/2} \quad (22a)\\ \dfrac{e^{-zP^{1/2}\Omega_P}}{\Omega}\sin(z\Omega_0+\Omega t)+\dfrac{\sin 2\Omega t}{2\Omega}\Phi_1(z,t)-\dfrac{\cos 2\Omega t}{2\Omega}\Psi_1(z,t)\ , & \text{if}\quad t\rangle z/(2\Omega)^{1/2}, \quad (22b)\end{cases}$$

$$\upsilon_1(z,t)=\begin{cases}\dfrac{\cos 2\Omega t}{2\Omega}\Phi_1(z,t)+\dfrac{\sin 2\Omega t}{2\Omega}\Psi_1(z,t)\ , & \text{if}\quad t\langle z/(2\Omega)^{1/2} \quad (22c)\\ \dfrac{e^{-zP^{1/2}\Omega_P}}{\Omega}\cos(z\Omega_0+\Omega t)+\dfrac{\cos 2\Omega t}{2\Omega}\Phi_1(z,t)+\dfrac{\sin 2\Omega t}{2\Omega}\Psi_1(z,t)\ , & \text{if}\quad t\rangle z/(2\Omega)^{1/2}, \quad (22d)\end{cases}$$

where the functions $\Phi_1(z,t)$ and $\Psi_1(z,t)$ denote

$$\Phi_1(z,t)=\begin{cases}e^{-z\Omega_0}H_{11}\left(|\xi_2|,\ \Omega_0 t^{1/2}\right)+e^{z\Omega_0}H_{12}\left(\xi_1,\ \Omega_0 t^{1/2}\right)\ , & \text{if}\quad t\langle z/(2\Omega)^{1/2} \quad (23a)\\ -e^{-z\Omega_0}H_{11}\left(|\xi_2|,\ \Omega_0 t^{1/2}\right)+e^{z\Omega_0}H_{12}\left(\xi_1,\ \Omega_0 t^{1/2}\right)\ , & \text{if}\quad t\rangle z/(2\Omega)^{1/2}, \quad (23b)\end{cases}$$

$$\Psi_1(z,t)\equiv\Omega_0 t^{1/2}\left[e^{-z\Omega_0}H_{21}\left(|\xi_2|,\ \Omega_0 t^{1/2}\right)-e^{z\Omega_0}H_{22}\left(\xi_1,\ \Omega_0 t^{1/2}\right)\right]\ , \qquad (23c)$$

with $\xi_{1,2}\equiv\tfrac{1}{2}zt^{-1/2}\pm\left(\tfrac{1}{2}\Omega t\right)^{1/2}$, $\Omega_0\equiv(\Omega/2)^{1/2}$; (23d,e)

$$u_2(z,t)=\begin{cases}\dfrac{\sin\Omega t}{2\Omega}\Phi_2(z,t)-\kappa\dfrac{\cos\Omega t}{2\Omega}\Psi_2(z,t)\ , & \text{if}\quad t\langle z\left(\dfrac{P|P-1|}{2\Omega}\right)^{1/2} \quad (24a)\\ \dfrac{e^{-zP^{1/2}\Omega_P}}{\Omega}\sin\left(z\Omega_P P^{1/2}+\kappa\dfrac{P-2}{P-1}\Omega t\right)+\dfrac{\sin\Omega t}{2\Omega}\Phi_{21}(z,t)-\kappa\dfrac{\cos\Omega t}{2\Omega}\Psi_2(z,t)\ , & \text{if}\quad t\rangle z\left(\dfrac{P|P-1|}{2\Omega}\right)^{1/2}, \quad (24b)\end{cases}$$

$$\upsilon_2(z,t)=\begin{cases}\dfrac{\cos\Omega t}{2\Omega}\Phi_2(z,t)+\kappa\dfrac{\sin\Omega t}{2\Omega}\Psi_2(z,t)\ , & \text{if}\quad t\langle z\left(\dfrac{P|P-1|}{2\Omega}\right)^{1/2} \quad (24c)\\ \dfrac{e^{-zP^{1/2}\Omega_P}}{\Omega}\cos\left(z\Omega_P P^{1/2}+\kappa\dfrac{P-2}{P-1}\Omega t\right)+\dfrac{\cos\Omega t}{2\Omega}\Phi_2(z,t)+\kappa\dfrac{\sin\Omega t}{2\Omega}\Psi_2(z,t)\ , & \text{if}\quad t\rangle z\left(\dfrac{P|P-1|}{2\Omega}\right)^{1/2}, \quad (24b)\end{cases}$$

where the functions $\Phi_2(z,t)$ and $\Psi_2(z,t)$ are obtained in the forms

$$\Phi_2(z,t)=\begin{cases}e^{-z\Omega_P P^{1/2}}H_{13}(|\zeta_2|,\ \Omega_P t^{1/2})+e^{z\Omega_P P^{1/2}}H_{14}(\zeta_1,\ \Omega_P t^{1/2})\ , & \text{if}\quad t\langle z\left(\dfrac{P|P-1|}{2\Omega}\right)^{1/2} \quad (25a)\\ -e^{-z\Omega_P P^{1/2}}H_{13}(|\zeta_2|,\ \Omega_P t^{1/2})+e^{z\Omega_P P^{1/2}}H_{14}(\zeta_1,\ \Omega_P t^{1/2})\ , & \text{if}\quad t\rangle z\left(\dfrac{P|P-1|}{2\Omega}\right)^{1/2}, \quad (25b)\end{cases}$$

$$\Psi_2(z,t)=\Omega_P t^{1/2}\left[e^{-z\Omega_P P^{1/2}}H_{23}(|\zeta_2|,\ \Omega_P t^{1/2})-e^{z\Omega_P P^{1/2}}H_{24}(\zeta_1,\ \Omega_P t^{1/2})\right]\ , \qquad (25c)$$

with $\zeta_{1,2}\equiv\dfrac{zP^{1/2}}{2t^{1/2}}\pm\left(\dfrac{\Omega t}{2|P-1|}\right)^{1/2}$, $\Omega_P\equiv\left(\dfrac{\Omega}{2|P-1|}\right)^{1/2}$; (25d.e)

$$u_3(z,t)=\begin{cases}-\left\{\dfrac{\sin 2\Omega t}{2\Omega}\Phi_3(z,t)-\kappa\dfrac{\cos 2\Omega t}{2\Omega}\Psi_3(z,t)\right\}\quad, & \text{if}\quad t\langle z\left(\dfrac{|P-1|}{2P\Omega}\right)^{1/2} \\ -\dfrac{e^{-zP^{1/2}\Omega_P}}{\Omega}\sin\left(z\Omega_P P^{1/2}+\kappa\dfrac{P-2}{P-1}\Omega t\right)-\left\{\dfrac{\sin 2\Omega t}{2\Omega}\Phi_3(z,t)-\kappa\dfrac{\cos 2\Omega t}{2\Omega}\Psi_3(z,t)\right\}, & \text{if}\quad t\rangle z\left(\dfrac{|P-1|}{2P\Omega}\right)^{1/2},\end{cases} \qquad \begin{matrix}(26a)\\ \\(26b)\end{matrix}$$

$$\upsilon_3(z,t)=\begin{cases}-\left\{\dfrac{\cos 2\Omega t}{2\Omega}\Phi_3(z,t)+\kappa\dfrac{\sin 2\Omega t}{2\Omega}\Psi_3(z,t)\right\}\quad, & \text{if}\quad t\langle z\left(\dfrac{|P-1|}{2P\Omega}\right)^{1/2} \\ -\dfrac{e^{-zP^{1/2}\Omega_P}}{\Omega}\cos\left(z\Omega_P P^{1/2}+\kappa\dfrac{P-2}{P-1}\Omega t\right)-\left\{\dfrac{\cos 2\Omega t}{2\Omega}\Phi_3(z,t)+\kappa\dfrac{\sin 2\Omega t}{2\Omega}\Psi_3(z,t)\right\}\quad, & \text{if}\quad t\rangle z\left(\dfrac{|P-1|}{2P\Omega}\right)^{1/2},\end{cases} \qquad \begin{matrix}(26c)\\ \\(26b)\end{matrix}$$

where the functions *Φ₃(z,t)* and *Ψ₃(z,t)* are given by

$$\Phi_3(z,t)=\begin{cases}e^{-z\Omega_P P^{1/2}}H_{15}(|\eta_2|,\ \Omega_P P^{1/2}t^{1/2})+e^{z\Omega_P P^{1/2}}H_{16}(\eta_1,\ \Omega_P P^{1/2}t^{1/2})\quad, & \text{if}\quad t\langle z\left(\dfrac{|P-1|}{2P\Omega}\right)^{1/2}, \\ -e^{-z\Omega_P P^{1/2}}H_{15}(|\eta_2|,\ \Omega_P P^{1/2}t^{1/2})+e^{z\Omega_P P^{1/2}}H_{16}(\eta_1,\ \Omega_P P^{1/2}t^{1/2})\quad, & \text{if}\quad t\rangle z\left(\dfrac{|P-1|}{2P\Omega}\right)^{1/2},\end{cases} \qquad \begin{matrix}(27a)\\ \\(27b)\end{matrix}$$

$$\Psi_3(z,t)\equiv\Omega_P P^{1/2}t^{1/2}\left[e^{-z\Omega_P P^{1/2}}H_{25}(|\eta_2|,\ \Omega_P P^{1/2}t^{1/2})+e^{z\Omega_P P^{1/2}}H_{26}(\eta_1,\ \Omega_P P^{1/2}t^{1/2})\right]\ , \qquad (27c)$$

with $$\eta_{1,2}\equiv\frac{z}{2t^{1/2}}\pm\left(\frac{\Omega P t}{2|P-1|}\right)^{1/2}\ . \qquad (27d)$$

In the expressions (20)-(27) is

$$\kappa=\begin{cases}1 & if\quad P\rangle 1\\ -1 & if\quad P\langle 1\end{cases}\ , \qquad (28)$$

and the functions H_{1j} and H_{2j} , with *j=1, 2, 3, 4, 5, 6,* are given in the Appendix (see, Eqs. A2,3).

It may be noted that the general results of expressions (9), (14), (16) and (20) include the results of the solution the same problem with vertical plate (or surface). Indeed, when the plate is vertical, the axis Oz′ is inclined at an angle $\phi=90^0$ from the vertically upward direction and all the above expressions have exactly the same forms with the characteristic length (cf., Eq. 3c)

$$L_0=\left[\frac{\nu^2}{g\beta'\ (T_w{}'-T_\infty{}')}\right]^{1/3}\ . \qquad (29)$$

Also, the general results (9), (14) and (20) include the results obtained by Chandran et al [11], with $\phi=90^0$ and the Prandtl number *P=1.* In this special case, we have $q_2(z,t)=0$ (from Eq. 8b) in Eq (9b) and, consequently, $u_2(z,t)=u_3(z,t)=\upsilon_2(z,t)=\upsilon_3(z,t)=0$ in Eqs. (20) and the last term of Eqs (14) is equal to zero. Also, for *P=1,* we get $A_2=B_2=0$ in Eqs. (16) and the last terms of Eqs (18) are obviously vanished. A detailed discussion for this simple case is given on the work of Chandran et al. [11].

The general conclusion for the exact solution of the present problem is that this general solution can be used readily for most gases, whose Prandtl number P is between 0,7 and 0,85, for liquids, whose P is generally greater than one; and for restricted classes of gas (namely, steam and ammonia) with P=1.

Appendix

On a complementary error function

The complementary error function of a complex variable can be separated into real and imaginary parts, in accordance with Strand's method [13]. For any complex number $z=x+iy$, with *x>0* and *y≥0*, we have

$$ercf\ (x+iy)=e^{-2ixy}\left[H_1(x,y)-iy\ H_2(x,y)\right]=f(x,y)+i\ h(x,y) \quad , \qquad \text{(A1)}$$

where

$$H_1(x,y)=\sum_{n=0}^{\infty}(xy)^{2n}\sigma_n(x) \quad , \qquad \text{(A2)}$$

$$H_2(x,y)=x\sum_{n=0}^{\infty}(n+1)\ (xy)^{2n}\sigma_{n+1}(x) \quad , \qquad \text{(A3)}$$

$$f(x,y)\ =H_1\cos 2xy-y\ H_2\sin 2xy \quad , \qquad \text{(A4)}$$

$$h(x,y)\ =-H_1\sin 2xy-y\ H_2\cos 2xy \quad , \qquad \text{(A5)}$$

$$\sigma_{n+1}(x)=\frac{2}{2n+1}\left[\frac{\exp(-x^2)}{\pi^{1/2}(n+1)!\,x^{2n+1}}-\frac{\sigma_n(x)}{n+1}\right], \quad n=0,\ 1,\ 2,\ ... \qquad \text{(A6)}$$

$$\sigma_0(x)=erfc(x) \quad . \qquad \text{(A7)}$$

Since $\overline{erfc(z)}=erfc(\bar{z})$, we also obtain

$$ercf\ (x-iy)=e^{2ixy}\left[H_1(x,y)+iy\ H_2(x,y)\right]=f(x,y)-i\ h(x,y) \quad , \qquad \text{(A7)}$$

It should be pointed out that for a complex number $z=x+iy$, with $\mathrm{Re}(z)=x<0$, one can use the relation $erfc(z)=2-erfc(-z)$. So, the above method of computation is employed for $erfc(-z)$ with $\mathrm{Re}(-z)=-x>0$.

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