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Ordinary differential equations

Existence of a Limiting Regime in the Sense of Demidovic for a Certain Class of Second Order Nonlinear Vector Differential Equations

[1]

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Abstract

In this paper, we employ a complete Lyapunov function, Demidovic theorem and the generalized theorems of Ezeilo to establish sufficient conditions for the existence of a limiting regime in the sense of Demidovic for certain second order nonlinear vector differential equation. We equally prove that the limiting regime is periodic or almost periodic with respect to variable t, uniformly in X, Y whenever the forcing term is periodic or almost periodic. The results of this paper are quiet new with respect to second order differential equations.

Keywords and phrases: second order nonlinear differential equation, limiting regime, uniformly periodic (or almost periodic) solution, Lyapunov function, convergence.

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1. Introduction

We shall consider the following second order nonlinear vector differential equation:

$$\ddot{X} + A\dot{X} + H(X) = P(t, X, \dot{X}), \tag{1.1}$$

where $t \in \mathbb{R}^+$, $X : \mathbb{R}^+ \to \mathbb{R}^n$, $H : \mathbb{R}^n \to \mathbb{R}^n$, $P : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, A is an $n \times n$ symmetric, positive definite matrix and the dots as usual indicate differentiation with respect to t. It is also assumed that the functions H and P are continuous in their respective arguments displayed explicitly.

On the qualitative properties of second order differential equations, many interesting results have been obtained. For results on stability [see: 1,2,3,10,11, 18,21,23,30,31], boundedness [see, 3,10,18,19,20,25,26,27,29] and convergence[17,22,28]. But on the subject of a limiting regime in the sense of Demidovic, as far as we know, nothing seems to have been done regarding second order differential equations. The followings are some of the results on existence of a limiting regime for third, fourth and fifth order differential equations.

In [15], Ezeilo used the ideas of Demidovic[12] and Ezeilo[16] to establish sufficient conditions on the existence of a limiting regime to the third order nonlinear differential equation of the form

$$x''' + ax'' + bx' + h(x) = p(t, x, x', x'')$$

where $a,\ b$ are constants and h, p are continuous functions of their arguments. Later, Afuwape and Omeike [8] considered a more general form of the equation above which is of the form

$$x''' + ax'' + g(x') + h(x) = p(t, x, x', x'')$$

the authors improved on the earlier results on the subject of discussion. Furthermore, Olutimo[24] extended the results of Afuwape and Omeike [8] to the corresponding vector version by considering a differential equation of the form

$$\dddot{X} + A\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X})$$

Afuwape[9] also extended the results of Ezeilo[15] to the fourth order nonlinear differential equation

$$x^{iv} + ax''' + bx'' + cx' + h(x) = p(t, x, x', x'', x''')$$

where a, b, c are constants. Much later, Adesina[5] went further to consider a more general fourth order nonlinear differential equation of the form

$$x^{iv} + \phi(x''') + f(x'') + g(x') + h(x) = p(t, x, x', x'', x''').$$

Adeina and Ukpera[4] on their part dealt with fifth order differential equation of the form

$$x^{v} + ax^{iv} + bx''' + cx'' + dx' + h(x) = p(t, x, x', x'', x''', x^{(iv)})$$

where a, b, c, d are constants.

Our goal in this paper is to establish sufficient conditions for the existence of a limiting regime in the sense of Demidovic and also prove that the limiting regime is periodic or almost periodic for a second order non-linear vector differential equations defined in (1.1) whenever the forcing function $P(t, X, \dot{X})$ is periodic or almost periodic in t uniformly with respect to X and $\dot{X}$. In establishing our results, we shall employ the direct method of Lyapunov coupled with the approach of Demidovic[12] and theorems of Ezeilo[16].

2. Preliminary results and definition

Demidovic[12] in 1961 considered a nonlinear system given by

$$\dot{X} = F(X) + G(t) \tag{2.1}$$

where $F(X)$ and $G(t)$ are continuous functions of their respective arguments displayed explicitly. He gave sufficient conditions which ensure the convergence of all solutions of equation (2.1) to a periodic solution (i.e limiting regime) for $t \to \pm\infty$. About four years later, Ezeilo[16] considered a more generalized differential system of the form

$$\dot{X} = f(t, X) + g(t, X) \tag{2.2}$$

and came up with the following results.

Let $f(t, X)$ in the equation (2.2) above satisfies either

$$\|f(t, 0)\| \leq m < \infty \quad \text{for all } t \in \mathbb{R}$$

or

$$\int_{-\infty}^{\infty} \|f(t, 0)\|^p dt < \infty, \ 1 \leq p < 2,$$

while $g(t, X)$ satisfies Lipschitz condition, with $g(t, 0) \equiv 0$. Then, Ezeilo in [16] stated and proved the following theorems for equation (2.2) above.

Theorem 2.1 [16]
Suppose that:

(i) there exists a positive definite $n \times n$ matrix A such that the eigenvalues of $\{D + D^T\}$, where $D = A\frac{\partial f}{\partial X}$, are all negative for all values of t and X.

(ii) $f(t, 0)$ satisfies either

$$\|f(t, 0)\| \leq m < \infty \quad \textit{for all } t \in \mathbb{R}$$

or

$$\int_{-\infty}^{\infty} \|f(t, 0)\|^p dt < \infty, \ 1 \leq p < 2.$$

(iii) $g(t,0) \equiv 0$ *and*

$$\|g(t,X) - g(t,Y)\| \leq \gamma(t)\|X - Y\|$$

for all $X,\ Y,\ t,$ *with* $\gamma(t)$ *satisfying*

$$\int_{-\infty}^{\infty} \gamma^q(t)dt < \infty,\ 1 \leq q \leq 2.$$

Then, there exists a unique solution $X^*(t)$ *of equation (2.2) such that*

$$\|X^*(t)\| \leq m,\ \textit{for } t \in \mathbb{R}, \tag{2.3}$$

and every solution $X(t)$ *of equation (2.2) converges to* $X^*(t)$ *as* $t \to +\infty$.

Theorem 2.2 [16]
Suppose conditions (i) and (ii) of Theorem 2.1 hold, and if in addition the following conditions hold

(i) if $f(t,X)$ *and* $g(t,X)$ *are uniformly almost periodic in* t *for* $\|X\| \leq m$, *then the unique solution* $X^*(t)$ *of equation (2.2) is uniformly almost periodic (u.a.p) in t;*

(ii) if $f(t,X)$ *and* $g(t,X)$ *are both periodic functions of* t, *for* $\|X\| \leq m$ *and have the same period* ω, *then* $X^*(t)$ *is periodic in* t, *with a least period* ω.

Definition 2.3 [8,12,15]
We say that a solution $X^*(t)$ *of equation (2.1) is a limiting regime in the sense of Demidovic, if there exists a constant* m, $0 < m < \infty$ *such that* $\|X^*(t)\| \leq m,\ -\infty < t < \infty$ *and if every other solution of equation(2.1)converges to* $X^*(t)$ *as* $t \to \infty$.

Definition 2.4 [24]
A continuous function $f : \mathbb{R} \to x$ *is called almost periodic if for each* $\epsilon > 0$ *there exists* $\varrho(\epsilon) > 0$ *such that every interval of length* $\varrho(\epsilon)$ *contains a number* τ *with property that*

$$|f(t+\tau) - f(t)| < \epsilon \textit{ for each } t \in \mathbb{R}.$$

Lemma 2.5 *Let* A *be an* $n \times n$ *real symmetric positive definite matrix. Then, for* $X \in \mathbb{R}^n$

$$\delta_a\|X\|^2 \leq \langle AX, X\rangle \leq \Delta_a\|X\|^2, \tag{2.4}$$

where δ_a *and* Δ_a *are, respectively, the least and greatest eigenvalues of the matrix* A.
Proof. See [6,7,13, 14] .

Lemma 2.6 [6,7,13,14]

Following the conditions earlier defined on $H(X)$ with $H(0) = 0$ and let $J_H(X)$ denotes the Jacobian matrix $\frac{\partial h_i}{\partial x_j}$ of $H(X)$, then,

$$\delta_h \|X\|^2 \leq \int_0^1 \langle H(sX), X \rangle ds \leq \Delta_h \|X\|^2,$$

where δ_h and Δ_h are the least and greatest eigenvalues of matrix $J_H(X)$ respectively.

Lemma 2.7 *Let Q and D be any two real $n \times n$ commuting symmetric matrices. Then,*

(i) the eigenvalues $\lambda_i(QD)$, $(i = 1, 2, \ldots, n)$ of the product matrix QD are real and satisfy:

$$\max_{1 \leq j,\ k \leq n} \lambda_j(Q)\lambda_k(D) \geq \lambda_i(QD) \geq \max_{1 \leq j,\ k \leq n} \lambda_j(Q)\lambda_k(D) \qquad (2.5)$$

(ii) the eigenvalues $\lambda_i(Q+D)$, $(i = 1, 2, \ldots, n)$ of the sum of matrices Q and D are real and satisfy:

$$\{\max_{1 \leq j \leq n} \lambda_j(Q) + \max_{1 \leq k \leq n} \lambda_k(D)\} \geq \lambda_i(Q + D) \geq \{\min_{1 \geq j \leq n} \lambda_j(Q) + \min_{1 \leq k \leq n} \lambda_k(D)\} \qquad (2.6)$$

where $\lambda_j(Q)$ and $\lambda_k(D)$ are, respectively, the eigenvalues of matrices Q and D.

Proof. See [6,7,13, 14].

Hence forth, it shall be assumed that vector function $P(t, X, \dot{X})$ is separable in the form

$$P(t, X, \dot{X}) = r(t) + Q(t, X, \dot{X})$$

with $r(t) = r(t) + Q(t, 0, 0)$ so that $Q(t, 0, 0) \equiv 0$. We shall write (1.1) in the equivalent form as

$$\dot{X} = Y + R(t), \quad \dot{Y} = -AY - H(X) + Q(t, X, Y + R(t)) - AR \qquad (2.7)$$

with $\|R(t)\| = \| \int_0^t r(\tau)\| d\tau \leq D,\ D > 0$.

3. Main result

The followings are the main theorems of this paper.

Theorem 3.1
Suppose that $H(0) = 0$ such that:

(i) the Jacobian matrix $J_H(X)$ of $H(X)$ and matrix A are symmetric and commute with each other and their eigenvalues $\lambda_i(J_H(X))$ and $\lambda_i(A)$, $(i = 1, 2, 3, \ldots, n)$ respectively satisfy:

$$0 < \delta_h \le \lambda_i(J_H(X)) \le \Delta_h$$

and

$$0 < \delta_a \le \lambda_i(A) \le \Delta_a$$

where δ_h and Δ_h are the least and greatest eigenvalues of matrix $J_H(X)$ and δ_a and Δ_a are the least and greatest eigenvalues of matrix A, such that δ_h, Δ_h, δ_a and Δ_a are all finites.

(ii)

$$\|Q(t, X_2, Y_2 + R) - Q(t, X_1, Y_1 + R)\| \le \gamma_0\{\|X_2 - X_1\| + \|Y_2 - Y_1\|\} \quad (3.1)$$

for all t and $X_i, Y_i \in \mathbb{R}^n$, $(i = 1, 2)$ and $\gamma_0 < \epsilon$, $\epsilon > 0$.

Then, there exists a unique solution $X^*(t)$ of (1.1) or (2.7) satisfying

$$\|X^*(t)\|^2 + \|\dot{X}^*(t)\|^2 \le D_0,$$

for $t \in \mathbb{R}^+$, where D_0 is a positive constant. Moreover, every other solution $X(t)$ of equation (1.1) converges to $X^*(t)$ as $t \to \infty$.

Theorem 3.2
Suppose that $H(0) = 0$ and conditions (i) and (ii) of Theorem 3.1 hold. Further, suppose that there exists a solution $X(t)$ of equation (1.1) such that

$$\|X(t)\|^2 + \|\dot{X}(t)\|^2 \le D_0.$$

Then,
(i) if $Q(t, X, Y)$ and $R(t)$ are almost periodic in t, for

$$\|X(t)\|^2 + \|\dot{X}(t)\|^2 \le D_0,$$

then $X^*(t)$ is almost periodic in t.
(ii) if $Q(t, X, Y)$ and $R(t)$ are periodic in t, with period η for

$$\|X(t)\|^2 + \|\dot{X}(t)\|^2 \le D_0,$$

then $X^*(t)$ is periodic in t, with period η.

Note, $X^*(t)$ is a limiting regime.

The main tool in proving the two theorems stated above is the scalar function known as Lyapunov functional defined by:

$$2V(X(t), Y(t)) = 2\int_0^1 \langle\{A + B^2\}H(sX), X\rangle ds + \langle B^2 Y, Y\rangle + \langle A^3 X, X\rangle + \langle AY, Y\rangle$$

$$+2\langle AX, AY\rangle \qquad (3.2)$$

where both A and B are $n\times n$ constant symmetric matrices which commute with each other. It is obvious that V(0,0) = 0.

Lemma 3.3
Assuming that all the conditions of Theorem 1 hold. Then we can find some positive constants δ_1 and Δ_1 such that

$$\delta_1\{\|X\|^2+\|Y\|^2\}\le V(X,Y)\le \Delta_1\{\|X\|^2+\|Y\|^2\} \qquad (3.3)$$

for any X, Y belonging to $\mathbb{R}^n$.

Proof of Lemma 3.3
On rearranging the function V defined above in equation (3.2), we obtain:

$$\begin{aligned} 2V(X(t),Y(t)) &= 2\int_0^1\langle\{A+B^2\}H(sX),X\rangle ds+\langle BY,BY\rangle \\ &+ \| A^{\frac{3}{2}}X+A^{\frac{1}{2}}Y\|^2 \\ &\ge 2\int_0^1\int_0^1\langle\{A+B^2\}J_H(s_1s_2X)X,X\rangle ds_1ds_2+\langle BY,BY\rangle. \end{aligned}$$

By applying the hypothesis (i) of the Theorem 3.1, Lemma 2.5 - 2.7, we have:

$$V\ge\{\delta_a+\delta_b^2\}\delta_h\|X\|^2+\frac{1}{2}\delta_b^2\|Y\|^2$$

If we let $\delta_1=\frac{1}{2}\min\{2\delta_h(\delta_a+\delta_b^2),\ \delta_b^2\}$, then we obtain the lower bound for V as:

$$V(X,Y)\ge\delta_1\{\|X\|^2+\|Y\|^2\}. \qquad (3.4)$$

The upper bound of V can also be obtained as follows.

$$\begin{aligned} 2V(X(t),Y(t)) &= 2\int_0^1\langle\{A+B^2\}H(sX),X\rangle ds+\langle BY,BY\rangle \\ &+ \| A^{\frac{3}{2}}X+A^{\frac{1}{2}}Y\|^2 \\ &= 2\int_0^1\int_0^1\langle\{A+B^2\}J_H(s_1s_2X)X,X\rangle ds_1ds_2+\langle BY,BY\rangle \\ &+ \langle A^3X,X\rangle+\langle AY,Y\rangle+2\langle AX,AY\rangle. \end{aligned}$$

Using Lemmas 2.5 - 2.7 and the fact that $2|\langle AY,AX\rangle|\le\langle AX,AX\rangle+\langle AY,AY\rangle$, we obtain

$$\begin{aligned} 2V(X(t),Y(t)) &\le 2\{\Delta_h(\Delta_a+\Delta_b^2)\}\|X\|^2+\{\Delta_a^2+\Delta_a+\Delta_b^2\}\|Y\|^2 \\ &+ \{\Delta_a^3+\Delta_a^2\}\|X\|^2 \\ &= \{2\Delta_h(\Delta_a+\Delta_b^2)+\Delta_a^3+\Delta_a^2\}\|X\|^2+\{\Delta_a^2+\Delta_a+\Delta_b^2\}\|Y\|^2. \end{aligned}$$

Letting $\Delta_1 = \frac{1}{2}\max\{2\Delta_h(\Delta_a + \Delta_b^2) + \Delta_a^3 + \Delta_a^2,\ \Delta_a^2 + \Delta_a + \Delta_b^2\}$, we obtain the upper bound of V as:

$$V \leq \Delta_1\{\|X\|^2 + \|Y\|^2\}. \tag{3.5}$$

Thus, inequality (3.3) follows on combining the estimates (3.4) and (3.5) together. This completes the proof of the Lemma 3.3.

Next, we find the derivative of $V(X,Y)$ with respect to t along the system (2.7) for all solutions (X(t), Y(t)). This gives:

$$\begin{aligned}
\dot{V} &= -\langle A^2X, H(X)\rangle - \langle B^2Y, AY\rangle + \langle\{A + B^2\}H(X) - AB^2Y, R(t)\rangle \\
&+ \langle\{B^2 + A\}Y + A^2X, Q\rangle \\
&= -\int_0^1 \langle A^2X, J_H(sX)X\rangle ds - \langle B^2Y, AY\rangle + \langle\{B^2 + A\}Y + A^2X, Q\rangle \\
&+ \int_0^1 \langle\{A + B^2\}J_H(sX)X - AB^2Y, R(t)\rangle ds
\end{aligned}$$

in view of the assumption (i) of the Theorem 3.1 and Lemmas 2.5 - 2.7 we have

$$\begin{aligned}
\dot{V} &\leq -\delta_a^2\delta_h\|X\|^2 - \delta_b^2\delta_a\|Y\|^2 + \{(\Delta_a + \Delta_b^2)\Delta_h\|X\| - \delta_a\delta_b^2\|Y\|\}D \\
&+ \{(\Delta_b^2 + \Delta_a)\|Y\| + \Delta_a^2\|X\|\}\|Q(t, X, Y + R)\| \\
&= -K_3\{\|X\|^2 + \|Y\|^2\} + K_4\{\|X\| + \|Y\|\} \\
&+ K_5\{\|X\| + \|Y\|\} \times \|Q(t, X, Y + R(t))\|
\end{aligned} \tag{3.6}$$

where $K_3 = \min\{\delta_a^2\delta_h,\ \delta_b^2\delta_a\}$, $K_4 = \max\{(\Delta_a + \Delta_b^2)\Delta_h,\ \delta_a\delta_b^2\}D$ and $K_5 = \max\{\Delta_b^2 + \Delta_a, \Delta_a^2\}$.
Now, by applying the condition (ii) of Theorem 3.1, we obtain

$$\begin{aligned}
\dot{V} &\leq -K_3\{\|X\|^2 + \|Y\|^2\} + K_6\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}} \\
&+ K_7\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}} \times \gamma_0\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}} \\
&\leq -K_3\{\|X\|^2 + \|Y\|^2\} + K_6\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}} + K_7\gamma_0\{\|X\|^2 + \|Y\|^2\} \\
&\leq -\{K_3 - K_7\gamma_0\}\{\|X\|^2 + \|Y\|^2\} + K_6\{\|X\|^2 + \|Y\|^2\}^{\frac{1}{2}}
\end{aligned}$$

The last inequality implies

$$\dot{V} \leq -\{K_3 - K_7\gamma_0\}V(t) + K_6V^{\frac{1}{2}}(t) \tag{3.7}$$

where $K_6 = K_4\sqrt{2}$ and $K_7 = K_5\sqrt{2}$. Thus, ϵ can be taken to be $\epsilon = K_7^{-1}K_3 > 0$. Hence, $\gamma_0 < \epsilon$ as indicated in Theorem 3.1.

According to Ezeilo[7], the following Lemma will be quite useful.

Lemma 3.4

Assuming that the conditions (i) and (ii) of Theorem 3.1 hold. Then, for arbitrary t_0, there exists positive constants K_8, K_9 depending on A, $H(X)$, Q and R such that for $t \geq t_0$,

$$V(X(t), Y(t)) \leq K_8 V(X(t_0), Y(t_0)) + K_9. \tag{3.8}$$

Moreover, there are finite constants η and K_{10}, also depending only on A, $H(X)$, Q and R such that if $V(X(t_0), Y(t_0)) \leq K_{10}$, then

$$V(X(t_0+\eta), Y(t_0+\eta)) \leq K_{10} \tag{3.9}$$

for every $\eta_0 \leq \eta < \infty$.

Proof
Let us set $W(t) = V(X(t), Y(t))^{\frac{1}{2}}$, then we have from inequality (3.7) that

$$\frac{d}{dt}\{W(t)\exp\{\frac{1}{2}[K_3 - K_7\gamma_0]t\}\} \leq \frac{1}{2}K_6 \exp\{\frac{1}{2}[K_3 - K_7\gamma_0]t\}. \tag{3.10}$$

Integrating (3.10) from t_0 to $t_0 + S$, $S \geq 0$, we have

$$W(t_0+S)\exp\{\frac{1}{2}[K_3 - K_7\gamma_0](t_0+S)\}$$

$$\leq W(t_0)\exp\{\frac{1}{2}[K_3 - K_7\gamma_0]t_0\} + \frac{1}{2}K_6\int_{t_0}^{t_0+S}\exp\{\frac{1}{2}[K_3 - K_7\gamma_0]t\}dt \tag{3.11}$$

It is obvious from the condition (ii) of Theorem 3.1 that the second term in the inequality (3.11) above is a constant and also finite since γ_0 is a constant. On some arrangements of terms in (3.11), we obtain

$$W(t_0+S) \leq K_{11}W(t_0)\exp\{-\frac{1}{2}K_3 S\} + K_{12}, \ S \geq 0 \tag{3.12}$$

where K_{12} is a positive constant depending on K_3, K_6 and K_7.
Now, if $K_{11}W(t_0) \leq K_{12}$, we have that

$$W(t_0+S) \leq 2K_{12}, \ \text{for } S \geq 0. \tag{3.13}$$

This means

$$V(t_0+S) \leq \{2K_{12}\}^2, \ \text{provided that } S \geq 0.$$

Also, if $K_{11}W(t_0) > K_{12}$, we have from (3.12) that

$$W(t_0+S) < 2K_{11}W(t_0), \ \text{for } S \geq 0.$$

This means

$$V(t_0+S) < \{2K_{11}\}^2 V(t_0), \ \text{provided that } S \geq 0.$$

Hence, in all cases, we have

$$V(t_0+S) \leq \{2K_{11}\}^2 V(t_0) + \{2K_{12}\}^2, \text{ provided that } S \geq 0,$$

which is equivalent to (3.8) with $K_8 = \{2K_{11}\}^2$ and $K_9 = \{2K_{12}\}^2$.
The concluding part of the proof of the Lemma is now to show that for some number, say η_0 (whose value will be determined later),

$$V(t_0+\eta) \leq K_{10}$$

for every $\eta_0 \leq \eta < \infty$ and K_{10} such that $V(t_0) \leq K_{10}$.

Let's define $K_{13} = K_9 = \{2K_{12}\}^2$.
First, if $V(t_0) \geq K_{13}$, we have that $K_{12} < \frac{1}{2}W(t_0)$.
Therefore, from (3.12), we have

$$W(t_0+S) < K_{11}W(t_0)\exp\{-\frac{1}{2}K_3 S\} + \frac{1}{2}W(t_0)$$

$$\leq W(t_0) \text{ provided } S \geq \frac{2\log 2K_{11}}{K_3} > \frac{\log 2K_{11}}{K_3}. \tag{3.14}$$

That is,

$$V(t_0+S) \leq V(t_0),$$

each time $V(t_0) \geq K_{13}$. Now, if $V(t_0) < K_{13}$, we have that $W(t_0) \leq K_{13}^{\frac{1}{2}}$. Thus, from (3.12), we have that

$$W(t_0+S) < K_{11}\exp\{-\frac{1}{2}K_3 S\}K_{13}^{\frac{1}{2}} + K_{13}^{\frac{1}{2}}$$

$$\leq 2K_{13}^{\frac{1}{2}}, \text{ provided that } S \geq \frac{2\log K_{11}}{K_3} > \frac{\log \frac{2}{3}K_{11}}{K_3}.$$

That is,

$$V(t_0+S) < 2K_{13}, \text{ provided that } S \geq \frac{\log 2K_{11}}{K_3}.$$

Thus, on choosing $\eta_0 = \frac{\log 2K_{11}}{K_3}$ and $K_{10} = 2K_{13}$ in the above inequality, inequality (3.9) of Lemma 3.4 is verified and this completes the proof of Lemma 3.4.

To prove Theorem 3.1 completely, we need to prove that any two solutions of (2.7) converge. This will be shown in the lemma below.

Lemma 3.5
Suppose that conditions (i) and (ii) of Theorem 3.1 hold. Suppose that in addition that there exists constants d_3, d_4, d_5 whose magnitude depend on A, $H(X), Q$, and R, then if (X_1, Y_1), (X_2, Y_2) are any two solutions of (2.7), then

$$U(t) \leq d_3 U(t_0)\exp\{-(d_4 - d_5\gamma_0)(t-t_0)\}, \tag{3.15}$$

where

$$U(t) = \{\|X_1(t) - X_2(t)\|^2 + \|Y_1(t) - Y_2(t)\|^2\}.$$

Proof

Given that $X_1(t)$ and $X_2(t)$ are any two solutions of (2.7), we define a function $W = W(t)$ by

$$W(t) = V((X_1(t) - X_2(t), (Y_1(t) - Y_2(t)))$$

where V is the function earlier defined in (3.2) but with X, Y replaced by $(X_1(t) - X_2(t))$ and $(Y_1(t) - Y_2(t))$ respectively. Then, by inequality (3.3), there exists positive constants say K_{14}, K_{15} such that

$$K_{14}U(t) \leq W(t) \leq K_{15}U(t). \tag{3.16}$$

Also by the inequality (3.16), it suffices to show that

$$W(t) \leq d_3 W(t_0) \exp\{-(d_4 - d_5\gamma_0)(t - t_0)\}, \ (t \geq t_0). \tag{3.17}$$

By the earlier calculation of $\dot{V}$ in (3.6), we have

$$\dot{W}(t) \leq -K_{16}\{\|X_1 - X_2\|^2 + \|Y_1 - Y_2\|^2\} + K_{17}^*\{\|X_1 - X_2\| + \|Y_1 - Y_2\|\}\|\theta\|,$$

where $\theta = Q(t, X_2, Y_2 + R) - Q(t, X_1, Y_1 + R)$. and $K_{17} = K_{17}^*\sqrt{2}$
Let us set $U(t) = \{\|X_1 - X_2\|^2 + \|Y_1 - Y_2\|^2\}$ then, we have

$$\dot{W}(t) \leq -K_{16}U(t) + K_{17}U^{\frac{1}{2}}(t)\|\theta\|. \tag{3.18}$$

Let β be any constant such that $1 \leq \beta \leq 2$ and set $2\alpha = 2 - \beta$, so that $0 \leq 2\alpha \leq 1$.
We write inequality (3.18) in the form

$$\dot{W} + K_{16}U(t) \leq K_{17}U^{\alpha}W^*,$$

where

$$W^* = (\|\theta\| - K_{16}K_{17}^{-1}U^{\frac{1}{2}})S^{(\frac{1}{2}-\alpha)}.$$

We will consider separately two possible cases as follow.
(i) $\|\theta\| \leq K_{16}K_{17}^{-1}U^{\frac{1}{2}}$ and
(ii) $\|\theta\| > K_{16}K_{17}^{-1}U^{\frac{1}{2}}$.
We find out that in either case, there exists some constants K_{18} such that $W^*(t) \leq K_{18}\|\theta\|^{2(1-\alpha)}$. Thus, we can rewrite the inequality (3.18) as

$$\dot{W} + K_{16}U(t) \leq K_{19}U^{\alpha}\gamma_0 U^{(1-\alpha)}$$

where $K_{19} \geq 2K_{17}K_{18}$. This immediately yields

$$\dot{W} + (K_{20} - K_{21}\gamma_0)W(t) \leq 0 \tag{3.19}$$

by (3.16), with positive constants K_{20} and K_{21}. On integrating (3.19) from t_0 to t_1, $(t_1 \geq t_0)$, we obtain

$$W(t_1) \leq W(t_0)\exp\{-(K_{20} - K_{21}\gamma_0)(t_1 - t_0)\}.$$

Again, by using (3.16), we obtain (3.17). Thus, inequality (3.15) implies that for all $t_1 - t_0 \geq 0$ and $\gamma_0 < d_4 d_5^{-1}$, $-(d_4 - d_5\gamma_0)(t - t_0)$ is negative and so, as $t = (t_1 - t_0) \to \infty$, we have $U(t) \to 0$. Which implies

$$\|X_1(t) - X_2(t)\| \to 0,\ \|Y_1(t) - Y_2(t)\| \to 0 \text{ as } t \to \infty.$$

So that, for the unique solution $X^*(t)$ of the equation (1), we have

$$\|X(t) - X^*(t)\| = 0,\ \|\dot{X}(t) - \dot{X}(t)\| = 0,$$

which implies that

$$X(t) = X^*(t),\ \dot{X}(t) = \dot{X}(t).$$

This completes the proof of Lemma 3.5.

Proof of Theorem 3.1
Having proved Lemma 3.4 and Lemma 3.5, the proof of Theorem 3.1 then follows exactly as in Theorem 1 of [7] with the obvious modifications as required.

Proof of Theorem 3.2
The method to be used in proving Theorem 3.2 is as outlined in Ezeilo [7] but with some modifications as a result of $Q(t, X, Y + R)$ which is almost periodic in t.
Let us consider the function defined as

$$\psi(t) = V(X(t + \eta) - X(t), Y(t + \eta) - Y(t))$$

where V is the function defined in equation (3.2) with $X,\ Y$ replaced by $X(t + \eta) - X(t), Y(t + \eta) - Y(t))$, respectively. Then, we easily have by the inequality (3.3) that there exists positive constants $c_1,\ c_2$ both positive such that

$$c_1 S(t) \leq \psi(t) \leq c_2 S(t) \tag{3.20}$$

with

$$S(t) = \{\|X(t + \eta) - X(t)\|^2 + \|Y(t + \eta) - Y(t)\|^2\}.$$

Following the approach used in proving Lemma 3.5, we have for some positive constants $c_3,\ c_4$ that

$$\begin{aligned} \dot{\psi} &\leq -c_3\{\|X(t + \eta) - X(t)\|^2 + \|Y(t + \eta) - Y(t)\|^2\} \\ &+ c_4\{\|X(t + \eta) - X(t)\| + \|Y(t + \eta) - Y(t)\|\}\|\theta\| \end{aligned} \tag{3.21}$$

with $\theta = Q(t + \eta, X(t + \eta), Y(t + \eta) + R(t + \eta)) - Q(t, X(t), Y(t) + R(t))$.

Now, we can rewrite (3.21) as

$$\begin{aligned}\dot{\psi} &\leq -c_3\{\|X(t+\eta)-X(t)\|^2+\|Y(t+\eta)-Y(t)\|^2\} \\ &+ \{\|X(t+\eta)-X(t)\|+\|Y(t+\eta)-Y(t)\|\}^{\frac{1}{2}} \times \\ &\quad \|Q(t+\eta, X(t+\eta), Y(t+\eta)+R(t+\eta))-Q(t, X(t)), Y(t)+R(t))\| \\ &+ c_4\{\|X(t+\eta)-X(t)\|+\|Y(t+\eta)-Y(t)\|\}\|\theta\|. \qquad (3.22)\end{aligned}$$

Assuming now that the function Q is uniformly almost periodic in t. Then for arbitrary number $\mu > 0$, we can find $\eta > 0$ such that

$$\|Q(t+\eta, X(t+\eta), Y(t+\eta)+R(t+\eta))-Q(t, X(t)), Y(t)+R(t))\| \leq \lambda\mu^2 \quad (3.23)$$

where λ is a constant whose value will be determined later to our credit. Thus, from (3.22), we obtain

$$\dot{\psi} \leq -c_3 S(t) + c_5 S^{\frac{1}{2}}(t)\|\theta\| + c_6 S^{\frac{1}{2}}(t)\lambda\mu^2 \quad (3.24)$$

where $c_5 = c_4\sqrt{2}$ and $c_6 = \sqrt{2}$. By condition (ii) of Theorem 3.1, we have

$$\{\|X(t+\eta)-X(t)\|^2+\|Y(t+\eta)-Y(t)\|^2\}^{\frac{1}{2}} \leq D_1 \quad (3.25)$$

then

$$\dot{\psi} + c_3 S(t) \leq c_5 S^{\frac{1}{2}}(t)\|\theta\| + c_6 D_1\lambda\mu^2. \quad (3.26)$$

Let β be any constant such that $1 \leq \beta \leq 2$ and set $\alpha = 1 - \frac{1}{2}\beta$, so that $0 \leq \alpha \leq 1$. Inequality (3.26) thus becomes,

$$\frac{d\psi}{dt} \leq c_5 S^{\alpha} U^* + c_6 D_1\lambda\mu^2 \quad (3.27)$$

where $U^* = S^{(\frac{1}{2}-\alpha)}\Big(\|\theta\| - c_5^{-1}c_3 S^{\frac{1}{2}}(t)\Big)$.

Now, if $\|\theta\| \leq c_5^{-1}c_3 S^{\frac{1}{2}}(t)$, we obtain

$$U^* \leq 0;$$

again, suppose that $\|\theta\| > c_5^{-1}c_3 S^{\frac{1}{2}}(t)$, that is,

$$S < (c_5 c_3^{-1}\|\theta\|)^2, \text{ we get}$$

$$U^* < c_7\|\theta\|^{2(1-\alpha)},$$

where $c_7 = (c_5 c_3^{-1})^{(2\alpha-1)}$.

Thus in the two cases, $U^* < c_7\|\theta\|^{2(1-\alpha)}$. Therefore, on using the fact that $\|\theta\| \leq \gamma_0 S^{\frac{1}{2}}$ from inequality (3.1), inequality (3.27) becomes,

$$\frac{d\psi}{dt} \leq c_7 c_5 \gamma_0^{2(1-\alpha)} S(t) + c_6 D_1\lambda\mu^2.$$

On using inequality (3.20), we have

$$\frac{d\psi}{dt} + c_8\gamma_0^\beta\psi \leq c_6 D_1 \lambda\mu^2 \tag{3.28}$$

where $c_8 = -c_7 c_5 \gamma^\beta$.
Integrating inequality 3.28 from t_0 to t with $t \geq t_0$ and letting

$$c_{11} = \int_{t_0}^{t} e^{c_8 s} ds,$$

we obtain

$$\begin{aligned} \psi(t) &\leq \psi(t_0)\exp\{c_8(t_0 - t)\} + c_{11}\exp\{-c_8 t\} D_1 \lambda\mu^2 \\ &\leq \psi(t_0)\exp\{c_8(t_0 - t)\} + c_{12}\lambda\mu^2 \end{aligned} \tag{3.29}$$

where $c_{12} = c_{11}\exp\{-c_8 t\}D_1$. By letting $t_0 \to -\infty$ in inequality (3.29) and noting that $\psi(t_0)$ is finite from (3.25), we then obtain

$$W(t) \leq c_{12}\lambda\mu^2$$

for arbitrary t. Now, by inequality (3.20) and the definition of $W(t)$, we obtain

$$\|X(t+\eta) - X(t)\|^2 + \|Y(t+\eta) - Y(t)\|^2 \leq c_{12}\lambda\mu^2 c_1^{-1}. \tag{3.30}$$

Taking $\lambda = c_1 c_{12}^{-1}$, inequality (3.30) thus becomes

$$\|X(t+\eta) - X(t)\|^2 + \|Y(t+\eta) - Y(t)\|^2 \leq \mu^2. \tag{3.31}$$

Multiplying inequality (3.31) by $\sqrt{2}$, we obtain

$$\sqrt{2}\{\|X(t+\eta) - X(t)\|^2 + \|Y(t+\eta) - Y(t)\|^2\} \leq \sqrt{2}\mu^2,$$

it then follows that

$$\|X(t+\eta) - X(t)\| + \|Y(t+\eta) - Y(t)\| \leq \mu \tag{3.32}$$

The proof of the first part of Theorem 3.2 is completes once we choose η to satisfy (3.23) and $\lambda = c_1 c_{12}^{-1}$.

The proof of the second part of Theorem 3.2 is as follows. Assuming that $Q(t, X, Y + R)$ is periodic in t with period ϵ and we fix the τ in the definition of $\psi(t)$. Then, the terms on the left hand side of (3.23) is identically zero, and if we proceed just as we did above, we shall obtain the following in place of (3.30)

$$\|X(t+\eta) - X(t)\|^2 + \|Y(t+\eta) - Y(t)\|^2 \leq 0.$$

But the above cannot be less than zero. Therefore,

$$\|X(t+\eta) - X(t)\|^2 + \|Y(t+\eta) - Y(t)\|^2 = 0.$$

This obviously implies that

$$X(t+\eta) = X(t) \text{ and } Y(t+\eta) = Y(t)$$

this therefore shows the periodicity as required and the proof of Theorem 3.2 is completed.

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