DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 4, 2008
Electronic Journal,
reg. N P2375 at 07.03.97
ISSN 1817-2172

http://www.neva.ru/journal
http://www.math.spbu.ru/diffjournal/
e-mail: jodiff@mail.ru

Weak maximal and minimal solutions for Hammerstein and Urysohn integral equations in reflexive Banach spaces

El-Sayed A.M.A & Hashem H.H.G
E-mail : amasayed@hotmail.com & hendhghashem@yahoo.com
Faculty of Science, Alexandria University, Alexandria, Egypt

Abstract: We present existence theorem of weak solutions for Hammerstein and Urysohn integral equations. Also, we shall define the weak maximal and weak minimal solutions for these integral equations and finally prove the existence of the weak maximal and minimal solutions.

Keywords: Weak solution; Urysohn integral equation ; Hammerstein integral equation; Weak maximal and weak minimal solutions.

1 Introduction and Preliminaries

Let $L_1(I)$ be the space of Lebesgue integrable functions defined on the interval $I = [0, 1]$. Let E be a reflexive Banach space with the norm $||.||$ and dual E^* and denote by $C[I, E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm $||.||_0$. The existence of weak solutions of the integral equation

$$x(t) \ = \ x_0 \ + \ \int_0^t \ f(s, x(s)) \ ds, \quad t \in I$$

was proved by O'Regan [9] where $f : I \times E \to E, \quad x_0 \in E.$.
The existence of weak solutions to the Hammerstein integral equation

$$x(t) \ = \ h(t) \ + \ \int_0^1 \ k(t, s) \ f(s, x(s)) \ ds, \quad t \in [0, 1]$$

was proved by O'Regan (see[8]) where x takes values in reflexive Banach spaces and f is weakly-weakly continuous.
In this paper, we shall study the existence of a weak solution to the Hammerstein integral equation

$$x(t) \;=\; g(t) \;+\; \int_0^1 k(t,s)\; f(s,x(s))\; ds, \qquad t \in [0,1] \tag{1}$$

and Urysohn integral equation

$$x(t) \;=\; g(t) \;+\; \int_0^1 u(t,s,x(s))\; ds, \qquad t \in [0,1] \tag{2}$$

Where x takes values in reflexive Banach spaces and $f : I \times E \to E$ is weakly measurable in t and weakly sequentially continuous in x .
The function $u : I \times I \times E \to E$ is weakly measurable in s and weakly sequentially continuous in x. Also, the existence of the weak maximal and minimal solutions will be proved.

Now, we shall present some auxiliary results that will be need in this work. Let E be a Banach space (need not be reflexive) and let $x : I \to E$, then

(1) $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in I$ if for every $\phi \in E^*$, $\phi(x(.))$ is continuous (measurable) at t_0.

(2) A function $h : E \to E$ is said to be sequentially continuous if h maps weakly convergent sequences in E to weakly convergent sequences in E.

If x is weakly continuous on I, then x is strongly measurable and hence weakly measurable (see[3] and [2]). Note that in reflexive Banach space weakly measurable functions are Pettis integrable if and only if $\phi(x(.))$ is Lebesgue integrable on I for every $\phi \in E^*$ (see[3] pp. 78). Now we state a fixed point theorem and some propositions which will be used in the sequel (see[9]).

Theorem 1.1. Let E be a Banach space and let Q be a nonempty, bounded, closed and convex subset of the space E and let $T : Q \to Q$ be a weakly sequentially continuous and assume that $TQ(t)$ is relatively weakly compact in E for each $t \in [0,1]$. Then, T has a fixed point in the set Q.
proposition 1.1. A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.
proposition 1.2. Let E be a normed space with $y \neq 0$. Then there exist a $\phi \in E^*$ with $||\phi|| = 1$ and $||y|| = \phi(y)$.

2 Hammerstein integral equation

Let E be a reflexive Banach space and $D \subset E$. Consider the following assumptions

(1) $g \in C[I, E]$;

(2) $f : I \times D \to E$ satisfies the following

(i) For each $t \in I$, $f_t = f(t, .)$ is weakly sequentially continuous;

(ii) For each $x \in D$, $f(., x(.))$ is weakly measurable on I ;

(iii) The weak closure of the range of $f(I \times D)$ is weakly compact in E (or equivalently: there exists an M such that $||f(t, x)|| \leq M$ $(t, x) \in I \times D$;)

(3) $k : I \times I \to R_+$ is integrable in s and continuous in t , the operator

$$Ky(t) = \int_0^1 k(t, s)\, y(s)\, ds$$

maps $L_1(I)$ into $L_1(I)$ and $\int_0^1 k(t, s)\, ds < M_1$, $t \in I$.

Definition 2.1. By a weak solution of (1) we mean a function $x \in C[I, E]$ such that

$$\phi(x(t)) = \phi(g(t)) + \int_0^1 k(t, s)\, \phi(f(s, x(s)))\, ds, \qquad t \in [0, 1]$$

for all $\phi \in E^*$.

Theorem 2.1. Let the assumptions (1)-(3) be satisfied. Then equation (1) has at least one weak solution $x \in C[I, E]$.

Proof: Define the operator T by

$$Tx(t) = g(t) + \int_0^1 k(t, s)\, f(s, x(s))\, ds, \qquad t \in I.$$

For any $x \in C[I, E]$ and since $f(., x(.))$ is weakly measurable on I and $||f(t, x)|| \leq M$, then $\phi(f(., x(.)))$ is Lebesgue integrable on I $\forall \phi \in E^*$ and since $k(t, .)$ is Lebesgue integrable on I, then we have $\phi(k(t, .)\, f(., x(.))) = k(t, .)\, \phi(f(., x(.)))$ is Lebesgue integrable on I $\forall \phi \in E^*$, then $k(t, .)\, f(., x(.))$ is Pettis integrable on I. Thus T is well defined.
We shall prove that $T : C[I, E] \to C[I, E]$.
Let $t_1,\ t_2 \in I$ and $t_1 < t_2$ (without loss of generality assume that $Tx(t_2) - Tx(t_1) \neq 0$)

$$Tx(t_2) - Tx(t_1) = g(t_2) - g(t_1) + \int_0^1 k(t_2, s)\, f(s, x(s))\, ds - \int_0^1 k(t_1, s)\, f(s, x(s))\, ds$$

$$= \ g(t_2) \ - \ g(t_1) \ + \ \int_0^1 \ [k(t_2, s) \ - \ k(t_1, s) \ ] f(s, x(s)) \ ds$$

Therefore as a consequence of proposition 1.2, we obtain

$$||Tx(t_2) - Tx(t_1)|| \ = \ \phi(Tx(t_2) - Tx(t_1))$$

$$= \ \phi(g(t_2) \ - \ g(t_1)) \ + \ \int_0^1 \ [k(t_2, s) \ - \ k(t_1, s) \ ]\phi(f(s, x(s))) \ ds$$

$$= \ ||g(t_2) \ - \ g(t_1)|| \ + \ \int_0^1 \ [k(t_2, s) \ - \ k(t_1, s) \ ] \ ||f(s, x(s))|| \ ds$$

$$\leq \ ||g(t_2) \ - \ g(t_1)|| \ + \ M \int_0^1 \ |k(t_2, s) \ - \ k(t_1, s) \ | \ ds.$$

let the set Q be defined as

$$Q \ = \ \{ \ x \in C[I, E] : \ ||x|| \leq \ M_2 \ \}, \qquad M_2 = ||g|| \ + \ M.M_1.$$

Let $x \in Q$, then we have

$$||Tx(t)|| \ = \ \phi(Tx(t)) \ = \ \phi(g(t)) \ + \ \int_0^1 \ k(t, s) \ \phi(f(s, x(s))) \ ds$$

$$||g|| + \int_0^1 k(t, s) \ ||f(s, x(s))|| \ ds \ \leq \ ||g|| + M \int_0^1 \ k(t, s) \ ds \ \leq \ ||g|| + M \ M_1$$

and hence $Tx \in Q$ implies $TQ \subset Q$ (i.e. $T : \ Q \to Q$)
Then Q is nonempty, uniformly bounded and strongly equi-continuous subset of $C[I, E]$. also it can be shown that Q is convex and closed.
As a consequence of Proposition 1.1, then TQ is relatively weakly compact.
It remains to prove that T is weakly sequentially continuous.
Let $\{x_n\}$ be a sequence in Q converges weakly to x $\forall t \in I$ and since $f(t, x(t))$ is weakly sequentially continuous in x, then $f(t, x_n(t))$ converges weakly to $f(t, x(t))$ thus $\phi(f(t, x_n(t)))$ converges strongly to $\phi(f(t, x(t)))$ (see assumption (iii)). Applying Lebesgue Dominated Convergence Theorem for Pettis integral, we get

$$\phi(\int_0^1 \ k(t, s) \ f(s, x_n(s)) \ ds) \ = \ \int_0^1 \ k(t, s) \ \phi(f(s, x_n(s))) \ ds$$

$$\to \int_0^1 \ k(t, s) \ \phi(f(s, x(s))) \ ds \qquad \forall \phi \in E^*, \quad t \in I.$$

Then T is weakly sequentially continuous. (i.e. $Tx_n(t) \to Tx(t)$ weakly $\forall t \in I$) Since all conditions of Theorem 1.1 are satisfied, then the operator T has at least one fixed point $x \in Q$ which competes the proof. ∎

3 Urysohn integral equation

Let E be a reflexive Banach space and $D \subset E$. Consider the following assumptions:

(1) $g \in C[I, E]$;

(2) $u : I \times I \times D \to E$ satisfies the following

(i) For each $t, s \in I \times I$, $u(t, s, .)$ is weakly sequentially continuous;

(ii) For each $x \in D$ and $t \in I$ $u(t, ., x(.))$ is weakly measurable on I ;

(iii) For each $x \in D$ and $s \in I$ $u(., s, x(s))$ is continuous on I ;

(3) $||u(t, s, x(s))|| \leq k(t, s)$, $k : I \times I \to R_+$ is integrable in s and continuous in t , the operator

$$Ky(t) = \int_0^1 k(t, s)\, y(s)\, ds$$

maps $L_1(I)$ into $L_1(I)$ and $\int_0^1 k(t, s) ds < M_1$, $t \in I$.

Definition 3.1. By a weak solution of (2) we mean a function $x \in C[I, E]$ such that

$$\phi(x(t)) = \phi(g(t)) + \int_0^1 \phi(u(t, s, x(s)))\, ds, \qquad t \in [0, 1]$$

for all $\phi \in E^*$.

Theorem 3.1. Let the assumptions (1)-(3) be satisfied. Then equation (2) has at least one weak solution $x \in C[I, E]$.

Proof: Define the operator T by

$$Tx(t) = g(t) + \int_0^1 u(t, s, x(s))\, ds, \quad t \in I.$$

For any $x \in C[I, E]$ since $u(t, ., x(.))$ is weakly measurable on I, then $\phi(u(t, ., x(.)))$ is strongly measurable on I $\forall \phi \in E^*$ and since $||u(t, s, x)|| \leq k(t, s)$, then $\phi(u(t, ., x(.)))$ is Lebesgue integrable on I $\forall \phi \in E^*$ and hence $u(t, ., x(.))$ Pettis integrable on I. Thus T is well defined.

We shall prove that $T : C[I, E] \to C[I, E]$.

Let $t_1, t_2 \in I$, $t_1 < t_2$ (without loss of generality assume that $Tx(t_2) - Tx(t_1) \neq 0$)and $x \in C[I, E]$

$$Tx(t_2) - Tx(t_1) = g(t_2) - g(t_1) + \int_0^1 u(t_2, s, x(s))\, ds - \int_0^1 u(t_1, s, x(s))\, ds$$

$$= g(t_2) - g(t_1) + \int_0^1 [u(t_2, s, x(s)) - u(t_1, s, x(s))] \, ds$$

Therefore as a consequence of proposition 1.2, we obtain

$$||Tx(t_2) - Tx(t_1)|| = \phi(Tx(t_2) - Tx(t_1))$$

$$= \phi(g(t_2) - g(t_1)) + \int_0^1 \phi[u(t_2, s, x(s)) - u(t_1, s, x(s))] \, ds$$

$$\leq ||g(t_2) - g(t_1)|| + \int_0^1 ||u(t_2, s, x(s)) - u(t_1, s, x(s))|| \, ds \qquad (3)$$

let the set Q be defined as

$$Q = \{ x \in C[I, E] : ||x|| \leq M_2 \}, \qquad M_2 = ||g|| + M.$$

Let $x \in Q$, then we have

$$||Tx(t)|| = \phi(Tx(t)) = \phi(g(t)) + \int_0^1 \phi(u(t, s, x(s))) \, ds$$

$$= \phi(g(t)) + \int_0^1 ||u(t, s, x(s))|| \, ds \leq ||g|| + \int_0^1 k(t, s) \, ds \leq ||g|| + M$$

and hence $Tx \in Q$ implies $TQ \subset Q$ (i.e. $T : Q \to Q$).
Then Q is nonempty, uniformly bounded and strongly equi-continuous subset of $C[I, E]$. also it can be shown that Q is convex and closed.
As a consequence of Proposition 1.1, then TQ is relatively weakly compact.
It remains to prove that T is weakly sequentially continuous.
Let $\{x_n\}$ be a sequence in Q converges weakly to x $\forall t \in I$ and since $u(t, s, x(s))$ is weakly sequentially continuous in x, then $u(t, s, x_n(s))$ converges weakly to $u(t, s, x(s))$ thus $\phi(u(t, s, x_n(s)))$ converges strongly to $\phi(u(t, s, x(s)))$. using assumption (3) and applying Lebesgue Dominated Convergence Theorem for Pettis integral, we get

$$\phi(\int_0^1 u(t, s, x_n(s)) \, ds) = \int_0^1 \phi(u(t, s, x_n(s))) \, ds$$

$$\to \int_0^1 \phi(u(t, s, x(s))) \, ds \qquad \forall \phi \in E^*, \quad t \in I.$$

Then T is weakly sequentially continuous. (i.e. $Tx_n(t) \to Tx(t)$ weakly $\forall t \in I$) Since all conditions of Theorem 1.1 are satisfied, then the operator T has at least one fixed point $x \in Q$ which competes the proof. ∎

4 The weak maximal and weak minimal solutions

Now we give the following definition

Definition 4.1 Let $q(t)$ be a weak solution of (1) Then $q(t)$ is said to be a weak maximal solution of (1) if every weak solution $x(t)$ of (1) satisfies the inequality

$\phi(x(t)) < \phi(q(t)), \quad \forall\phi \in E^*$. A weak minimal solution $s(t)$ can be defined by similar way by reversing the above inequality i.e. $\phi(x(t)) > \phi(s(t)), \quad \forall\phi \in E^*$.

In this section f assumed to satisfy the following assumption:

(4) for any $x,\ y \in E$ satisfying $\phi(x(t)) < \phi(y(t)), \quad \forall\phi \in E^*$ implies that $\phi(f(s,x(s))) < \phi(f(s,y(s)))$

Lemma 4.1 Let $f(t,x),\ k(t,s)$ satisfy assumptions of Theorem 2.1 and let $x(t),\ y(t) \in C[I,E]$ on I satisfying

$$\begin{aligned}
\phi(x(t)) &\leq \phi(g(t)) + \int_0^1 k(t,s)\ \phi(f(s,x(s)))\ ds \\
\phi(y(t)) &\geq \phi(g(t)) + \int_0^1 k(t,s)\ \phi(f(s,y(s)))\ ds, \quad \forall\phi \in E^*
\end{aligned}$$

where one of them is strict.

If $(f(t,x))$ satisfies assumption (4). Then

$$\phi(x(t)) < \phi(y(t)). \tag{4}$$

proof: Let the conclusion (4) be false, then there exists t_1 such that

$$\phi(x(t_1)) = \phi(y(t_1)) \qquad t_1 > 0$$

and

$$\phi(x(t)) < \phi(y(t)) \qquad 0 < t < t_1.$$

Since $(f(t,x))$ satisfies assumption (4), we get

$$\begin{aligned}
\phi(x(t_1)) &\leq \phi(g(t_1)) + \int_0^1 k(t_1,s)\ \phi(f(s,x(s)))\ ds \\
&< \phi(g(t_1)) + \int_0^1 k(t_1,s)\ \phi(f(s,y(s)))\ ds \\
&< \phi(y(t_1)).
\end{aligned}$$

Which contradicts the fact that $\phi(x(t_1)) = \phi(y(t_1))$, then

$$\phi(x(t)) < \phi(y(t)).$$

Theorem 4.1 Let the assumptions of Theorem 2.1 be satisfied. If $f(t,x)$ satisfies assumption (4), then there exist a weak maximal and weak minimal solutions of (1).

Proof Firstly we shall prove the existence of the weak maximal solution of (1). Let $\epsilon > 0$ be given. Now consider the integral equation

$$x_\epsilon(t) \;=\; g(t) \;+\; \int_0^1 \; k(t,s)\; f_\epsilon(s, x_\epsilon(s))ds, \tag{5}$$

where

$$f_\epsilon(t, x_\epsilon(t)) \;=\; f(t, x_\epsilon(t)) \;+\; \epsilon.$$

Clearly the function $f_\epsilon(t, x_\epsilon)$ satisfies the conditions (1)-(3) of Theorem 2.1 and

$$||\; f_\epsilon(t, x_\epsilon)\; || \;\leq\; M \;+\; \epsilon \;=\; M'.$$

Therefore equation (5) has a weak solution $x_\epsilon \in C[I,E]$ according to Theorem 2.1. Let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then

$$x_{\epsilon_1}(t) \;=\; g(t) \;+\; \int_0^1 \; k(t,s)\; f_{\epsilon_1}(s, x_{\epsilon_1}(s))\; ds,$$

$$x_{\epsilon_1}(t) \;=\; g(t) \;+\; \int_0^1 \; k(t,s)\; (\; f(s, x_{\epsilon_1}(s)) \;+\; \epsilon_1)\; ds,$$

implies that

$$\phi(x_{\epsilon_1}(t)) \;>\; \phi(g(t)) \;+\; \int_0^1 \; k(t,s)\; (\; \phi(f(s, x_{\epsilon_1}(s))) \;+\; \epsilon_2)\; ds, \tag{6}$$

$$\phi(x_{\epsilon_2}(t)) \;=\; \phi(g(t)) \;+\; \int_0^1 \; k(t,s)\; (\; \phi(f(s, x_{\epsilon_2}(s))) \;+\; \epsilon_2)\; ds \tag{7}$$

Using Lemma 4.1, then (6) and (7) implies

$$\phi(x_{\epsilon_2}(t)) < \phi(x_{\epsilon_1}(t)) \qquad for\; t \in \;[0,1].$$

As shown before in the proof of Theorem 2.1 the family of functions $x_\epsilon(t)$ defined by (5) is uniformly bounded and of strongly equi-continuous functions. Hence by Arzela-Ascoli Theorem , there exists a decreasing sequence ϵ_n such that $\epsilon \to 0$ as $n \to \infty$, and $lim_{n\to\infty}\, x_{\epsilon_n}(t)$ exists uniformly in $[0,1]$ and denote this limit by $q(t)$. from the weakly sequentially continuity of the function f_{ϵ_n} in the second argument and applying Lebesgue Dominated Convergence Theorem for Pettis integral, we get

$$q(t) \;=\; \lim_{n\;\to\;\infty} x_{\epsilon_n}(t) \;=\; g(t) \;+\; \int_0^1 \; k(t,s)\; f(s, q(s))\; ds$$

which proves that $q(t)$ as a solution of (1).
Finally, we shall show that $q(t)$ is the weak maximal solution of (1). To do this, let $x(t)$ be any weak solution of (1). Then

$$\begin{aligned}\phi(x_\epsilon(t)) &= \phi(g(t)) + \int_0^1 k(t,s)\,(\phi(f(s,x_\epsilon(s))) + \epsilon)\,ds \\ &> \phi(g(t)) + \int_0^1 k(t,s)\,\phi(f(s,x_\epsilon(s)))\,ds\end{aligned}$$

and

$$\phi(x(t)) = \phi(g(t)) + \int_0^1 k(t,s)\,\phi(f(s,x(s)))\,ds$$

applying Lemma 4.1, we get

$$\phi(x_\epsilon(t)) > \phi(x(t)) \qquad for\ t \in [0,1].$$

from the uniqueness of the maximal solution (see [2]), it is clear that $x_\epsilon(t)$ tends to $q(t)$ uniformly in $t \in [0,1]$ as $\epsilon \to 0$.
By similar way as done above we can prove that s(t) is the weak minimal solution of (1). ∎
The weak maximal and minimal solutions of (2) can be defined in the same fashion as done above.
Now, the function u assumed to satisfy the following assumption:

(4^*) for any $x,\ y \in E$ satisfying $\phi(x(t)) < \phi(y(t)), \quad \forall \phi \in E^*$ implies that $\phi(u(t,s,x(s))) < \phi(u(t,s,y(s)))$

Now the following lemma can be proved.
Lemma 4.2 Let $u(t,s,x)$ satisfies assumptions of Theorem 3.1 and let $x(t),\ y(t) \in C[I,E]$ on I satisfying

$$\begin{aligned}\phi(x(t)) &\le \phi(g(t)) + \int_0^1 \phi(u(t,s,x(s)))\,ds \\ \phi(y(t)) &\ge \phi(g(t)) + \int_0^1 \phi(u(t,s,y(s)))\,ds, \quad \forall \phi \in E^*\end{aligned}$$

where one of them is strict.
If $(u(t,s,x))$ satisfies assumption (4^*). Then

$$\phi(x(t)) < \phi(y(t)).$$

Theorem 4.2 Let the assumptions of Theorem 3.1 be satisfied. If $u(t,s,x)$ satisfies assumption (4^*), then there exists a weak maximal and weak minimal solutions of (2).

Proof Firstly we shall prove the existence of the weak maximal solution of (2). Let $\epsilon > 0$ be given. Now consider the integral equation

$$x_\epsilon(t) \ = \ g(t) \ + \ \int_0^1 \ u_\epsilon(t, s, x_\epsilon(s))ds, \tag{8}$$

where

$$u_\epsilon(t, s, x_\epsilon(t)) \ = \ u(t, s, x_\epsilon(t)) \ + \ \epsilon.$$

Clearly the function $u_\epsilon(t, s, x_\epsilon)$ satisfies the conditions (1)-(3) of Theorem 3.1 and

$$|| \ u_\epsilon(t, s, x_\epsilon) \ || \ \leq \ k(t, s) \ + \ \epsilon \ = \ k_1(t, s) \in L_1.$$

Therefore equation (8) has a weak solution $x_\epsilon \ \in C[I, E]$ according to Theorem 3.1. Let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then

$$x_{\epsilon_1}(t) \ = \ g(t) \ + \ \int_0^1 \ u_{\epsilon_1}(t, s, x_{\epsilon_1}(s)) \ ds,$$

$$x_{\epsilon_1}(t) \ = \ g(t) \ + \ \int_0^1 \ (u(t, s, x_{\epsilon_1}(s)) \ + \ \epsilon_1) \ ds,$$

implies that

$$\phi(x_{\epsilon_1}(t)) \ > \ \phi(g(t)) \ + \ \int_0^1 \ (\phi(u(t, s, x_{\epsilon_1}(s))) \ + \ \epsilon_2) \ ds, \tag{9}$$

$$\phi(x_{\epsilon_2}(t)) \ = \ \phi(g(t)) \ + \ \int_0^1 \ (\phi(u(t, s, x_{\epsilon_2}(s))) \ + \ \epsilon_2) \ ds \tag{10}$$

Using Lemma 4.2, then (9) and (10) implies

$$\phi(x_{\epsilon_2}(t)) < \phi(x_{\epsilon_1}(t)) \qquad for \ t \in \ [0, 1].$$

As shown before in the proof of Theorem 3.1 the family of functions $x_\epsilon(t)$ defined by (8) is uniformly bounded and of strongly equi-continuous functions. Hence by Arzela-Ascoli Theorem , there exists a decreasing sequence ϵ_n such that $\epsilon \to 0$ as $n \to \infty$, and $lim_{n \to \infty} \ x_{\epsilon_n}(t)$ exists uniformly in $[0, 1]$ and denote this limit by $q(t)$. from the weakly sequentially continuity of the function u_{ϵ_n} in x and applying Lebesgue Dominated Convergence Theorem for Pettis integral, we get

$$q(t) \ = \ \lim_{n \to \infty} x_{\epsilon_n}(t) \ = \ g(t) \ + \ \int_0^1 \ u(t, s, q(s)) \ ds$$

which proves that $q(t)$ as a solution of (2).

Finally, we shall show that $q(t)$ is the weak maximal solution of (2). To do this, let $x(t)$ be any weak solution of (2). Then

$$\begin{aligned}\phi(x_\epsilon(t)) &= \phi(g(t)) + \int_0^1 (\phi(u(t,s,x_\epsilon(s))) + \epsilon)\, ds \\ &> \phi(g(t)) + \int_0^1 \phi(u(t,s,x_\epsilon(s)))\, ds\end{aligned}$$

and

$$\phi(x(t)) = \phi(g(t)) + \int_0^1 \phi(u(t,s,x(s)))\, ds$$

applying Lemma 4.2, we get

$$\phi(x_\epsilon(t)) > \phi(x(t)) \quad for\ t \in [0,1].$$

from the uniqueness of the maximal solution (see [2]), it is clear that $x_\epsilon(t)$ tends to $q(t)$ uniformly in $t \in [0,1]$ as $\epsilon \to 0$.
By similar way as done above we can prove that s(t) is the weak minimal solution of (2). ∎

References

[1] M. Cichon, Weak solutions of ordinary differential equations in Banach spaces, *Discuss. Math. Differ.Incl. Control Optim.*, 15(1995), 5-14.

[2] N. Dunford, j.T. Schwartz, Linear operators, *Interscience, Wiley*, New York, 1958.

[3] E. Hille, R.S. Phillips, Functional Analysis and Semi-groups *Amer. Math. Soc. Colloq. Publ. 31*,Amer. Math. Soc., Providence, R. I., 1957.

[4] A.M.A. EL-Sayed, F.M. Gaafar, H.G.H. Hashem, On the maximal and minimal solutions of arbitrary-orders nonlinear functional integral and differenbtial equations, *MATH. SCi, RES.J.*, 8(11)(2004), 336-348.

[5] A.M.A. EL-Sayed, H.H.G. Hashem, Carathéodory type theorem for a nonlinear quadratic integral equation, *MATH. SCI. RES. J.* 12(4),(2008)71-95.

[6] A.M.A. EL-Sayed, H.H.G. Hashem, Integrable and continuous solutions of nonlinear quadratic integral equation , *Electronic Journal of Qualitative Theory of Differential Equations* 25(2008), 1-10.

[7] A.R. Mitchell, CH. Smith, An existence Theorem for weak solutions of differential equations in Banach spaces, *In: Nonlinear Equations in Abstract Spaces. Proc. Int.Symp., Arlington* ,(1977) 387-40314.

[8] D. O'Regan, Integral equations in reflexive Banach spaces and weak topologioes *Amer. Math. Soc.*, Vol. 124, No. 2, (1996), 607-614.

[9] D. O'Regan, Fixed point theory for weakly sequentially continuous mapping *Math. Comput. Modeling* , 27(1998), 1-14.

[10] H.A.H. Salem, A.M.A. EL-Sayed, Fractional order integral equations in reflexive Banach spaces *Math. Slovaca.*, 55 , No. 2(2005), 169-181.

[11] B. Ross, K. S. Miller, An Introduction to Fractional Calculus and Fractional Differential Equations. *John Wiley, New York* (1993).

[12] I.Podlubny . Fractional Differential equations . *San Diego-NewYork-London.*1999.

[13] S. G. Samko, A. A. Kilbas, O. Marichev, Integrals and Derivatives of Fractional Orders and Some of their Applications. *Nauka i Teknika, Minsk* (1987).