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On oscillation of neutral second order differential equations

Oscillation criteria for second order nonlinear neutral differential equations.

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Abstract

In this paper we are concerned with the oscillatory behavior of solutions of two general classes of second order nonlinear neutral differential equations. The obtained results improve and extend some known criteria in the literature. Two illustrative examples are given to justify our results.

1. Introduction

The aim of this paper is to study the oscillatory behavior of solutions of the nonlinear differential equations of the type: $[r(t)\psi(x(t))(x(t)+p(t)x(\sigma(t)))']'+\sum_{j=1}^{n}q_j(t)f_j(x(t),x(\tau_j(t)))=0,\ t\ge t_0,$

$$j=1,2,\dots,n. \tag{1.1}$$

and

$$(r(t)|z'(t)|^{\alpha-1}z'(t))'+\sum_{i=1}^{m}f_j(t,x(\sigma_j(t)))=0,\quad t\ge t_0 \tag{1.2}$$

where $z(t)=x(t)+\sum_{i=1}^{n}p_i(t)x(\tau_i(t))$ and $\alpha>0$.

Here $r(t),\sigma\in C^1(I,(0,\infty))$, $p(t),q_j(t)\in C^1(I,R)$ and $f_j\in C(R^2,R)$.

Throughout the paper we assume that

(H_1) $\int^{\infty} r^{-1}(s)\, ds = \infty$ for $t \in I, I = [t_0, \infty)$.

(H_2) $0 \le p(t) \le 1$, and $q_j(t) > 0$ for $j = 1,2,...,n.$

(H_3) $\forall\, t \ge t_0, \sigma'(t) > 0$, $\sigma(t) \le t$, and $\lim_{t\to\infty} \sigma(t) = \infty.$

(H_4) $\tau_j(t) \in C(I, R), \tau_j(t) \le t$, $\lim_{t\to\infty} \tau_j(t) = \infty$ for $t \ge t_0$, and $j = 1,2,...,n.$

(H_5) $\psi(u) \in C^1(R, R), \psi(u) > 0$, and for any $t \ge t_0$ there exist two positive constants C and C_1 such that $C \le \psi(u(t)) \le C_1$.

(H_6) $\dfrac{f_j(x, y)}{y} \ge \mu_j > 0$ for $y \ne 0$, and $x f_j(x, y) > 0$ for $x y > 0$ and $j = 1,2,...,n.$

Following Philos [13], we shall define a class of functions X. So we first define

$D = \{(t, s) : t_0 \le s < \mathrm{t} < \infty\}$ and $D_0 = \{(t, s) : t_0 \le s \le \mathrm{t} < \infty\}$.

We say that a continuous function $H : D \to [0, \infty)$ belongs to the class X denoted by $H \in X$ if

(i) $H(t, t) = 0, H(t, s) > 0$ for $(t, s) \in D_0$.

(ii) $H(t, s)$ has a continuous partial derivative with respect to s defined by

$$\frac{\partial H(t, s)}{\partial s} = -h(t, s)\sqrt{H(t, s)} \quad \text{for some } h \in C(D_0, R).$$

By a solution of Eq.(1.1), or Eq.(1.2), we mean a continuously differentiable function $x(t)$ which has the property $r(t)\psi(x(t))[x(t) + p(t)x(\sigma(t))]' \in C^1([T, \infty), R)$, $T \ge t_0$, and satisfies Eq.(1.1),or eq.(1.2), for all $t \ge T$.

We restrict our attention to those solutions $x(t)$ which exist on some half-line $[T, \infty)$, $T \ge t_0$ and satisfy the condition $\sup_{t\ge T}\{|x(t)| : t \ge T\} > 0$. A solution $x(t)$ is called oscillatory if it has arbitrarily large zeros, otherwise, it is called nonoscillatory. Eq.(1.1),or eq.(1.2) is called oscillatory if all of its solutions are oscillatory.

Recently a notable interest in obtaining sufficient conditions for oscillation of different forms of neutral differential equations increased due to the importance of this class in many applications in science and technology. Many contributions appeared in the literature to discuss the oscillation of second order equations which are special cases of Eq.(1.1) and Eq. (1.2) (see [1] ,[2],[3],[5],[7],and [10]),and references therein.

For some related works, *Travis [14] and Waltman [15]* discussed the oscillstion problem of neutral delay differential equation of the type

$$(x(t)+p(t)x(t-\tau))''+q(t)x(t-\sigma)=0 \tag{1.3}$$

Many other contributions were recently offered about the oscillatory behavior of such problems of second order differential equations (see [1]-[15]). In [4], *Grammatikopoulos and Ladas* extended the results of [14,15]. They proved that, if

$\int^{\infty} q(s)(1-p(s-\tau))\, ds=\infty,$ then (1.3) is oscillatory.

Recently, *Rogovchenko and Tuncay [9]*, studied the oscillatory behavior of solutions of second order nonlinear differential equation with a damping term of the form

$$(r(t)x'(t))'+p(t)x'(t)+q(t)f(x(t))=0.$$

Very recently, Hassanbulli and Rogovchenko [7] studied the oscillation of nonlinear neutral differential equations of the type

$$[r(t)(x(t)+p(t)x(t-\tau))']'+q(t)f(x(t),x(\sigma(t)))=0 \tag{1.4}$$

Džurina et al.[2] and Sun et al.[10] discussed the oscillation criteria for the differential equation:

$$(r(t)\left|x'(t)\right|^{\alpha-1}x'(t))'+q(t)\left|x(\sigma(t))\right|^{\alpha\text{-}1}x(\sigma(t))=0 \tag{1.5}$$

Lui et al. [8] and Xu et al. [11,12] extended the results of ([2] and [10]) to Eq.(1.5).

In this paper, we give some new sufficient conditions for Eq. (1.1) and Eq. (1.2) to be oscillatory .We illustrate our results by two examples. The key idea in the proofs makes use of the idea used in [7] and [1].

2. Oscillation criteria for Eq. (1.1)

Suppose that there exists a function $g\in C^{1}(I,(0,\infty))$, where $I=[t_0,\infty)$, such that

$$0<\inf_{s\geq t_0}[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}]\leq\infty \tag{2.1}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{r(\tau_j(s))g(s)}{\tau_j'(s)}\,h^2(t,s)\,ds<\infty \tag{2.2}$$

Theorem 2.1. Let the conditions (H_1)-(H_6) hold . Suppose that there exists a function $g\in C^{1}(I,(0,\infty))$, such that for some $m\geq 1$ and $H\in X$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\varphi(s)-\frac{mC_1v_j(s)r(\tau_j(s))}{4\tau_j'(s)}\,h^2(t,s)\,ds=\infty,\ j=1,...,n. \tag{2.3}$$

where

$$\varphi(t)=v_j(t)[\sum_{j=1}^{n}\mu_j q_j(t)(1-p(\tau_j(t)))+\frac{\tau_j'(t)r^2(t)g^2(t)}{C_1r(\tau_j(t))}-[r(t)g(t)]'],$$

and

$$v_j(t) = \exp[-2\int^t \frac{\tau_j'(s) r(s) g(s)}{C\, r(\tau_j(s))}\, ds],$$

then Eq. (1.1) is oscillatory.

Proof. Suppose the contrary that $x(t)$ be a nonoscillatory solution of Eq. (1.1). Then there exists $t_1 \geq t_0$ such that $x(t) \neq 0 \ \ \forall t \geq t_1$. Assume that $x(t) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1 \geq t_0$. Define $z(t) = x(t) + p(t)x(\sigma(t)) \ \ \forall t \geq t_1$. It is clear that $z(t) \geq x(t) > 0$. Now since

$$[r(t)\psi(x(t))z'(t)]' = -\sum_{j=1}^{n} q_j(t) f_j(x(t), x(\tau_j(t))) \leq 0,$$

then $r(t)\psi(x(t))z'(t)$ is nonincreasing. Going through as in Theorem 2 of [6] we see that

$z'(t) > 0 \ \ \forall \mathrm{t} \geq \mathrm{t}_1$.

But since from (H_6) we note that

$$[r(t)\psi(x(t))z'(t)]' + \sum_{j=1}^{n} \mu_j q_j(t) x(\tau_j(t)) \leq 0 \tag{2.4}$$

Hence since $x(t) = z(t) - p(t)x(\sigma(t)) \geq z(t) - p(t)z(\sigma(t)) \geq [1-p(t)]z(t)$, then there exists $t_2 \geq t_1$ such that $x(\tau_j(t)) \geq [1-p(\tau_j(t))]z(\tau_j(t)) \quad \forall\ t \geq t_2$. (2.5)

Therefore

$$[r(t)\psi(x(t))z'(t)]' \leq -\sum_{j=1}^{n} \mu_j q_j(t)[1-p(\tau_j(t))]\, z(\tau_j(t)) \tag{2.6}$$

Define $w(t) = v_j(t) r(t)[\dfrac{\psi(x(t))z'(t)}{z(\tau_j(t))} + g(t)]$ for $j = 1, 2, \ldots, n.$ (2.7)

Then using (2.6), (H_3) and (H_6), we obtain

$$\begin{aligned} w'(t) \leq & -\sum_{j=1}^{n} \mu_j q_j(t)[1-p(\tau_j(t))] v_j(t) + \frac{v_j'(t)}{v_j(t)}[w(t) - r(t)v_j(t)g(t)] \\ & - \frac{r(t)v_j(t)\psi(x(t))z'(\tau_j(t))\tau_j'(t)}{z'(t)}\left(\frac{z'(t)}{z(\tau_j(t))}\right)^2 + r(t)g(t)v_j'(t) + v_j(t)[r(t)g(t)]' \end{aligned} \tag{2.8}$$

Since $r(t)\psi(x(t))z'(t)$ is nonincreasing and $\tau_j(t) \leq t$, then

$$z'(\tau_j(t)) \geq \frac{r(t)\psi(x(t))z'(t)}{r(\tau_j(t))\psi(x(\tau_j(t)))} \tag{2.9}$$

Thus substituting from (2.8) in (2.9) we get

$$w'(t) \le -v_j(t)[\sum_{j=1}^{n} \mu_j q_j(t)(1-p(\tau_j(t))) + \frac{\tau_j^{'}(t) r^2(t) g^2(t)}{C_1 r(\tau_j(t))} - [r(t)g(t)]']$$
$$-\frac{\tau_j^{'}(t)}{C_1 v_j(t) r(\tau_j(t))} w^2(t)$$

i.e.

$$w'(t) \le -\varphi(t) - \frac{\tau_j^{'}(t)}{C_1 v_j(t) r(\tau_j(t))} w^2(t) \tag{2.10}$$

Multiplying (2.10) by $H(t,s)$ and integrating with respect to s from t_2 to t, we get for all $m \ge 1$

$$\int_{t_2}^{t} w'(s) H(t,s)\, ds \le -\int_{t_2}^{t} \phi(s) H(t,s)\, ds - \int_{t_2}^{t} \frac{\tau_j^{'}(s) w^2(s)}{C_1 v_j(s) r(\tau_j(s))} H(t,s)\, ds$$

i.e.

$$\int_{t_2}^{t} \phi(s) H(t,s)\, ds \le w(t_2) H(t,t_2) - \int_{t_2}^{t} [h(t,s)\sqrt{H(t,s)}\, w(s) + \frac{\tau_j^{'}(s) w^2(s)}{C_1 v_j(s) r(\tau_j(s))} H(t,s)] ds$$

Thus

$$\int_{t_2}^{t} \phi(s) H(t,s)\, ds \le -\int_{t_2}^{t} \frac{(m-1)\tau_j^{'}(s) H(t,s)}{m C_1 v_j(s)\, r(\tau_j(s))} w^2(s)\, ds + \int_{t_2}^{t} \frac{m C_1 v_j(s)\, r(\tau_j(s))}{4\tau_j^{'}(s)} h^2(t,s)\, ds$$
$$-\int_{t_2}^{t} \left(\sqrt{\frac{H(t,s)\tau_j^{'}(s)}{m C_1 v_j(s)\, r(\tau_j(s))}}\; w(s) + \sqrt{\frac{m C_1 v_j(s)\, r(\tau_j(s))}{4\tau_j^{'}(s)}}\; h(t,s) \right)^2 ds \tag{2.11}$$

Therefore

$$\int_{t_2}^{t} [\varphi(s) H(t,s) - \frac{m C_1 v_j(s) r(\tau_j(s))}{4\tau_j^{'}(s)} h^2(t,s)] ds \le |w(t_2)|\, H(t,t_2)$$

$$\int_{t_2}^{t} [\varphi(s) H(t,s) - \frac{m C_1 v_j(s) r(\tau_j(s))}{4\tau_j^{'}(s)} h^2(t,s)] ds \le |w(t_2)|\, H(t,t_0)$$
$$\le H(t,t_0)[|w(t_2)| + \int_{t_0}^{t_2} |\varphi(s)|\, ds]$$

Hence

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_2}^{t} [\varphi(s) H(t,s) - \frac{m C_1 v_j(s) r(\tau_j(s))}{4\tau_j^{'}(s)} h^2(t,s)] ds$$
$$\le |w(t_2)| + \int_{t_0}^{t_2} |\varphi(s)|\, ds < \infty.$$

(2.12)

This contradicts (2.3). Hence Eq. (1.1) is oscillatory.

Remark 2.1. Theorem 2.1 extends Theorem 2 of [7].

Theorem 2.2. Let (2.1) holds . Assume that there exist functions $H \in X$, $g \in C^1(I,R)$ and $\Phi \in C(I,R)$ such that for all $T \ge t_0$ .If for some $m \ge 1$,we have

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\varphi(s) - \frac{mC_1 v_j(s) r(\tau_j(s))}{4\tau_j'(s)} h^2(t,s)\, ds \ge \Phi(T),\ j = 1,\ldots,n. \tag{2.13}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^t \frac{\tau_j'(s)\Phi_+^2(s)}{v_j(s) r(\tau_j(s))}\, ds = \infty \tag{2.14}$$

where $\Phi_+(t) = \max(\Phi(t), 0)$, then Eq.(1.1) is oscillatory.

Proof. Suppose the contrary that $x(t)$ is a nonoscillatory solution of Eq. (1.1) such that $x(t) > 0$, and $x(\tau_j(t)) > 0$. Then proceeding as in the proof of Theorem 2.1 until we arrive Eq. (2.11), then on the light of (2.13) it follows that which implies with Eq. (2.13) that

$$\Phi(t_2) \le \limsup_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^t [\varphi(s) H(t,s) - \frac{mC_1 v_j(s) r(\tau_j(s))}{4\tau_j'(s)} h^2(t,s)] ds$$

$$\le w(t_2) - \liminf_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^t [\frac{(m-1)H(t,s)\tau_j'(s)}{mC_1 v_j(s) r(\tau_j(s))} w^2(s)] ds$$

Then

$$\Phi(t_2) + \liminf_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^t [\frac{(m-1)H(t,s)\tau_j'(s)}{mC_1 v_j(s) r(\tau_j(s))} w^2(s)] ds \le w(t_2)$$

Consequently,

$$\Phi(t_2) \le w(t_2) \tag{2.15}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^t [\frac{H(t,s)\tau_j'(s)}{v_j(s) r(\tau_j(s))} w^2(s)] ds \le \frac{mC_1}{m-1}[w(t_2) - \Phi(t_2)] < \infty. \tag{2.16}$$

Now, assume that $\int_{t_2}^\infty [\frac{\tau_j'(s) w^2(s)}{v_j(s) r(\tau_j(s))}] ds = \infty$ for $j = 1,2,\ldots,n.$ (2.17)

Then from (2.17) it follows that for any positive constant β , there exists $t_3 \ge t_2$ such that

$$\int_{t_2}^t [\frac{\tau_j'(s) w^2(s)}{v_j(s) r(\tau_j(s))}] ds \ge \frac{\beta}{\alpha} \quad \text{for } j = 1,2,\ldots,n. \tag{2.18}$$

Thus by the integration by parts we obtain

$$\frac{1}{H(t,t_2)}\int_{t_2}^{t}[H(t,s)\frac{\tau_j'(s)w^2(s)}{v_j(s)r(\tau_j(s))}]ds=\frac{1}{H(t,t_2)}\int_{t_2}^{t}H(t,s)\,d\,[\int_{t_2}^{s}\frac{\tau_j'(l)w^2(l)}{v_j(l)r(\tau_j(l))}dl]$$

$$=\frac{1}{H(t,t_2)}\int_{t_2}^{t}[\int_{t_2}^{s}\frac{\tau_j'(l)w^2(l)}{v_j(l)r(\tau_j(l))}dl][-\frac{\partial H(t,s)}{\partial s}]ds$$

$$\geq\frac{1}{H(t,t_2)}\int_{t_3}^{t}[\int_{t_2}^{s}\frac{\tau_j'(l)w^2(l)}{v_j(l)r(\tau_j(l))}dl][-\frac{\partial H(t,s)}{\partial s}]ds$$

$$\geq\frac{\beta}{\alpha}\frac{1}{H(t,t_2)}\int_{t_3}^{t}[-\frac{\partial H(t,s)}{\partial s}]ds$$

$$=\frac{\beta}{\alpha}\frac{H(t,t_3)}{H(t,t_2)}\geq\frac{\beta}{\alpha}\frac{H(t,t_3)}{H(t,t_0)} \qquad (2.19)$$

Since from Eq. (2.1), it is clear that there exists some $\alpha>0$ such that

$$\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}.>\alpha>0$$

Then (2.19) becomes

$$\frac{1}{H(t,t_2)}\int_{t_2}^{t}[H(t,s)\frac{\tau_j'(s)w^2(s)}{v_j(s)r(\tau_j(s))}]ds\geq\beta \qquad (2.20)$$

But since β is an arbitrary positive constant, then

$$\liminf_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^{t}[\frac{H(t,s)\tau_j'(s)}{v_j(s)r(\tau_j(s))}w^2(s)]ds=\infty.$$

This contradicts (2.16).

Now since from (2.17) we have

$$\int_{t_2}^{\infty}[\frac{\tau_j'(s)w^2(s)}{v_j(s)r(\tau_j(s))}]ds=\infty$$

While from (2.15) we get

$$\int_{t_2}^{\infty}\frac{\tau_j'(s)\Phi_+^2(s)}{v_j(s)r(\tau_j(s))}ds\leq\int_{t_2}^{\infty}\frac{\tau_j'(s)w^2(s)}{v_j(s)r(\tau_j(s))}ds<\infty$$

THis contradicts (2.14). So Eq. (1.1) is oscillatory.

Remark 2.1. Theorem 2.2 extends Theorem 5 in [7].

Example 2.1. Consider the neutral differential equation

$$[(2-\sin x)(x(t)+e^{-t}\ x(\sigma(t)))']'+\sum_{j=1}^{2}q_j(t)f_j(x(t),x(\tau_j(t)))=0,\qquad t\geq 0. \qquad (2.21)$$

Here $r(t)=1,\ p(t)=e^{-t},\ g(t)=1,\ \tau_1(t)=t/2,\ \tau_2(t)=t/4,\ q_1(t)=e^{t/2},\ q_2(t)=e^{t/4}$

Choosing $\mu_1=\mu_2=1$, then it is clear that

$$C=1,\ C_1=3,\ v_1(t)=\exp[-2\int_{t_0}^{t}\frac{\tau_1'(s)r(s)g(s)}{C\,r(\tau_1(s))}\,ds]=e^{-t},\quad v_2(t)=e^{-t/2},$$

$$\varphi_1(t)=v_1(t)[\sum_{j=1}^{2}\mu_j\,q_j(t)(1-p(\tau_j(t)))+\frac{\tau_1'(t)r^2(t)g^2(t)}{C_1r(\tau_1(t))}-[r(t)g(t)]']$$

$$=e^{-t/2}+e^{-3t/4}-\frac{11}{6}e^{-t}$$

$$\varphi_2(t)=1-\frac{23}{12}e^{-t/2}+e^{-t/4}$$

Let $H(t,s)=(t-s)^2$, then $h(t,s)=2$. Thus, from (2.3)

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t}[H(t,s)\varphi_1(s)-\frac{mC_1v_1(s)r(\tau_1(s))}{4\tau_1'(s)}h^2(t,s)]\,ds$$

$$=\frac{1}{t}[\frac{9}{6}t^2-\frac{138}{54}t-16e^{-t/2}-\frac{14}{6}e^{-t}-\frac{128}{27}e^{-3t/4}+\frac{922}{54}]$$

and

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t}[H(t,s)\varphi_2(s)-\frac{mC_1v_2(s)r(\tau_2(s))}{4\tau_2'(s)}h^2(t,s)]\,ds$$

$$=\frac{1}{t}[\frac{1}{3}t^3+\frac{1}{6}t^2-\frac{100}{6}t+\frac{800}{12}e^{-t/2}+16t\,e^{-t/4}-64e^{-t/4}+\frac{848}{12}]$$

Hence

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}[H(t,s)\varphi_j(s)-\frac{mC_1v_j(s)r(\tau_j(s))}{4\tau_j'(s)}h^2(t,s)]\,ds=\infty,\quad j=1,2.$$

Therefore, Eq. (2.12) is oscillatory by Theorem 2.1.

3. Oscillation criteria of Eq. (1.2) in the case $0\le p_i(t)\le 1$.

In this section we are concerned with oscillatory behavior of solutions of Eq. (1.2) Assume that

(I_1) $p_i(t)\in C([t_0,\infty),R)$,and $-\mu\le p_i(t)\le 1$, for $\mu\in(0,1)$ and $i=1,2,\dots,n.$

(I_2) $\sigma_j(t)\in C([t_0,\infty),R)$, $\sigma_j(t)\le t$, $\sigma_j'(t)>0$, $\lim_{t\to\infty}\sigma_j(t)=\infty$ for $j=1,2,\dots,m.$

(I_3) $f_j(t,x)\in C([t_0,\infty)\times R,R)$, such that

We assume further that there exist $q_j(t)\in C([t_0,\infty),(0,\infty))$ such that

$$Q(t)=\int_t^{\infty}\sum_{j=1}^{m}q_j(s)[1-\sum_{i=1}^{n}p_i(\sigma_j(s))]^{\alpha}ds\ ,\quad\text{and}\quad R(t)=\sum_{j=1}^{m}\frac{\alpha\ \sigma_j'(t)}{r^{1/\alpha}(\sigma_j(t))}$$

Theorem 3.1. Suppose that Eq. (1.2) is nonoscillatory, then there exists a positive function $u(t)$ on $[t_1,\infty)$ such that

$$Q(t) < \infty, \int_t^{\infty} R(s)\, u^{\frac{\alpha+1}{\alpha}}(s)\, ds < \infty,$$

$$u(t) \ge Q(t) + \int_t^{\infty} R(s)\, u^{\frac{\alpha+1}{\alpha}}(s)\, ds\ , \tag{3.1}$$

and $$\limsup_{t\to\infty} u(t)\left[\int_{t_0}^{\sigma_j(t)} r^{-1/\alpha}(s)\, ds\right]^{\alpha} \le 1 \tag{3.2}$$

Proof. Let $x(t) > 0$ be a nonoscillatory solution of Eq. (1.2), $x(\tau_i(t)) > 0$ and $x(\sigma_j(t)) > 0$ for $t \ge t_1 \ge t_0$. Then $z(t) > 0$.

Since from Eq. (1.1) we have

$$(r(t)|z'(t)|^{\alpha-1} z'(t))' = -\sum_{i=1}^{m} f_j(t, x(\sigma_j(t))) \le -\sum_{i=1}^{m} q_j(t)\, x^{\alpha}(\sigma_j(t)) \le 0. \tag{3.3}$$

Thus $r(t)|z'(t)|^{\alpha-1} z'(t)$ is nonincreasing function. We have two cases for $z'(t)$

(i) $z'(t) < 0$ or (ii) $z'(t) > 0$

(i) Suppose that $z'(t) < 0$, then there exists $t_2 \ge t_1 \ge t_0$ such that $z'(t_2) < 0$. But since

$r(t)|z'(t)|^{\alpha-1} z'(t)$ is nonincreasing, then

$r(t)|z'(t)|^{\alpha-1} z'(t) \le r(t_2)|z'(t_2)|^{\alpha-1} z'(t_2)$ for $t \ge t_2$

Thus

$$z(t) \le z(t_2) - r^{1/\alpha}(t_2)|z'(t_2)| \int_{t_2}^{t} r^{-1/\alpha}(s)\, ds$$

Hence, $\lim_{t\to\infty} z(t) = -\infty$ which contradicts the fact that $z(t) > 0$.

(ii) If that $z'(t) > 0$ for $t \ge t_2 \ge t_1 \ge t_0$, then since $\sigma_j(t) \le t$, it follows that

$r(t)(z'(t))^{\alpha} \le r(\sigma_j(t))[z'(\sigma_j(t))]^{\alpha}$

Thus

$$\frac{z'(\sigma_j(t))}{z'(t)} \ge \left[\frac{r(t)}{r(\sigma_j(t))}\right]^{1/\alpha} \tag{3.4}$$

Now since from (3.3), we have

$$[r(t)(z'(t))^{\alpha}]' + \sum_{i=1}^{m} q_j(t)\, x^{\alpha}(\sigma_j(t)) \le 0 \tag{3.5}$$

Then by (I_3), we get

$$x(\sigma_j(t)) \ge z(\sigma_j(t))[1 - \sum_{i=1}^{n} p_i(\sigma_j(t))] \tag{3.6}$$

Thus by (3.5) and (3.6), it follows that

$$[r(t)(z'(t))^{\alpha}]' + \sum_{i=1}^{m} q_j(t)\, z^{\alpha}(\sigma_j(t))[1-\sum_{i=1}^{n} p_i(\sigma_j(t))]^{\alpha} \le 0$$

Thus

$$\frac{[r(t)(z'(t))^{\alpha}]'}{z^{\alpha}(\sigma_j(t))} \le -\sum_{j=1}^{m} q_j(t)[1-\sum_{i=1}^{n} p_i(\sigma_j(t))]^{\alpha} \tag{3.7}$$

Now define

$$u(t) = r(t)[\frac{z'(t)}{z(\sigma_j(t))}]^{\alpha} \quad \text{for} \quad t \ge t_2,$$

Then $u(t) > 0$. By differentiation using (3.4) and (3.7), we get

$$u'(t) \le -\sum_{j=1}^{m} q_j(t)[1-\sum_{i=1}^{n} p_i(\sigma_j(t))]^{\alpha} - \alpha \frac{r(t)(z'(t))^{\alpha}\sigma_j'(t)}{z^{\alpha+1}(\sigma_j(t))} z'(\sigma_j(t))[\frac{r(t)}{r(\sigma_j(t))}]^{1/\alpha}$$

i.e.

$$u'(t) + \sum_{j=1}^{m} q_j(t)[1-\sum_{i=1}^{n} p_i(\sigma_j(t))]^{\alpha} + R(t)\, u^{\frac{\alpha+1}{\alpha}}(t) \le 0$$

Integrating the above inequality, it follows that

$$u(t_3) - u(t) + \int_t^{t_3} \sum_{j=1}^{m} q_j(s)[1-\sum_{i=1}^{n} p_i(\sigma_j(s))]^{\alpha}\, ds + \int_t^{t_3} R(s)\, u^{\frac{\alpha+1}{\alpha}}(s)\, ds \le 0 \tag{3.8}$$

But since $Q(t) < \infty$, then

$u(t_3) \le u(t) - Q(t) \to -\infty$ as $t \to \infty$

This contradicts the fact that $u(t) > 0$.

Similarly we can show that

$$\int_t^{\infty} R(s)\, u^{\frac{\alpha+1}{\alpha}}(s)\, ds < \infty \quad \text{for} \quad t \ge t_2 .$$

Letting $t_3 \to \infty$ in (3.8), we get (3.1).

Now to prove (3.2), we note that

$$\frac{1}{u(t)} = \frac{1}{r(t)}[\frac{z(\sigma_j(t))}{z'(t)}]^{\alpha}$$

i.e. $$\frac{1}{u(t)} = \frac{1}{r(t)}\left[\frac{z(t_2) + \int_{t_2}^{\sigma_j(t)} r^{-1/\alpha}(s) r^{1/\alpha}(s) z'(s)\, ds}{z'(t)}\right]^{\alpha}$$

$$\frac{1}{u(t)} \ge [\int_{t_2}^{\sigma_j(t)} r^{-1/\alpha}(s) ds]^{\alpha}$$

Hence

$$u(t)[\int_{t_2}^{\sigma_j(t)} r^{-1/\alpha}(s)\,ds]^{\alpha} \le 1$$

This implies Eq. (3.2).

Remark 3.1. Theorem 3.1 extends Theorem 2.1 of Dong [1].

Let $\{y_n(t)\}_{n=0}^{\infty}$ be a sequence of continuous functions on $[T,\infty)$ defined as

$y_0(t) = Q(t)$ for $t \ge t_0$

and

$$y_n(t) = \int_t^{\infty} R(s)[y_{n-1}(s)]^{\frac{\alpha+1}{\alpha}}\,ds + Q(t) \quad \text{for } n = 1,2,\ldots \text{ and } t \ge t_0. \tag{3.9}$$

Lemma 3.1. If Eq. (1.2) is nonoscillatory, then $y_n(t) \le u(t)$ where $u(t)$ be as defined in Theorem 3.1 and there exists a positive function $y(t)$ on $[T,\infty)$ such that $\lim_{t\to\infty} y_n(t) = y(t)$ for $t \ge T \ge t_0$. In addition,

$$y(t) = \int_t^{\infty} R(s)[y(s)]^{\frac{\alpha+1}{\alpha}}\,ds + Q(t) \quad \text{for } t \ge T \tag{3.10}$$

Proof. The proof is similar to the proof of Lemma 2.4 in [1].

Corollary 3.1. Let $y_n(t)$ be defined as in Eq. (3.9). If there exists some $y_n(t)$ such that

$$\limsup_{t\to\infty} y_n(t)[\int_{t_2}^{\sigma_j(t)} r^{-1/\alpha}(s)\,ds]^{\alpha} > 1 \quad \text{for } n = 0,1,2,\ldots \quad \text{and} \quad j = 1,2,\ldots,m, \tag{3.11}$$

then Eq. (1.2) is oscillatory.

Proof. Suppose that Eq. (1.2) is nonoscillatory, then Eq. (3.2) holds by Theorem 3.1. Moreover by Lemma 3.1 it follows that $y_n(t) \le u(t)$. Thus from Eq. (3.2) we get

$\limsup_{t\to\infty} y_n(t)[\int_{t_2}^{\sigma_j(t)} r^{-1/\alpha}(s)\,ds]^{\alpha} \le 1$ which contradicts Eq. (3.11). Hence Eq. (1.2) is oscillatory.

Putting $n = 1$ in Corollary 3.1, we have the following corollary

Corollary 3.2. Assume that

$$\limsup_{t\to\infty} y_n(t)[\int_{t_2}^{\sigma_j(t)} r^{-1/\alpha}(s)\,ds]^{\alpha}[\int_t^{\infty} R(s)Q^{\frac{\alpha+1}{\alpha}}(s)\,ds] + Q(t) > 1,$$

then Eq. (1.2) is oscillatory.

Remark 3.2. Corollaries 3.1 and 3.2 extend and improve Corollaries 2.5 and 2.6 of [1].

Theorem 3.2. Let $y_n(t)$ be defined as in Eq. (3.9). If there exists some $y_n(t)$ such that we have

$$\int_t^{\infty} \sum_{j=1}^{m} q_j(t)[1 - \sum_{i=1}^{n} p_i(\sigma_j(t))]^{\alpha} \exp[\int_t^{\infty} R(s)\,y_n^{1/\alpha}(s)\,ds]\,dt = \infty$$

(3.12)

or

$$\int_t^\infty R(t)\, y_n^{1/\alpha}(t)\, Q(t) \exp[\int_t^\infty R(s)\, y_n^{1/\alpha}(s)\, ds]\; dt = \infty \tag{3.13}$$

then Eq. (1.2) is oscillatory.

Proof. Suppose that Eq. (1.2) is nonoscillatory, then from Eq. (3.10), we have

$$y'(t) = -R(t) y^{\frac{\alpha+1}{\alpha}}(t) - \sum_{j=1}^{m} q_j(t)[1 - \sum_{i=1}^{n} p_i(\sigma_j(t))]^\alpha \quad \text{for } t \geq T.$$

But since $y_n(t) \leq y(t)$, then

$$y'(t) \leq -R(t) y_n^{1/\alpha}(t)\, y(t) - \sum_{j=1}^{m} q_j(t)[1 - \sum_{i=1}^{n} p_i(\sigma_j(t))]^\alpha$$

i.e.

$$y(t) \leq \exp[-\int_T^t R(s)\, y_n^{1/\alpha}(s)\, ds] \left(y(T) - \int_T^t [\sum_{j=1}^{m} q_j(s)[1 - \sum_{i=1}^{n} p_i(\sigma_j(s))]^\alpha \exp[-\int_T^t R(u)\, y_n^{1/\alpha}(u)\, du]]\; ds \right)$$

Then

$$\infty > y(T) \geq \int_T^t \sum_{j=1}^{m} q_j(s)[1 - \sum_{i=1}^{n} p_i(\sigma_j(s))]^\alpha \exp[\int_T^s R(u) y_n^{\frac{1}{\alpha}}(u) du] ds,$$

which contradicts Eq. (3.12)

Now, define

$$v(t) = \int_t^\infty R(s) y^{\frac{\alpha+1}{\alpha}}(s) ds \;\; \text{for } t \geq T.$$

Thus

$$v'(t) = -R(t) y_n^{\frac{\alpha+1}{\alpha}}(t) \geq -R(t) y_n^{\frac{1}{\alpha}}(t) y(t) = -y_n^{\frac{1}{\alpha}}(t) R(t)[v(t) + Q(t)]$$

Therefore

$$v(t) \leq \exp[\int_{t_0}^t R(s) y_n^{\frac{1}{\alpha}}(s) ds[v(t) - \int_{t_0}^\infty R(t) y_n^{\frac{1}{\alpha}}(t) Q(t) dt]$$

Hence

$$\infty > v(t) \geq \int_{t_0}^\infty R(t) y_n^{\frac{1}{\alpha}}(t) Q(t) \exp[\int_{t_0}^t R(s) y_n^{\frac{1}{\alpha}}(s)\; ds]\; dt$$

This contradicts Eq. (3.13) and therefore Eq. (1.2) is oscillatory.

Remark 3.3. Theorem 3.2 extends Corollary 2.7 of [2].

Note that if $n = 0$ in Theorem 3.2, we get the following result which improves corollary 2.8 of [1].

Corollary 3.3. Let $y_n(t)$ be defined in Eq. (3.9) and either

$$\int_{t_0}^\infty \sum_{j=1}^{m} q_j(t)[1 - \sum_{i=1}^{n} p_i(\sigma_j(t))]^\alpha \exp[\int_{t_0}^t R(s) Q^{\frac{1}{\alpha}}(s) ds] dt = \infty$$

or

$$\int_{t_0}^{\infty} R(t) Q^{\frac{\alpha+1}{\alpha}}(t) \exp[\int_{t_0}^{t} R(s) Q^{\frac{1}{\alpha}}(s) ds] dt = \infty .$$

Then Eq. (1.2) is oscillatory.

Example 3.1. Consider the D.E.

$$[(t^{-\alpha} |z'(t)|^{\alpha-1} z'(t)]' + \sum_{i=1}^{2} f_j(t, x(\sigma_j(t))) = 0, \quad \mathrm{t} \geq 0 . \tag{3.14}$$

Here, $r(t) = t^{-\alpha}$, $q_1(t) = \frac{1}{t^2}$, $q_2(t) = \frac{2}{t^2}$, $\sigma_1(t) = \frac{t}{2}$, $\sigma_2(t) = \frac{t}{3}$, $p_1(t) = \frac{1}{4}$, $p_2(t) = \frac{2}{4}$.

And we will let $\alpha = 1$.

It is clear that

$$Q(t) = \int_{t}^{\infty} \sum_{j=1}^{2} q_j(s)\ ds = \int_{t}^{\infty} [q_1(s) + q_2(s)]\ ds$$

$$= \int_{t}^{\infty} [\frac{1}{s^2} + \frac{2}{s^2}]\ ds = \frac{3}{t}$$

and

$$R(t) = \sum_{j=1}^{2} \frac{\alpha \sigma_j'(t)}{r^{1/\alpha}(\sigma_j(t))} = \alpha\ [\frac{\sigma_1'(t)}{r^{1/\alpha}(\sigma_1(t))} + \frac{\sigma_2'(t)}{r^{1/\alpha}(\sigma_2(t))}]$$

$$= \frac{1/2}{(t/2)^{-1}} + \frac{1/3}{(t/3)^{-1}} = \frac{13}{36} t$$

So

$$I = \int_{t_0}^{t} R(s) Q^{\frac{1}{\alpha}}(s)\ ds = \int_{0}^{t} \frac{13 s}{36} \frac{3}{s}\ ds = \frac{13}{12} t ,$$

and

$$\sum_{j=1}^{2} q_j(t)[1 - \sum_{i=1}^{2} p_i(\sigma_j(t))]^{\alpha}$$

$$= q_1(t)[1 - [p_1(\sigma_1(t)) + p_2(\sigma_1(t))]] + q_2(t)[1 - [p_1(\sigma_2(t)) + p_2(\sigma_2(t))]]$$

$$= \frac{1}{t^2}[1 - \frac{3}{4}] + \frac{2}{t^2}[1 - \frac{3}{4}] = \frac{3}{4t^2}$$

Thus

$$\int_{t_0}^{\infty} \sum_{j=1}^{2} q_j(t)[1 - \sum_{i=1}^{2} p_i(\sigma_j(t))]^{\alpha} \exp[\int_{t_0}^{t} R(s) Q^{\frac{1}{\alpha}}(s)\ ds]\ dt$$

$$\int_{0}^{\infty} \frac{3}{4t^2} e^{I}\ dt = \int_{0}^{\infty} \frac{3}{4t^2} e^{\frac{13}{12}t} dt = \int_{0}^{\infty} \frac{3}{4t^2}[1 + \frac{13}{12} t + (\frac{13}{12})^2 \frac{t^2}{2!} + (\frac{13}{12})^3 \frac{t^3}{3!} + (\frac{13}{12})^4 \frac{t^4}{4!} + \ldots]\ dt = \infty$$

Hence, by Corollary 3.3 Eq. (2.14) is oscillatory.

4. Oscillation criteria of Eq. (1.2) for $-\mu \leq p_i(t) \leq 0$

Define $Q(t)=\int_t^{\infty}\sum_{j=1}^{m} q_j(s)ds$. We give The following theorem which partially generalizes Theorem 3.1 of [1].

Theorem 4.1. Assume that every solution of Eq. (1.2) is neither oscillatory nor tends to zero, then there exists a positive function $u(t)$ on $[T,\infty)$ such that

$$Q(t) < \infty, \int_t^{\infty} R(s)\, u^{\frac{\alpha+1}{\alpha}}(s)\, ds < \infty,$$

$$u(t) \geq Q(t)+\int_t^{\infty} R(s)\, u^{\frac{\alpha+1}{\alpha}}(s)\, ds \text{ for } t \geq t_0, \tag{4.1}$$

$$\text{and } \limsup_{t\to\infty} u(t)\left[\int_{t_0}^{\sigma_j(t)} r^{-1/\alpha}(s)\, ds\right]^{\alpha} \leq 1 \tag{4.2}$$

Proof. Let $x(t)$ be a solution of Eq. (1.2) which is neither oscillatory nor tends to zero. Such that $x(t) > 0,\ x(\tau_i(t)) > 0$ and $x(\sigma_j(t)) > 0$ for $t \geq t_1 \geq t_0$. Then $z(t) > 0$.

Now from Eq. (1.2), we have

$$(r(t)|z'(t)|^{\alpha-1} z'(t))' = -\sum_{i=1}^{m} f_j(t, x(\sigma_j(t))) \leq -\sum_{i=1}^{m} q_j(t)\, x^{\alpha}(\sigma_j(t)) \leq 0.$$

Thus $r(t)|z'(t)|^{\alpha-1} z'(t)$ is nonincreasing function. Then $z(t)$ and $z'(t)$ are eventually of one signe.

Now we have one of the two possible cases

(i) $z(t) > 0$ (ii) $z(t) < 0$

(i) Assume that $z(t) > 0$, then the proof will be as the proof of Theorem 3.1 until we reach Eq. (3.5) in the form

$$[r(t)(z'(t))^{\alpha}]' + \sum_{i=1}^{m} q_j(t)\, z^{\alpha}(\sigma_j(t)) \leq 0 \tag{4.3}$$

This completes the proof as in Theorem 3.1.

(ii) Assume that $z(t) < 0$ eventually for $t \geq t_2 \geq t_1 \geq t_0$, then we have two cases,

(a) $x(t)$ is unbounded (b) $x(t)$ is bounded

(a) Suppose that $x(t)$ is unbounded, then

$$x(t) = z(t) - \sum_{i=1}^{n} p_i(t) x(\tau_i(t)) < -\sum_{i=1}^{n} p_i(t) x(\tau_i(t)) < \sum_{i=1}^{n} x(\tau_i(t)) \tag{4.4}$$

Further, since $x(t)$ is unbounded, then we can choose a sequence $\{T_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} T_n = \infty$, from which $\lim_{n\to\infty} x(T_n) = \infty$ and $\max_{T_1 \leq t \leq T_n} x(t) = x(T_n)$

By choosing N so large such that $\tau_i(T_N) > T_1$ for $T_N > t_2$. Thus $\max\limits_{\tau_i(T_N)\le t\le T_N} x(t) = x(T_N)$ which contradicts Eq. (4.4).

(b) Suppose that $x(t)$ is bounded, then it follows that $x(t) \to 0$ as $t \to \infty$.

Since $\limsup\limits_{t\to\infty} z(t) \le 0$, then

$$\limsup_{t\to\infty}[x(t) + \sum_{i=1}^{n} p_i(t)\, x(\tau_i(t))] \le 0$$

$$\limsup_{t\to\infty} x(t) + \liminf_{t\to\infty} \sum_{i=1}^{n} p_i(t)\, x(\tau_i(t)) \le 0$$

$$(1-\mu)\limsup_{t\to\infty} x(t) \le 0$$

This shows that $x(t) \to 0$ as $t \to \infty$, and so the proof is completed.

Let $\{y_n(t)\}_{n=0}^{\infty}$ be a sequence of continuous functions on $[T,\infty)$ defined as follows

$$y_0(t) = Q(t) \quad \text{for} \quad t \ge t_0$$

and

$$y_n(t) = \int_t^{\infty} R(s)[y_{n-1}(s)]^{\frac{\alpha+1}{\alpha}}\, ds \ + Q(t), \quad \text{for} \quad n = 1,2,... \text{ and } t \ge t_0.$$

Corollary 4.1. Let $y_n(t)$ be defined as before. If there exists some $y_n(t)$ such that

$$\limsup_{t\to\infty} y_n(t)[\int_{t_0}^{\sigma_j(t)} r^{-1/\alpha}(s)\, ds]^{\alpha} > 1 \ , \ \text{for } n = 0,1,2,... \quad \text{and} \quad j = 1,2,...,m,$$

then every solution of Eq. (1.2) is either oscillatory or tends to zero.

Proof. The proof is similar to the proof of Corollary 3.1.

Corollary 4.2. Assume that

$$\limsup_{t\to\infty}[\int_{t_0}^{\sigma_j(t)} r^{-1/\alpha}(s)\, ds]^{\alpha}[\int_t^{\infty} R(s) Q^{\frac{\alpha+1}{\alpha}}(s)\, ds \ + Q(t)] > 1 \quad \text{for } n = 0,1,2,... \quad \text{and} \quad j = 1,2,...,m, \text{ then}$$

every solution of Eq. (1.2) is either oscillatory or tends to zero.

Remark 4.1. Corollaries 4.1 and 4.2 extend and improve Corollaries 3.4 and 3.5 of [2].

Corollary 4.3. Let $y_n(t)$ be defined as before. If there exists some $y_n(t)$ such that either

$$\int_{t_0}^{\infty} \sum_{j=1}^{m} q_j(t) \exp[\int_{t_0}^{t} R(s)\, y_n^{1/\alpha}(s)\, ds]\, dt = \infty$$

or

$$\int_{t_0}^{\infty} R(t)\, y_n^{1/\alpha}(t)\, Q(t) \exp[\int_{t_0}^{t} R(s)\, y_n^{1/\alpha}(s)\, ds]\, dt = \infty,$$

Proof. The proof is similar to that of corollary 3.3

Remark 4.2. Corollary 4.3 improves and extends corollary 3.6 of [1]

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