DIFFERENTIAL EQUATIONS
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Dynamical systems on manifolds

Hedgehogs hunting with Cantor, Hausdorff and Liouville

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Abstract

We show the equivalence between Borel's regular sets model and the Liouville formula for the approximation of irrationals through rational numbers. The Diophantine-Liouville dichotomy can be resolved via sequences of transfinitely many approximants.

The existence of non-linearizable hedgehogs is purely theoretical and their electronic visualization is not feasible.

We show that the non-linearizable hedgehog for polynomials $e^{2\pi i\theta}z + \mathcal{O}(z^k), k \geq 2$ is a locally connected, plane-filling Julia set with Hausdorff dimension 2, spreading radially everywhere inside a bounded non-empty region.

Contents

"Ciascuna stella ne li occhi mi piove
del lume suo e de la sua vertute;
le mie bellezze sono al mondo nove,
però che di là su mi son venute:
le quai non posson esser canosciute
se non da canoscenza d'omo in cui
Amor si metta per piacer altrui."

DANTE ALIGHIERI,
Io mi son pargoletta bella e nova,
Rime, 34, verses 11–17.

1 Introduction

1.1 An overview of indifferent dynamics

Let $f(z), z \in \mathbb{C}$ be any complex function over the Riemann sphere $\mathbb{C}_\infty$. The study of the dynamical systems governed by the iterates $f_n(z)$ $(-\infty \leq n \leq +\infty)$ is termed *Complex Dynamics.* One speaks more properly of *Holomorphic Dynamics* when $f_n(z)$ are analytic maps. A remarkable attention is payed to the invariant elements, especially to the cycles of periodic points: the solutions of the equations

$$f_n(\delta) \equiv \delta.$$

The cycle *period* is rated by n. Just for simplicity, one likes to focus on the case of $n = 1$, when the cycle includes one periodic point δ, which is termed the *fixed point.* One is also interested in studying the neighboring dynamics near cycles: this approach begins with the classification of cycles through an indicator value, the modulus of the *multiplier* M: $|M| = \prod_1^n |f'(\delta_n)|$. Cycles could be *super-attracting*, *attracting*, *indifferent* and *repelling*, depending on the

[1] *"Every star showers into these my glances\some of its light and of its lofty virtue;\your world can hardly recognize my beauty,\which down from heaven has descended on me:\at man alone is privileged to know it,\who, blessed with great discerning wisdom, harbors\Love in his heart to please another person."* [3]

modulus $|M| = 0, |M| < 1, |M| = 1, |M| > 1$, respectively. When $|M| \neq 1$, the modulus is sufficient for determining the dynamics about δ. But the case $M = 1$ might be read ambiguously, disguising behind this Euler form

$$|f'(\delta)| = |e^{2\pi i\theta}| = 1, \qquad \theta \in \mathbb{R}. \tag{1.1.1}$$

Here one drops M and focuses on the argument θ. The expression *indifferent dynamics* refers to the local behavior of $f_n(z)$ inside a sufficiently small neighborhood of δ, if (1.1.1) holds: these studies became a respectful branch in the whole theory of Holomorphic Dynamics. The role of the multiplier and of the first derivative naturally introduces the application of Taylor series to the local study of any fixed point. In the indifferent case, the linear term of the expansion for the iterates f_n at δ formally writes as

$$f'_n(\delta) = e^{2\pi i\theta n}. \tag{1.1.2}$$

But it does not necessarily follow that

$$|f'_n(\delta)| = |e^{2\pi i\theta n}|, \tag{1.1.3}$$

when the local dynamics of $f_n(z)$ are[1] isomorphic (or geometrically equivalent) to the rigid θ-rotation. The folklore and the literature in Holomorphic Dynamics divides this branch into two main subfields, characterized by the following two numerical conditions enjoyed by the argument:

$$\theta \in \mathbb{Q} \qquad \text{and} \qquad \theta \in \mathbb{R}\backslash\mathbb{Q}.$$

We speak of *rational* and *irrational dynamics* respectively. Although we will apply these standard definitions in what follows, we stress that it would be more convenient (1°) to re-frame the indifferent dynamics under the viewpoint of Lyapunov (asymptotical) stability of the iterated orbits about δ, (2°) to remove the modulus operator from (1.1.3) and (3°) to focus on (1.1.2). One can then re-edit (1.1.2) and distinguish these two situations:

$$e^{2\pi i\theta n} = 1 \quad \text{or} \quad e^{2\pi i\theta n} \neq 1, \tag{1.1.4}$$

[1]Unlike all the other case, the indifferent dynamics can stand as one evidence of the possible gap between the formal expressions and their geometrical meanings.

when neighboring orbits are locally unstable or stable, respectively. These latter conditions are not quite the same as $\theta \in \mathbb{Q}$ and $\theta \in \mathbb{R} \backslash \mathbb{Q}$. Given a generic real θ, we will see that stability holds for the union set of rationals and of a subset of irrationals exclusively, while instability holds in its complement.

Joining Lyapunov's and Taylor's view together may help to *introduce* the action performed by the higher order derivatives in the Taylor series of $f_n(z)$ for one of the cases (1.1.3) [or equivalently (1.1.4)] to hold. But it does not help to *figure out why and how* this could happen or not. One such discussion belongs to the domain of Number Theory: loosely speaking, the above action may be stopped or not, according to the numerical conditions of θ. This suggests why the argument θ is essential here. Again, higher order derivatives, belonging to the Taylor series, can be compared to 'breakers' [keepers] of the Lyapunov instability [stability]. The argument θ, if [not] endowed with a proper numerical condition, is [not] able to stop the breaking action.

Indifferent dynamics focus on how the numerical properties of θ guarantee that the linear term $e^{2\pi i\theta n}$ is the only one ruling the local dynamics. Whichever way one likes to follow (dynamical or numerical), the two-fold numerical split is just artificial and lessens the study thanks to disjoint categories. It holds according to how large one's mathematical viewpoint is. If it is the largest, distinct cases do merge into one. Otherwise the rational dynamics, relatively easier to study, differ from the irrational ones, extremely complicated and thus legitimately settling at the hard side of the Holomorphic Dynamics. Such dynamics are intimately related to analogous questions in the theory of conservative (Hamiltonian) systems and in Celestial Mechanics (KAM theory).

There is some fashinating aura around the close relationships between geometrical and numerical features, but their level of complication may have spawned an unpleasant counter–effect among the audience, in fact the irrational case did not taste as very attractive and intriguing to specialists during the last 60 years. It follows that the historical flow of these developments is quite winding. The former results consisted of existence statements and formalized during early 1900s, after Kasner's attempts [51] and Pfeiffer's remarks in U.S. [76]. Independently in France, Fatou and Julia focused on the same problem, consolidating the related mathematical groundwork as follows.

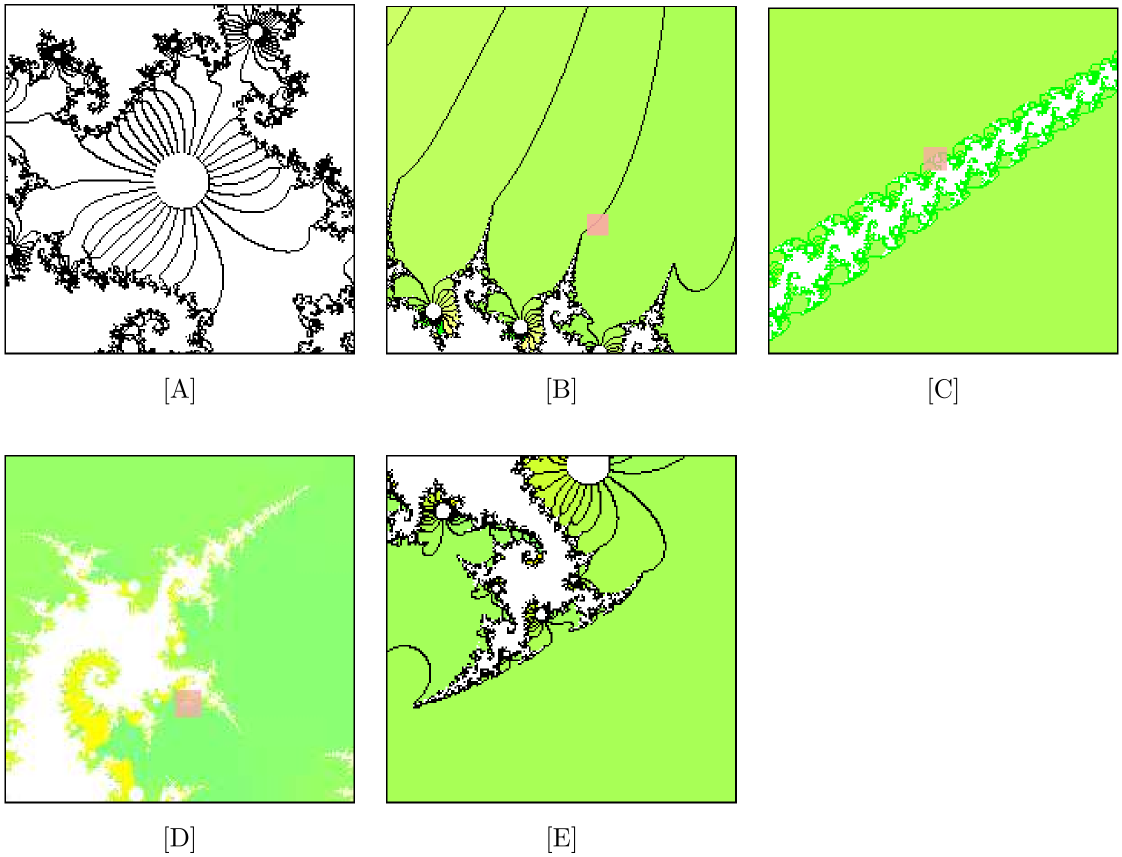


Figure 1.1.1: **Hedgehog with non-maximal Siegel compactum.** If blown-up, the 'filaments' in [A] show the wedging action of the basin to infinity into the bounded one; the central disc is the Siegel compactum of positive area. [B] Such filaments are not truly uni-dimensional: they look like lines in small figures because of their very narrow width. [C,D] If magnified (red squares), the true nature, as well as the wedging action, gets clearer. Finally [E] shows filaments again, highlighting another aspect of hedgehog behavior: the plane-filling rate or, more technically, its Hausdorff dimension.

Analogously to the (super-)attracting/repelling ($|f_n'(\delta)| \neq 1$) cases, existence theorems of local type focus on the conditions for the iterates of rational maps $f_n(z)$ to turn into the holomorphic germ

$$g_n(z) : e^{2\pi i\theta n} z + \mathcal{O}(z^k), \quad z \in \mathbb{C}, \qquad k \geq 2, n \geq 1. \quad [^2] \tag{1.1.5}$$

inside a bounded neighborhood of δ. One such transformation is termed '*linearization*' and it is formulated into the so-called *Schröder's functional equation*

[2] A robust theory has been developed for such quadratic type germs ($k = 2$); conditions may be formally different for germs of higher degree (see [99]).

model, generalized for iterates here below:

$$\phi[f_n(z)] = a_n\phi(z), \qquad n \geq 1. \quad [^3] \tag{1.1.6}$$

If this equation holds, there exists an invertible function $\phi(z)$ mapping the iterates $f_n(z)$ into the Taylor series (1.1.5). Again, if (1.1.6) holds, $f_n(z)$ are *locally linearized* into $f_n(z) \mapsto a_n z$ about δ, i.e. the Taylor expansion of $f_n(z)$ includes the linear term $a_n z$. We speak of *topological conjugation*, locally exclusively, from $f_n(z)$ to $a_n z$. When $a_n = e^{2\pi i\theta n}$, the linearization turns $f_n(z)$ into the germs $g_n(z)$: topologically, there exists a neighborhood centered at δ, isomorphic to a disc, the *Siegel disc* $\mathcal{S}$ – like in (1.1.3) – where local dynamics are governed by the linear term of (1.1.5) exclusively. Their behavior is analogous to aperiodic, rigid rotations. The original Fatou's and Julia's statements can be filed under the so-called *center-problem*, the expression pointing out to the study of the conditions for the linearization (1.1.6) to hold. 'Center' is in fact a dated expression[4], still in use to indicate the equidistant interior point from the circle-shaped boundary of the Siegel disc $\mathcal{S}$, where iterates $f_n(z)$ rotate.

After the early developments by Kasner, Pfeiffer, Fatou, Julia, Ritt and Cremer (in chronological order), it was clear that the conditions, for (1.1.6) to hold, are numerical. Siegel finally showed they are Diophantine [90]. Since this result settled a long standing question, historians liked to close the first stage of developments here.

During 1960s and 1970s, the branch of indifferent dynamics entered the second stage thanks to some advancements inherited from the results in KAM theory, from Arnold's works on small divisors [1], from Brjuno's on differential equations [16] and from Cherry's on the iterates with indifferent multiplier [27]. This short list indicates the real number of few isolated productions, scattered through the Time and concerning of continuous systems governed by analytic differential equations. They indirectly tie to iterated complex maps: in fact these two fields were already known to keep lots of affinities in general. The indifferent dynamics were rather sleepy until late 1980s, except for few works

[3]This parametric version shows how the Schröder functional equation can apply to the (super-)attracting and the repelling case (when $a < 1$ or $a > 1$).

[4]Julia pulled it from Poincaré's terminology in the theory of linear differential equations, where the rotational behavior in a neighborhood of one such fixed point also happens.

by the French [48, 61, 62] and by the Russian school [67, 95], where authors attacked the problem in local terms, discussing on the linearization conditions and on the Siegel disc properties $\mathcal{S}$.

A third stage begun when one radically new look was brought in during 1990s: specialists[5] realized that the numerical nature of θ not just affects the Siegel disc size – locally speaking – but even the topology of the Julia set J_θ at a larger view. The theory of hedgehogs roots into the investigation on the dynamical and geometrical properties of $\partial\mathcal{S}$ and of J_θ, featuring what follows.

(1°) Because the local linearization and the existence theorems were already clear in general, the study of the properties for the argument θ naturally followed as one further elaboration of the former results. This topic became one dominant trend during the new wave. One notices that the complex iterated map $f(z)$ was studied as a polynomial exclusively and in the form (1.1.5). The latter can be regarded as a Taylor series and it can still (and at best) support the study of the extended version of an older problem (to be explained in the next sections); or equivalently, specialists looked out at a newer, bigger problem keeping the same local features as of the original and more restricted version, since when it was formulated (back in late 1910s). *Here the Diophantine order of θ and the metrics of Siegel discs are concerned.*

(2°) On another side, the new trend started to extend the investigation to the *semi–local scope*: this means to drop the solely local analysis of the Siegel disc and to investigate on the Julia set J_θ too. In view of (1°), it seems that one shall exclude the linear term from the Taylor series and consider the higher orders terms. Anyway it is not a correct view: the Taylor series are obsolete to work on problems of not exclusively local scope. Tools were naturally forged elsewhere: in the realms of Topology and of Number Theory, the closest fields to the essence of the problem. *Here the dynamics and the geometries of Julia sets J_θ are discussed.*

To summarize, the discoveries showed that the numerical properties of θ either affect the size of $\mathcal{S}$ and the Hausdorff dimension of J_θ in a mutual fashion: as the former shrink to 0, the latter increases up to 2 in the complex plane. Because of the particular shapes taken during this process, the Julia sets J_θ

[5]Headed by R. Pérez-Marco and by J.-Ch. Yoccoz, who introduced several outstanding techniques.

were termed *hedgehogs*.

1.2 The hedgehogs

In the Holomorphic Dynamics, the 'hedgehog' is nowadays the expression collecting the family $\mathcal{H}$ of J_θ when $\theta \in \mathbb{R}\backslash\mathbb{Q}$. Equivalently, it is the visual version which resumes all the semi–local dynamics in the irrational case. The theory of hedgehogs does not replace the older center-problem, but it extends the latter in the same terms as in the previous two points.

The center-problem was introduced during the earliest studies on the global dynamics for complex rational maps all over the Riemann sphere, i.e. inside the independent works by Fatou and by Julia during 1906–1920. At the eyes of these two pioneers, it immediately showed as a very hasch question. History tells that Fatou's and Julia's false positions did not only root to their lack of background[6], but there were intrinsic difficulties in grasping an overall view because several essential concepts were not clear to them yet. Scratch by scratch, difficulties have been slowly removed during the later decades, until the core became visible in the recent years. These events had been showing that an interdisciplinary approach, joining Topology, Analysis and Number Theory, was strongly required. The introduction of the term 'center' by Julia was likewise imitated by Cremer, who coined the opposite expression *non-center* and a number of derived terms, in order to have at hand a basis of preliminary concepts to discuss the opposite side of the question. The center-problem original statement discards the case of the geometries and the neighboring dynamics when (1.1.6) does not hold and centers do not exist. But, alike the center-problem, this counterpart also requires the combined approach via Number Theory and Topology. Equivalently to Cremer's terms, we call this counterpart as the *non-center-problem* [28]. This is the point break with the theory of continuous systems and ordinary differential equations, where the non-center situation cannot happen.

Following Cremer's production, the only paper on non-centers was written by T. M. Cherry in 1964 [27], where a geometrical object – whose topology

[6] They were not Number Theorists.

was very similar to modern hedgehogs – was sketched out. One shall wait until the early 1990s for seeing more mathematical works on non-centers, illustrating the new theory of hedgehogs as the modern way to approach, resume and solve definitely the center-problem. It was grabbed and placed where deserved: just as a dot inside one bigger puzzle! Hedgehogs in fact delivered a corpus of concepts for watching all irrational dynamics coherently, where the rational and irrational dynamics are tied together [73].

For what can be known about the iterates of polynomials (1.1.5), hedgehogs relate $\mathcal{S}$ to J_θ, classifying $\mathcal{S}$ into 3 types, differing for the numerical properties of θ and for the radius r of the disc $\mathcal{S}$. The range of r is as follows:

1°) the *maximal* value R, so $r = R$;

2°) *non-maximal* r, when $0 < r < R$;

3°) the *null* value, when $r = 0$.

As r shrinks from R to 0, J_θ wedges into the bounded basin. There is no wedging action in 1°) and it is maximal in 3°) when $\delta \in J_\theta$ is a non-center and termed today as *Cremer point*: the two-fold nature of this theory gets clearer as one runs this list backwards from 3°) to 1°) in terms of linearization regions or, from 1°) to 3°), in terms of hedgehogs if the rate of wedging action is accounted: the Hausdorff dimension of Julia sets goes way up to 2 as r decreases to 0. The cases 1°) and 2°) involve *linearizable hedgehogs*, while the *non-linearizable* type belongs to 3°). If the linearization region is maximal, $\mathcal{S}$ is a *Siegel disc*; otherwise these smaller discs are defined *Siegel compacta* ([73], p. 245).

Despite the power of covering all subcases in indifferent dynamics, we opine that the theory of hedgehogs did not get to the mature stage yet. We mean to the sense of most results based upon the dynamics of polynomials exclusively. This could still stand a restriction. As we will remark in the closing sections of this work, one further advancement could want to 'get back' to the original terms: the rational maps $R(z)$ with indifferent points. The dynamics of $R(z)$ should be directly investigated at the semi-local scope, searching out of the Siegel disc $\mathcal{S}$ and moving towards the Julia set. The promising results on the hedgehogs for the iterates of polynomials (1.1.5) already suggested that the Schröder's functional equation (1.1.6) is no longer useful outside $\partial\mathcal{S}$.

Question 1.2.1. *Could a full picture of hedgehogs geometries be taken from the study of the iterated polynomials exclusively? Would it more adequate to study the singularities distribution for a general rational map $R(z)$ and to know if such latter points exist and how they could affect these geometries? For example to know whether they [do not] spread radially along privileged directions? How many and where?*

These are just questions, perhaps indications for further researches, not discussed here but just hinted. The goal of this article is to deepen the mathematical meaning of some results that already appeared in [83] but were developed in terms of graphics exclusively.

1.3 The problem of drawing hedgehogs

We are first interested in understanding if computers may capture, and at what degree, the shapes of $\mathcal{H}$. While (1°) is trivial, the display of cases (2°) or (3°) gets much harder, due to the *delicate numerical properties* of $\theta \in \mathbb{R}\backslash\mathbb{Q}$, together with *machine approximation* and the *slow convergence speed* in each type of indifferent dynamics. Both aspects drastically slow down the performance of most graphical methods: these are clear evidences that they do not fit in such dynamics. Not surprising anyway, since they had originally been developed for particular shapes of Julia sets.[7] The global character of these (widespread) methods collides with the local dynamics around Siegel compacta.

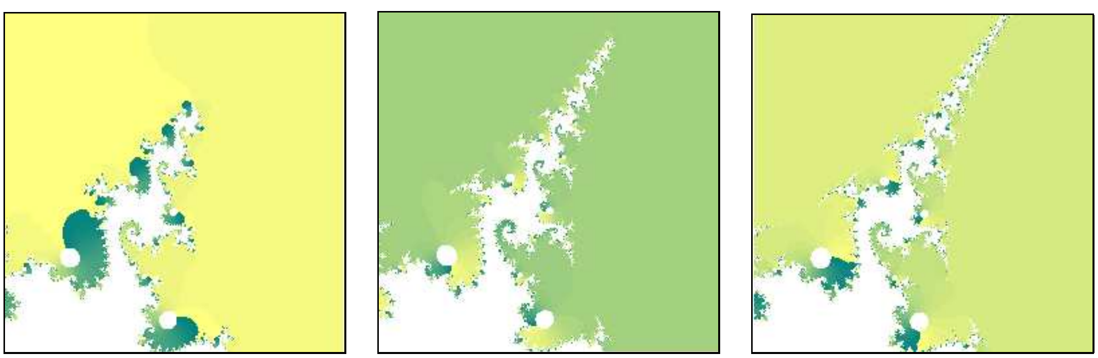
So we are long for new strategies. Old methods may work sufficiently fine for rational dynamics when $\theta \in \mathbb{Q}$. But, when $\theta \in \mathbb{R}\backslash\mathbb{Q}$, the efforts to obtain fine drawings of J_θ require to be customized.

A *first question* of interest was to know *whether and how* a detailed display of $J_\theta, \theta \in \mathbb{R}$ can be accomplished in a reasonable time: if $\theta \in \mathbb{Q}$, the convergence rate gets slower as iterates $f_n(z)$ get closer to δ. If $\theta \in \mathbb{R}\backslash\mathbb{Q}$ is Diophantine of

[7]The display of particular topologies could require dedicated methods, as the author showed for Quaternionic Julia sets [82].

any finite order κ, there exists a sufficiently small neighborhood where such rate is null, (1.1.6) holds and iterates rotate around the fixed point δ. Therefore any candidate method shall not take into account the convergence or the divergence.

One way to crack this problem was suggested in [83]. It is especially useful while dealing with case (2°) when the Siegel radius is non-maximal and when $\theta \in \mathbb{Q}$. This method is not free from flaws: figures 1.3.1 include white discs pointing out to a lack of accuracy. On the other hand, it runs faster: if compared to the aforementioned methods, the gap is significant. Results show that the hedgehogs topology gets clearer as well as its wedging action inside the bounded basin. Our original goal was to offer a strategy for drawing pictures regarding the indifferent dynamics through a exhaustive set of parameters. Despite such detailed results, this is not a total victory, because our method still enjoys a same flaw as most of the known methods: the impossibility of drawing the extremal case of non-linearizable hedgehogs. The feeling of reaching so close to it let us wonder *if*, rather than *how* (like we did before). It was then natural to focus on a *second question*: are non-linearizable hedgehogs computable[8] ? It is evident that the response demands a deeper analytical approach than our achievements in [83], where only empirical methods were applied. Just as



[A] [B] [C]

Figure 1.3.1: **The wedging action.** Here we blew up fig. 1.1.1/B. Even with a relatively small, but progressively increasing number of iterates, this method emphasizes the wedging action inside the bounded Fatou component. White discs acknowledge the lack of accuracy in the visual reproduction of the geometry: a side effect from the main feature which allows the method to be very fast.

anticipation here, we sketch out that the response is negative: the characters of irrational dynamics depend on delicate numerical conditions and computers

[8]A set is said to be *computable* if it can be drawn on the computer screen [15].

can just approximate them. Although this latter statement is quite obviously true, the approach we followed begun a prolific discussion on Diophantine Approximations, Measure Theory and Topology. Flattering results are retrieved for linearizable hedgehogs. Definitely, approximation equals failure in the non-linearizable situation: any algorithmic[9] approach cannot crack the problem: this is the definitive obstruction stopping the wished performance. The question necessarily splits into two cases, if hedgehogs can be algorithmically managed or not. This particular topic belongs to a wider range of questions, classified as '*computability of Julia sets*'.

2 Gathering the required background

In the next two sections we will sketch out some background concepts from the Measure Theory, Diophantine Approximations and Topology. Where required, some additional concepts will be developed as well, in order to have the appropriate tools for the final proof. The reader is directed to the books [40, 35, 53, 58, 64] for prerequisite background information not provided here.

2.1 Diophantine and Liouville irrationals

Let

$$R(z) = \frac{m(z)}{n(z)} \qquad \nexists w_1 \in \mathbb{C}, R(w_1) : m(w_1) = n(w_1) = 0,$$

be a complex rational map. $R(z)$ is the quotient of two non-zero polynomials with rational coefficients, so that

$$\deg(R(z)) = \max\{\deg(m), \deg(n)\} \geq 2.$$

In [28], where the dynamics of the iterates $R_n(z)$ about an irrationally indifferent fixed point δ are investigated, Hubert Cremer showed that the closed unit interval $\mathcal{I} \equiv [0, 1]$ includes a zero Lebesgue measure set of irrational values α, not allowing the conjugation of $R_n(z)$ into the holomorphic germ (1.1.5),

[9] Mostly applying to cases (1°) and (2°).

thus obstructing the *local linearization* $R_n(z) \mapsto e^{2\pi i\alpha n}$ around δ. Topologically, there exists no neighborhood $U \supset \delta$, centered at δ and being conformally isomorphic to a disk under $R_n(z)$. We say that $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is a *Siegel value* [*Cremer value*] if the linearization applies [does not apply] and call δ a *Siegel point* [*Cremer point*]. Long before Cremer's 1927 paper [28], Joseph Liouville (1844) showed that α may 'resist', more or less strongly, their approximation by sequences of rational numbers p_n/q_n. This strength is rated by an integer value κ in the following *Diophantine condition*: [10]

$$\left|\alpha - \frac{p_n}{q_n}\right| > \frac{\epsilon}{q_n^\kappa}, \qquad \forall\kappa \geq 2 \quad \kappa \in \mathbb{Z}^+. \tag{2.1.1}$$

(A refined version, to be applied to other problems in Number Theory, wants $\kappa \in \mathbb{R}$.) This condition is satisfied by Siegel values. At this point, for sake of comfort, we remanage the standard operator for defining the closest integer p to $q\alpha$ (check Lang's book [58] for original usage). Let $\|q_n\alpha\|$ be the left-hand side of the inequality (2.1.1):

$$\|q_n\alpha\| \equiv \left|\alpha - \frac{p_n}{q_n}\right|;$$

if α satisfies (2.1.1), α is a *Diophantine* irrational and it can be classified by the order $\kappa < +\infty$, $\alpha \in \mathcal{D}(\kappa)$. Given a non-zero polynomial of finite degree κ,

$$P_\kappa(x) = \sum_{i=0}^{\kappa} b_i x^i, \quad b_i \in \mathbb{Q},$$

α is *algebraic of order* κ for $P_\kappa(\alpha) = 0$ holds, given $\inf \kappa < +\infty$. Liouville applied the remarkable formula (2.1.1) to retrieve an equivalent definition for algebraic numbers in terms of rational approximations: in fact, the expressions '*Algebraic*' and 'Diophantine' mean the same in this context. There exists a class of irrationals, the *transcendental numbers*, which are not algebraic, so not the solutions of any such polynomial $P_\kappa(x)$.

Now let $\mathcal{D}(2)$ be Diophantine irrationals of quadratic order, then

$$\mathcal{D}(2+) \equiv \bigcap_{k>2} \mathcal{D}(\kappa) \tag{2.1.2}$$

[10] Any relation where only integer solutions are allowed. Here they are p_n, q_n and κ.

is the topological representation for Diophantine irrationals of order $\kappa > 2$. One knows that $\mathcal{D}(2+)$ and $\mathcal{D}(\infty)$ are of *full Lebesgue measure* ([64], pp. 120, 222).[11] These two classes of irrationals play a special role in the theory of Diophantine Approximations. Surprisingly[12] $\mathcal{D}(\infty)$ is not analogously applied to *Diophantine irrationals of infinite order*, but it is defined as

$$\mathcal{D}(\infty) \equiv \bigcup_{k<+\infty} \mathcal{D}(\kappa) \tag{2.1.3}$$

i.e. the union set of all Diophantine numbers of any order κ, according to this nesting rule ([64], p. 222):

$$\mathcal{D}(m) \subset \mathcal{D}(n), \qquad 2 \leq m < n < +\infty. \tag{2.1.4}$$

The set $\mathbb{L} \equiv \mathbb{R}\backslash(\mathbb{Q} \cup \mathcal{D}(\infty))$ of *Liouville numbers* is the complement of $\mathcal{D}(\infty)$ in $\mathbb{R}\backslash\mathbb{Q}$ and consists of the Cremer values not satisfying (2.1.1). One knows that $\mathbb{L}$ has *zero Hausdorff dimension* ([64], p. 222): [13]

Lemma 2.1.1. *The set $\mathbb{R}\backslash\mathcal{D}(\kappa)$ has Hausdorff dimension $\leq 2/\kappa$. Hence the set $\mathbb{L}$ of Liouville numbers has zero Hausdorff dimension.*

Unless otherwise indicated, α and θ will denote an arbitrary irrational number and an arbitrary Liouville number respectively.

2.2 Khintchine's theorems

The sense from the whole corpus of the previous results just grants the existence of sequences of rational approximants to algebraic irrationals α of arbitrarily large order κ, but cannot give any information on how or if one can reach α. We move the focus from α to the behavior of the sequence p_n/q_n. Several results were developed in this direction. We first mention two metric theorems by Khintchine, termed as *Khintchine's convergence* and *divergence theorems* respectively, concerning of the interesting property 2 of theorem 2.5.1 (p. 31) ([52, 53], cf. [58], pp. 23–24):

[11] Check the definition of Lebesgue measure at p. 35.

[12] We keep this definition until sharper results will be shown, where the ambiguous symbol ∞ is replaced by the least transfinite ordinal ω_0. The development of a proper and more extensive environment will lessen this exception, letting it match the original meaning.

[13] Check the definition of Hausdorff dimension at p. 38.

Theorem 2.2.1 (Khintchine's Convergence Theorem). *Let ψ be a positive function such that*

$$\sum_{q=1}^{\infty} \psi(q)$$

converges. Then for almost all numbers α (i.e. outside a set of zero Lebesgue measure), there is only a finite number of solutions to the inequality

$$\|q\alpha\| < \frac{\psi(q)}{q}. \quad (2.2.1)$$

Theorem 2.2.2 (Khintchine's Divergence Theorem). *Let ψ be a positive function such that*

$$\sum_{q=1}^{\infty} \psi(q)$$

diverges. Then for almost all numbers α, there are infinitely many solutions to the inequality $\|q\alpha\| < \frac{\psi(q)}{q}$.

The terms '*convergence*' and '*divergence*' refer to functions which converge asymptotically to a finite bounded value and to ∞ respectively.

These two Khintchine's theorems are based upon the boolean-like performance through 'convergence-divergence', intuitively setting up the existence of one threshold value for determining if the inequality (2.2.1) has finitely or infinitely many solutions. These theorems portray the early and rough status of the metric Theory of Numbers before the achievements by Roth and alia. The information retrieved here are insufficient to allow any direct contact with Diophantine-Liouville dichotomy: for example, theorem 2.2.1 may allude to Liouville numbers (because of the zero measure set), while 2.2.2 cannot help to determine if the values in question are Diophantine or Liouville. From merging theorems together, one can find out a '*strip of doubt*' close to α and where one cannot determine exactly if solutions are finitely or infinitely many. Despite these comments and flaws in metric terms, we will make some observations about the sequences of approximants p_n/q_n.

2.3 A first look on the sequences of approximants

With regard to these Khintchine's theorems, let $q > 1$. If $\sum_{q=1}^{\infty} \psi(q)$ diverges [converges], $\psi(q)/q$ diverges [converges]. Let $\deg(f)$ be the degree of the function f in parentheses.

From $\psi(q)\frac{1}{q}$, $1/q \to 0$ as q grows: so the convergence and divergence of $\psi(q)/q$ involve the growth order of $\psi(q)$, because of $\deg(\frac{1}{q}) = \log_q(\frac{1}{q}) = -1$ is constant. With no loss of generality,[14] let $\psi(q) : aq^n, n \in \mathbb{Z}$ and $\log_q \psi(q) = n$. Let $\deg(\psi(q)) \equiv \log_q \psi(q)$, we will first discuss the exponents n, when a is constant. Let the maximum operator:

$$M_\psi = \max\left\{ |\log_q \psi(q)|, \left| \log_q \left(\frac{1}{q}\right) \right| \right\}, \tag{2.3.1}$$

which becomes

$$M_\psi = \max\{|\log_q \psi(q)|, 1\} \quad or \quad M_\psi = \max\{\log_q \psi(q), -1\}.$$

According to (2.3.1), we find that $M_\psi = |\log_q \psi(q)|$ holds when:

1. $\log_q \psi(q) = n \geq 2$. In fact, given

   $$\log_q \psi(q) + \log_q \left(\frac{1}{q}\right) \equiv n - 1 \geq 1,$$

   one has $\psi(q) > q$ and $\psi(q)/q > 1$. For example, $\psi(q) \equiv q^{n\geq 2}$ and $\psi(q)/q \Rightarrow \psi(q) \equiv q^{n\geq 1}$. Thus $\sum_{q=1}^{\infty} \psi(q)$ and $\psi(q)/q$ diverges;

2. $\log_q \psi(q) = n < 1$. In fact, given

   $$\log_q \psi(q) + \log_q \left(\frac{1}{q}\right) \equiv n - 1 \leq -1.$$

[14] We may also assume the polynomial form

$$P(q) : \sum_{i=0}^{n<+\infty} a_i q^i,$$

where $\deg(P(q)) = n$. From our usage of the maximal operator, one sees that the conclusions of this section depend on the value n of the largest exponent, thus from the monomial aq^n. Along these lines we can take on $\psi(q)$ as a general Taylor series, opening to the cases of a polynomial or transcendental $\psi(q)$.

One has $\log_q \psi(q) < \log_q q$ and $\psi(q)/q$ rewrites into $1/q^{n+1} \to 0$. Thus $\sum_{q=1}^{\infty} \psi(q)$ and $\psi(q)/q$ converge;

3. $\log_q \psi(q) = n = 1 = \log_q(\frac{1}{q})$. In fact, given

$$\log_q \psi(q) + \log_q \left(\frac{1}{q}\right) \equiv n - 1 = 0,$$

the discussion moves to the parameter a: in fact, $\psi(q)/q \equiv a$. Given $\log_a(a) \geq 1$, $\sum_{q=1}^{\infty} \psi(q)$ diverges; otherwise, if $\log_a(a) < 1$, $\sum_{q=1}^{\infty} \psi(q)$ converges;

4. Finally, one might like to discuss a and n *simultaneously*. This is easy as $\frac{\psi(q)}{q}$ splits into $a\frac{\varphi(q)}{q}$, where $\psi(q) \equiv a\varphi(q)$. Given

$$m = \min\{\log_a(a), \log_q \varphi(q)/q\}, \quad M = \max\{\log_a(a), \log_q \varphi(q)/q\}$$

$\sum_{q=1}^{\infty} \psi(q)/q$ converges [diverges] for $M+m < 0$ [$M+m > 0$]. If $M+m = 0$, $\frac{\psi(q)}{q}$ is constant.

Khintchine's theorems assume that the sequence of q_i is *a priori* determined, $q_i = 1, 2, 3, \ldots$, according to the sum operator $\sum_{q=1}^{\infty}$. These sequences are trivial anyway and we are mostly interested in studying the general case, where there exists a map $q_{i+1} = \phi(q_i)$. Here we focus on the coefficient a or on the exponent n of ϕ: both elements govern the approximation rate.

2.4 On the strip of doubt

According to section 2.3, it seems that a general discussion on the behavior of the map $\psi(q)$ should take into account either its speed of convergence [of divergence]. But the rational form of $\psi(q)/q$ brings in one more delicate discussion, involving the combinations resulting from a comparison between the growing [shrinking] rates of the map $\psi(q)$ at the numerator and of the identity map q at the denominator; except when both maps shrink and the resulting $\psi(q)/q$ shrinks too, the two speed rates shall be compared so that $\psi(q)/q$ grows [shrinks] as their algebraic sum is positive [negative]. The Khintchine's two theorems represent the natural environment for this discussion. In fact, the different speed rates of the numerator $\psi(q)$ and the denominator q affect the convergence and the divergence of $\psi(q)/q$ and thus may require the application of theorem 2.2.1 or 2.2.2, *where* required. Our goal is to set up one scenario where these two theorems are complementary in topological terms, so to have a combined version. One such new statement may be even weaker, because of the behavior would depend on the sequence q_n as well as on the map $\psi(q)$. Fractions with different speed rates in the numerator and in the denominator may give rise to a larger number of cases than we need. So we set some restrictions to $\psi(q)$, for crunching that number to only one which fits our purposes.

Let $\psi(q_n) : \mathbb{N} \to \mathbb{R}$ be a convergent sequence. Let $q_n = 1, 2, \ldots$ and $c, k \neq \infty$, where $c, k \in \mathbb{R}$, are constants. Consequently,

$$\psi(q_n) \to c \quad \Rightarrow \quad \sum_{q_n=1}^{\infty} \psi(q_n) \to k \neq \infty.$$

It is not intended that we have convergence for all $q > 0$. Now suppose the hypotheses of the two theorems hold simultaneously for a same rational map $\psi(q_n)/q_n$. Loosely speaking, there may exist one or more values of q_n such that $\psi(q_n)/q_n$ may diverge or not. So one can speak of one domain $\mathcal{R}_C$ of convergence and one $\mathcal{R}_D$ of divergence. As $n = 1, 2, 3, \ldots$, the denominators q_n increase sufficiently fast that we have

$$\liminf \frac{\psi(q_n)}{q_n} = \liminf \frac{c}{q_n} = 0.$$

The two theorems under consideration imply that $\|q\alpha\| \to 0$. Let the triplet of points $q_s \leq q_{s+1} \leq q_{s+2}$ be a subsequence of q_n so that $s \in N$ ($N = \{n \in \mathbb{N} : 1, 2, 3, \ldots\}$) with the (here, purely artificial) condition that no couple of points q_s belong to a same domain $\mathcal{R}_C$ or to $\mathcal{R}_D$ or to the boundary $\mathcal{R}_B \equiv \mathbb{R}\backslash(\mathcal{R}_C \cup \mathcal{R}_D)$. At least two members of the subsequence q_s must lie inside two different regions. Let q_s and q_{s+2} belong to the region of divergence and convergence, respectively, while q_{s+1} lies on their common boundary. Fig. 2.4.1 illustrates the situation for $s = 0$. Regarding the region of convergence, note that theorem 2.2.2 implies $\|q_n\alpha\| \to 0$ as q_n increases without bound; if $\psi(q_n)/q_n$ is not everywhere convergent, it seems plausible that there exists one region of

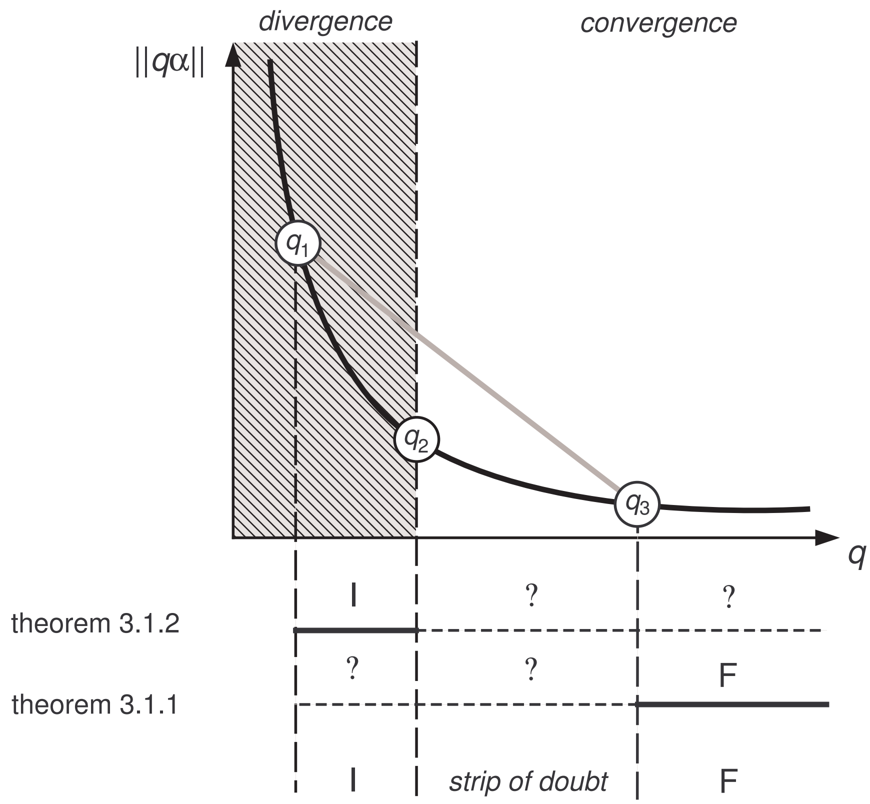


Figure 2.4.1: **Khintchine's theorems.**

divergence where theorem 2.2.1 holds. The bottom of fig. 2.4.1 shows the intervals where theorems hold (solid segment). The question marks and the dashed segments point to intervals[15] where each theorem gives no information on the number of solutions for the inequality

$$\|q_n\alpha\| < \frac{\psi(q_n)}{q_n}, \tag{2.4.1}$$

depending on the values q_n. If we combine theorem 2.2.1 to 2.2.2 into one new theorem, we obtain a result on the number of solutions for the inequality (2.4.1) within a small neighborhood of the irrational α. Given $q_1 \neq q_3$ (and it could not be otherwise if they belong to distinct domains of convergence), one obtains a gap where the combined version cannot help on the number of solutions of (2.4.1). We call this gap the *strip of doubt*. One of the goals for the theory of

[15]The letters F and I indicate the ranges where (2.4.1) has *finitely* and *infinitely* many solutions, respectively.

Diophantine Approximations is to compute the optimal value q_2 for which the gap closes when

$$\liminf |q_1 - q_3| = 0.$$

The lower bound vanishes identically when one can approximate q_1 and q_2 so sharply to have one same value q_2 on the boundary $\mathcal{R}_B$ for $q_1 = q_2 = q_3$. Each region $\mathcal{R}_C$ and $\mathcal{R}_D$ is well-defined and their union covers $\mathbb{R}\backslash\{\alpha\}$. This situation raises some natural questions of both topological and of numerical nature: *could $\mathcal{R}_C \cap \mathcal{R}_D \equiv \emptyset$ hold or not? Is $\mathcal{R}_C$ open or closed? Since q_n is not assumed to be an integer necessarily, what is the nature of q_n (or of $\log q_n$) when it lies on the boundary? Should $\frac{\psi(q_n)}{q_n}$ meet particular integrability conditions? And, if so, what should the nature of set of discontinuities be?*

Some have been crucial in the historical development of Diophantine Approximations. They will be considered later again here, to elucidate our main problem and to ground the further results.

2.5 On the accessibility to Liouville numbers

According to the theory of Diophantine Approximations, one special class includes the algebraic irrationals *of constant type* (equivalently, *of bounded type*), enjoying these two[16] equivalent properties ([58], p. 24):

Theorem 2.5.1. *Given an algebraic irrational α of constant type:*

1. *There exists a constant $\epsilon > 0$ such that (2.1.1) holds for all integers $q_n > 0$ and $\kappa = 2$.*
2. *For any positive function ψ with convergent sum $\sum \psi(q_n)$, the inequality $\|q_n\alpha\| < \psi(q_n)/q_n$ has only a finite number of solutions.*

From property 1 one understands the reason why the expression *Diophantine irrationals of constant type* refers to $\mathcal{D}(2)$, the irrationals of quadratic order. On the contrary, Liouville numbers θ satisfy this complimentary formula instead:

[16] More equivalent properties could be listed. But were omitted because these listed ones best fit the sense of our discussion.

given $p, q \in \mathbb{N}$ where $gcd(p, q) = 1$ and $\epsilon > 0$, then

$$\liminf \|q\alpha\| \leq \frac{\epsilon}{q^\kappa}, \quad \forall \kappa \geq 2. \tag{2.5.1}$$

Liouville numbers are *transcendental* (the converse does not hold). The previous discussion introduced the existence of a Diophantine-Liouville dichotomy, coming straight from the so-called *Liouville's approximation theorem*:

Theorem 2.5.2 (Liouville's approximation theorem). *Let α be any algebraic number of finite order $\kappa \geq 2$. Then there exists a positive constant $\epsilon = c(\alpha)$ [17], such that*

$$\|q_n\alpha\| > \frac{\epsilon}{q_n^\kappa} \tag{2.5.2}$$

for all rational numbers p_n/q_n.

Diophantine and Liouville numbers differ if the lower bound for the infinitesimal ϵ may be positive or vanish identically, if (2.5.2) holds or not, respectively. This sets a break-point for the Diophantine-Liouville dichotomy. There is also one such split for Diophantine irrationals, between $\mathcal{D}(2)$ and $\mathcal{D}(2+)$: while property 2 of theorem 2.5.1 *always* holds for $\mathcal{D}(2+)$, it holds for $\mathcal{D}(2)$ *only if* the denominators q_n meet one condition, discussed below.

One such generalization for $\mathcal{D}(2+)$ was shown during the mid 1950s and represents one of the most intriguing results in Diophantine Approximations. Sharper versions of theorem 2.5.2 followed by Thue [93], by Siegel [89], by Dyson [34], focusing on how κ affects the number of solutions p_n/q_n of (2.5.2) inside a neighborhood of α. These results culminated into the celebrated[18] Thue-Siegel-Dyson-Roth's theorem [84], giving a sharper estimation of κ and elucidating the question on the number of these solutions in a much more[19] extensive manner:

Theorem 2.5.3 (Thue-Siegel-Roth's theorem). *Given an algebraic irrational number α, there exist only finitely many solutions p_n/q_n of*

$$\|q_n\alpha\| < \frac{1}{q_n^{2+\epsilon}},$$

with $\epsilon > 0$.

[17]The value of ϵ depends only on α.

[18]K. F. Roth was awarded a Fields medal in 1958 for this result.

[19]There is still much to learn. All the later production after Liouville's theorem lacked of *non-effectiveness*, because the related methods of proof do not make it possible to determine *how ϵ can be computed from α.*

From theorem (2.5.3), Roth deduced the existence of a least upper bound $\mu(\alpha) \equiv \inf_{\mu \in \mathbb{R}} \mu$, called *irrationality measure*[20], indicating how closely α can be *well-approximated* by p_n/q_n. The value $\mu(\alpha)$ plays as a threshold for $\|q_n\alpha\|$, inviting to focus on this version of the above inequality:

$$\|q_n\alpha\| < \frac{1}{q_n^{\mu(\alpha)}}. \tag{2.5.3}$$

Hopefully, comparing (2.5.3) to Roth's theorem will not confuse the reader: κ and $\mu(\alpha)$ do not keep the same meaning. The operator $\mu(\alpha)$ indicates the upper bound to be taken on by κ for a neighborhood of α to include infinitely or finitely many solutions of (2.5.1). If $\kappa < \mu(\alpha)$, there exist infinitely many solutions inside a neighborhood of α and converging *arbitrarily close* to α, which is said *well-approximable*. Otherwise, solutions are finitely many for $\kappa > \mu(\alpha)$: one such number cannot guarantee arbitrarily close convergence to α, which is thus a *badly approximable* value. The existence of infinitely many approximants p_n/q_n, together with their *equi-distribution*[21] inside intervals of arbitrary width, are necessary conditions for assuming that approximants may get arbitrarily close to α.

Roth found a striking classification of all reals into three categories: $\mu(\alpha) = 1$ if $\alpha \in \mathbb{Q}$, $\mu(\alpha) = 2$ if $\alpha \in \mathcal{D}(\kappa)$ for $\kappa \geq 2$ and $\mu(\alpha) > 2$ if $\alpha \in \mathbb{L}$. According to Roth's sense, while any neighborhood of $\mathcal{D}(2)$ and $\mathcal{D}(2+)$, with smaller radius than $1/q^{2+\epsilon}$, cannot include infinitely many rational approximants, the unbounded interval $\mu(\alpha) > 2$ indicates the infinite irrationality measure of Liouville numbers. Roth gave a definitive response to the natural question arising from some weak theorems in Number Theory (see for example, section 2.4). Thus $\mu(\theta)$ cannot be exceeded by finite κ and Liouville numbers are well-approximable. Reaching to θ via sequences of rational approximants with arbitrarily sharp precision is effective. We can equivalently state that:

Proposition 2.5.1. *For every $\theta \in \mathbb{L}$, there exist arbitrarily small neighborhoods $U(\theta)$ of θ such that $U(\theta)$ contains solutions of* (2.5.1) *that accumulate arbitrarily close to θ.*

[20] Alternatively, 'Liouville-Roth constant' or 'irrationality exponent'.

[21] Related results appear in [4] by Behnke, [71] by Ostrowski and [96] by Weyl.

Roth's classification helps to earn the global understanding of several results, developed years before into separate theorems, but not arranged into a systematic corpus yet. One of them is this classic theorem ([68], p. 95):

Theorem 2.5.4. *Given any irrational number α, there are infinitely many rational numbers $\frac{p_n}{q_n}$ in lowest terms (or coprime, i.e.* $\gcd(p_n, q_n) = 1$*) such that*

$$-\frac{1}{q_n^h} < \alpha - \frac{p_n}{q_n} < \frac{1}{q_n^h}. \tag{2.5.4}$$

From elementary calculus, one finds that (2.5.4) can be rewritten as follows:

$$\|q_n\alpha\| < \frac{1}{q_n^h}. \tag{2.5.5}$$

According to Roth's theorem (p. 32), $h \leq 2 + \epsilon$ relates to infinitely many solutions of (2.5.5) for the algebraic α. The estimation of the bounding denominator value in the right-hand side of (2.5.5) was sharpened by the Hurwitz theorem (cf. [50]), stating:

Theorem 2.5.5 (Hurwitz's theorem). *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and A be a constant satisfying $0 < A \leq \sqrt{5}$. Then there are infinitely many solutions p_n/q_n of*

$$\|q_n\alpha\| < \frac{1}{Aq_n^2}.$$

The optimality[22] of $A = \sqrt{5}$, as upper bound for the existence of such infinitely many solutions, stands out as another intriguing result: for $A > \sqrt{5}$, one finds finitely many solutions of the above inequality, according to Roth's theorem. Hurwitz's theorem fits the same sense as of the questions concluding the section 2.4. As Roth's and Hurwitz's theorems melt together, one can check the conditions met by q_n for knowing how many approximants converge *arbitrarily close* to θ. Applying the notation of Roth's theorem, one writes

$$\|q_n\alpha\| < \frac{1}{q_n^{2+\epsilon}} < \frac{1}{\sqrt{5}q_n^2}.$$

[22]Hermite already found it was $0 < A \leq \sqrt{3}$.

Equivalently, in terms of q_n sequence, one seeks the maximal value q_{n+1} for infinitely or finitely many solutions to exist. So one writes

$$\|q_n\alpha\| < \frac{1}{q_{n+1}q_n^2} < \frac{1}{\sqrt{5}q_n^2} \quad \textit{where} \quad q_n^\epsilon \equiv q_{n+1}.$$

The magnitude of two consecutive denominators q_n, q_{n+1} is ruled by

$$q_{n+1}q_n^2 > \sqrt{5}q_n^2 \Rightarrow q_{n+1} > \sqrt{5}.$$

Alternatively, taking each denominator q_n at once, one writes

$$q_n > \sqrt[2\epsilon]{5},$$

regardless of the value n. The existence of infinitely many approximants is thus guaranteed by

$$q_n \leq \sqrt[2\epsilon]{5}.$$

More interestingly, the sequences $0 \xrightarrow{\epsilon} \infty$ and $\infty \xrightarrow{\epsilon} 0$ hold for $\alpha \in \mathcal{D}(\kappa), \kappa \geq 2$ and lead to $q_n \to 1$, according to the limits

$$\lim_{\epsilon\to+\infty} \sqrt[2\epsilon]{5} = 1, \qquad \lim_{\epsilon\to 0^+} \sqrt[2\epsilon]{5} = \infty.$$

With regard to the class $\mathcal{D}(2+)$, the existence of infinitely many solutions of (2.5.2) is a feature implying that $q_n \leq 1$. But it cannot hold because of $q_n \in \mathbb{Z}^+, q_n \geq 2$ by hypothesis. Since $\gcd(p_n, q_n) = 1$, q_n always exceeds the bounding exponent $2+\epsilon$: for Roth's theorem, no neighborhood of $\alpha \in \mathcal{D}(2+)$ can include infinitely many solutions.

2.6 Background on regular sets

We take a short break from Diophantine Approximations and move to Topology. We focus on some concepts in Measure Theory. Let $E \subset \mathbb{R}$. Since $\mathbb{R}$ is a metric space, we introduce the following theorem ([59], p. 5):

Theorem 2.6.1. *Every non-empty open set $E \in \mathbb{R}$ can be uniquely expressed as a finite or countably infinite union of pairwise disjoint open intervals.*

Émile Borel introduced[23] a different definition of the set E through a similar model based upon a more general construction of sequences of sets E_h.

Borel first gave the definition of the *regular* set R of points belonging to $\bigcap E_h$ $(h = 1, 2, \dots)$. He distinguished the role of R from the pointwise *fundamental* set F of limit points, ([11], pp. 2 and 5) for the sequence E_h. *Unlike Lebesgue*, Borel did not require E_h to be pairwise disjoint (see proposition 2.6.1: Borel's definition requires the intersection of E_h to be non-empty). He also developed this model for limit sets F of any countability: following the early version where F was countable (Ch. I, p. 2), Borel extended F to any countability with *infinitely many* or even *uncountably many points* (Ch. III, pp. 14–15). More explicitly, Borel stated ([11], p. 2):

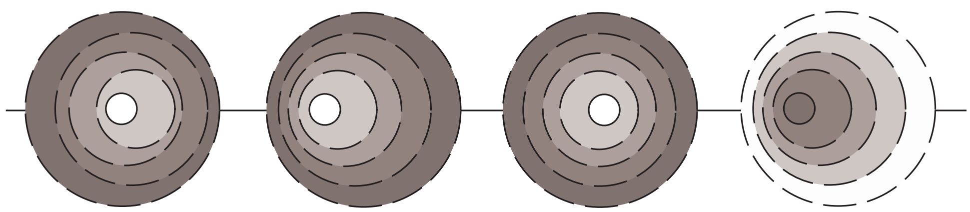


Figure 2.6.1: **Regular sets generation.** Configurations of nested and pairwise disjoint subsets E_h, here depicted as discs. They may also be segments, squares or any geometrical entity which is nestable. Convergence is not required to be uniform (A) or running inwards (B-C): it may also run outwards (D).

Proposition 2.6.1. *The set E of interior points for all E_h $(h = 1, 2, \dots)$ is regular (evidently of zero Lebesgue measure). Every set of zero Lebesgue measure belongs to a regular set.*

Since $\mathbb{L}$ is of zero measure, it can be reached via regular sets. This latter model could retrieve fundamental sets with different topologies, according to the initial conditions: for example, the nature of the sequence Γ of the indexes h or the countability of F. If Γ is bounded from above $(\limsup h \le i, 0 < i < +\infty)$,

[23]In a vaguely polemical footnote, Borel claimed the independent authorship of such concepts and attested that the '*zero Lebesgue measure*' definition became more popular than to the expression he coined: '*null sets*'. See [11], p. 1.

the regular set is not necessarily of zero measure nor could it be the same as its limit set. So $R \equiv E \equiv F$ may not necessarily hold for this model.

2.7 Independence of the regular sets model

We now return to Diophantine approximation and discuss another basic result, which follows from lemma 2.1.1 ([64], p. 223):

Lemma 2.7.1. *The set $\mathcal{D}(2)$ of Diophantine numbers of constant type has zero Lebesgue measure.*

From (2.1.4), one notices that the set of Diophantine numbers of order m enjoys a *cumulative* definition, due to the inclusion of subsets of order $2 \leq m < +\infty$ ([64], p. 222):

$$\mathcal{D}(m) = \bigcup_{\kappa \leq m} \mathcal{D}(\kappa).$$

Now we introduce the set of Diophantine irrationals *of strict order* m, i.e. with order $\kappa = m$. Given $2 \leq \kappa < m$, this definition reformulates into

$$\mathcal{D}[m] = \bigcup_{\kappa \leq m} \mathcal{D}(\kappa) \setminus \bigcup_{\kappa \leq (m-1)} \mathcal{D}(\kappa),$$

and the above lemma generalizes as follows ([64], p. 225):

Lemma 2.7.2. *Every set $\mathcal{D}[\kappa]$ of Diophantine numbers of strict order κ has zero Lebesgue measure and positive Hausdorff dimension.*

Proof. According to classic Diophantine Analysis, the definition (2.1.1) of a Diophantine irrational θ of order m implies that θ does not belong to a union of intervals of diameters $2\epsilon/q^m$ (refer to fig. 3.2.1). Likewise the class of Diophantine ζ of order $m-1$, does not belong to a union of intervals of length $2\epsilon/q^{m-1}$. Finally the Diophantine irrationals θ of strict order m do not belong to the interval of width

$$\frac{2\epsilon}{q^{m-1}} - \frac{2\epsilon}{q^m} = \frac{2\epsilon q - 2\epsilon}{q^m} = \frac{2\epsilon(q-1)}{q^m},$$

tending to 0 for $\epsilon > 0$. So there exists a sequence of decreasing and positive widths and thus of infinitely many nested sets: it follows readily that every set $\mathcal{D}[m]$ of strict order $m < +\infty$ has zero Lebesgue measure. The limit (2.1.1) never vanishes identically. Hence every set $\mathcal{D}[m]$ has positive Hausdorff dimension. □

Following the previous lemma 2.7.2 and our previous discussion on Liouville numbers, we can definitely apply the notion of regular sets to Diophantine irrationals of any strict order κ:

Proposition 2.7.1. *Diophantine irrationals of any strict order $\kappa < +\infty$ can be set as fundamental points of the limit set F in the regular sets model.*

2.8 Hunting with Liouville and Hausdorff

In the end of section 2.1, we quoted a lemma stating that the s-dimensional Hausdorff dimension $H^s(E)$ of a set E is 0. We will go over the definition of $H^s(E)$, in order to settle the analogy between the regular sets model and the set of Liouville numbers. Let (X, d) be a metric space. Given $E \subset X$, the diameter and the radius of E is defined as follows:

$$\mathrm{diam}(E) := \sup_{x,y\in E} d(x,y), \quad \mathrm{radius}(E) := \frac{\mathrm{diam}(E)}{2}, \tag{2.8.1}$$

where $d(x, y)$ is the distance between two points $x, y \in E$. Let the operator

$$H^s_\delta(E_n) = \inf\left\{\sum_{n=0}^{\infty}\left(\mathrm{diam}(E_n)\right)^s\right\}, \tag{2.8.2}$$

where $E_n \subset X, \bigcup_{n=0}^{\infty} E_n \supset E, \mathrm{diam}(E_n) \leq \delta, n \geq 0$, where E_n are again pairwise disjoint. Hence the *s-dimensional Hausdorff measure* can be formulated as follows:

$$H^s(E) = \lim_{\delta\to 0^+} H^s_\delta(E). \tag{2.8.3}$$

In order to meet the topological and metric conditions (2.8.1) for computing the Hausdorff dimension of $\mathbb{L}$, we illustrate some concepts involving the Hausdorff measure first. Let $E \equiv \mathbb{L}$ and $X \equiv \mathbb{R}$, in fact $\mathbb{L} \subset \mathbb{R}$ and $\mathbb{R}$ is endowed with

Set Theory				**Number Theory**
Regular sets	$\leftrightarrow$	*Nesting process*	$\leftrightarrow$	Decreasing modulus
Exclusion sets width	$\leftrightarrow$	*Approximation*	$\leftrightarrow$	Rational approximants
Fundamental points	$\leftrightarrow$	*Limit set*	$\leftrightarrow$	Cremer value

the Euclidean metric. The definition of s-dimensional Hausdorff measure can be applied to $\mathbb{L}$ via regular sets E_n. The lower bound of (2.8.2) is again 0 if one rewrites it into:

$$H^s_\delta(E) = \inf\left\{\sum_{n=0}^{\infty}\left(\frac{\text{diam}(E_n)}{2}\right)^s\right\} = \inf\left\{\sum_{n=0}^{\infty}\Big(\text{radius}(E_n)\Big)^s\right\} = 0. \tag{2.8.4}$$

With regard to the Liouville model, where E_n are nested discs, all centered at θ and whose radius is the distance from the Liouville θ to p_n/q_n, the n^{th}-convergent located on ∂E_n (see fig. 3.2.1). So we can merge (3.2.3) with (2.8.4) and obtain this new version of the latter:

$$H^s_\delta(\mathbb{L}) = \inf\left\{\sum_{n=0}^{\infty}\left|\theta - \frac{p_n}{q_n}\right|^s\right\}; \tag{2.8.5}$$

it turns out that the Hausdorff measure of $\mathbb{L}$ is 0 according to the limit (2.5.1) which holds at each point of $\mathbb{L}$:

$$H^s(\mathbb{L}) = \liminf_{|\theta - \frac{p_n}{q_n}|\to 0^+}\left\{\sum_{n=0}^{\infty}\left|\theta - \frac{p_n}{q_n}\right|^s\right\} = 0. \tag{2.8.6}$$

The definition of Hausdorff dimension d_H is in general as follows,

$$d_H(E) = \inf\{s|H^s(E) = 0\} \quad or \quad d_H(E) = \sup\{s|H^s(E) = \infty\}. \tag{2.8.7}$$

The formula (2.8.6) satisfies the left-hand side of (2.8.7), so that the second part of lemma 2.1.1 at p. 25 can be restated:

Theorem 2.8.1. *The Hausdorff dimension d_H of set $\mathbb{L} \subset \mathbb{R}$ of Liouville numbers θ is zero: $d_H(\mathbb{L}) = 0$.*

From (2.8.5) and the theory of regular sets, $E_n \to E \equiv \mathbb{L}$. For theorem 2.8.1, $\mathbb{L}$ consists of *isolated points* and it is *totally disconnected*. There is a respected and wide literature about $\mathbb{L}$ (belonging to the most general family of Cantor-like sets). See [70] by Olsen and Renfro for deeper results on the metrics of $\mathbb{L}$: the dimension functions there introduced show that $\mathbb{L}$ enjoys the curious property of not having a positive finite measure relative to any generalized Hausdorff measure. We also suggest the joint works by Cabrelli, Molter at Alia [22, 23, 24, 41]. With no loss of generality, our discussion gets easier if just one sequence of E_n, converging to one limit point $\theta \in E$, is assumed at once: approximating one Liouville number finds to be analogous to the regular sets convergence towards a totally disconnected limit set (see table 2.8).

3 Towards the equivalence of models

In this section we deepen the analogies between the topological model of regular sets and the Liouville numbers definition.

3.1 Faster than polynomials

Both $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ are dense. Since $\mathcal{D}(\infty) \subset \mathbb{R}\backslash\mathbb{Q}$ is of full Lebesgue measure on $\mathcal{I} \equiv [0,1]$ ([64], p. 120), it is also dense in $\mathbb{R}$; so $\mathbb{L} \equiv \mathbb{R}\backslash(\mathbb{Q} \cup \mathcal{D}(\infty))$ is co-dense in $\mathbb{R}$. From Transcendental Numbers Theory, $\mathbb{L} \subset \mathbb{R}$ is everywhere-dense set. Because of $\mathcal{D}(\infty)$ is the complement of a zero measure set and $\mathbb{L}$ is the complement of a first category set, each set has the cardinality of continuum in every interval of $\mathbb{R}$. So $\mathbb{L}$ is uncountable too. In terms of regular sets, the second question we posed at p. 22 amounts to inquiring about the possibility of moving along the sequence of nested sets E_h up to $E \equiv \mathbb{L}$. It comes natural to wonder

Question 3.1.1. *What are the conditions governing the possibility of finding a sequence of nested sets E_h which converge to the zero measure set E of fundamental points, so that $E \equiv F \equiv X\backslash \bigcup E_h$ holds?*

We first need to recall this theorem obtained by Paul du Bois-Reymond in his researches on convergent and on divergent series ([33], Ch. IV or [11], p. 16 and [38]):

Theorem 3.1.1 (du-Bois Reymond). *Given a sequence of positive and monotonically increasing functions $\varphi_h(n)$ of polynomial type, $h \to \infty, n \in \mathbb{N}^+$ at arbitrarily high speed, there always exists another such function $\varphi(n)$ growing faster than the sequence $\varphi_h(n)$.*

Owing to its monotonic and increasing behavior, φ is order-preserving (isotone) and $\varphi : \mathbb{R} \to \mathbb{R}$. Now we can exploit the shrinking sequence of nested sets in the regular sets model. Consider the exterior of the discs in fig. 3.2.1: this is the sequence of pairwise disjoint sets E_h for $h = 1, 2, \ldots$, called *exclusion sets*; analogously, if the interior is assumed, we speak of *inclusion sets* I_h (depicted in shades of grey). From this same figure, the visual analogy with concepts in Diophantine Approximations is straightforward. We can drop the Lebesgue assumption that $E_n \cap E_{n+1} \equiv \emptyset$ and get close to Borel's more intuitive viewpoint:

$$E \equiv F \equiv X \backslash \bigcup E_h \equiv \bigcap_{h=1,2,\ldots} I_h.$$

Inclusion sets can be restated as the *intersection* $I \equiv \bigcap\limits_{h=1,2,\ldots} I_n$, while exclusion sets are the *union* $E \equiv \bigcup\limits_{h=1,2,\ldots} E_n$; the actions of inclusion sets and of exclusion sets suggest the approximation rules followed by classes of Liouville numbers and the topology of Diophantine irrationals of different order κ respectively. According to theorem 3.1.1, there are no finite upper bounds for the speed of the family $\mathcal{P}$ of polynomials $\varphi(n)$: given any subfamily of $\mathcal{P}$ and *whose speed is rated by a finite ordinal* $|s_h| < +\infty$, one can find another such subfamily whose speed is rated by a finite ordinal $|s_{h+1}| > |s_h|$. Along with the polynomial formula $\sum_{i=0}^{k} a_i x^i$, one understands that s_h is determined by the magnitude of the coefficients a_i. This reinforces the "relative" character of polynomial speed, because faster and faster rates when a_i are arbitrarily large ordinals but of finite magnitude. In the next few pages, we will investigate on the existence of "*absolutely fast*" formulas. Because of the 'ongoing' construction of regular sets, it is easy to apply the theorem 3.1.1, in order to deduce the following:

Corollary 3.1.2. *Let $|s_1|$ be the speed rate for shrinking, nested sequence of exclusion sets E_{h_1}. There exists another sequence E_{h_2} with speed rate $|s_2| > |s_1|$ and* *which decreases faster. In general, one has*

$$\text{radius } (E_h) < \frac{1}{\varphi_h(n)}, \tag{3.1.1}$$

where $\varphi_h(n)$ is a sequence of polynomial maps in the form $\sum_i^{d<+\infty} a_i x^i, a_i \in \mathbb{R}$.

The width of *inclusion sets* I_h gets smaller as $\varphi_h(n)$ increases. Let the sequence of points $a_n \in D_h \equiv \partial E_h$ (fig. 3.2.1). Given $a_n \in \mathbb{Q}$, we find a more comfortable version of (3.1.1) as the width of I_h are expressed in terms of θ:

$$|\theta - a_n| < \frac{1}{\varphi_h(n)}, \qquad \lim a_n = \theta, \tag{3.1.2}$$

or

$$\|q_h\theta\| < \frac{1}{\varphi_h(n)} \tag{3.1.3}$$

holds for an a priori determined sequence of indexes n or, equivalently, for a sequence of functions φ_h (we need to go over the nature of φ_h and investigate their coefficients, according to the remarks in section 2.3).

The polynomial φ_h is the so-called *approximation function* (see [30], p. 14) and θ is *φ-approximable*. After the discussion about $d_H(\mathbb{L}) = 0$, we will strengthen the connection between the metrics of $\mathbb{L}$ and the regular sets model via Khintchine's theorems (p. 26). Linking the relations (3.1.1) and (3.1.3) together with (2.2.1), the regular sets model leads back again to Khintchine's theorems and to this formula approximating the Liouville numbers:

$$\|q_n\theta\| < \frac{\psi(q_n)}{q_n},$$

where, according to section 2.3, $\psi(q_n)$ is convergent. Since we need to investigate the nature of the functions approximating θ, we focus on $\psi(q_n)$. According to Khintchine theorems, we assume they converge, so that $\sum_{q=1}^{\infty} \psi(q)$ converges too. Therefore we can set

$$\psi(q) \simeq \frac{1}{\varphi(n)} \quad \Rightarrow \quad \sum_{q=1}^{\infty} \psi(q) \simeq \sum_{h=1}^{\infty} \frac{1}{\varphi_h(n)}$$

in (3.1.3) for the regular sets model (which is assumed to converge). *In these new terms, we turned the problem of reaching $\theta \in \mathbb{L}$ via rational approximants into the equivalent investigation on the nature of the functions φ_h.* While the sum $\sum_{q=1}^{\infty} \psi(q)$ is assumed to deal with one *a priori* determined sequence of denominators q_n and ψ. Now we want to study the sequence for different functions $\varphi_h(n)$ where, after giving just one input value n, the sequence of convergents is retrieved.

From another equivalent viewpoint, the behavior of the convergent sequence of summands $\sum_{q=1}^{\infty} \psi(q)$ is analogous to $\sum_{h=1}^{\infty} 1/\varphi_h(n)$. Since $q, n \in \mathbb{N}^+, q \geq 0$, regardless of the values, we choose q_n to be input instead of n and switch from 1 to ϵ in the numerator. In fact, from Number Theory, we know the latter cannot be a constant for Diophantine irrationals of order $\kappa \geq 3$. We find that

$$\|q_h\theta\| < \frac{\epsilon}{q_h\varphi_h(n)}, \qquad \psi(q_h) \simeq \frac{\epsilon}{\varphi_h(n)}. \tag{3.1.4}$$

The relation (3.1.4) comes straight from the following theorem (cf. [58], p. 21):

Theorem 3.1.3. *Let $\{p_n/q_n\}$ be the sequence of principal convergents to $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and let φ be an increasing function ≥ 1, such that for all n sufficiently large,*

$$\frac{\epsilon}{q_n^2\varphi(q_n)} \leq \|q_n\alpha\|. \tag{3.1.5}$$

Then α is said to be 'of type $\leq \varphi$'.

The *type* $\leq \varphi$ refers to the existence of a solution of the inequality $\|q\alpha\| < 1/q^2$ for all sufficiently large numbers B, assumed $B/\varphi(B) \leq q < B$.

The φ-sequence $\varphi_h(n)$ in the left-hand inequality of (3.1.4) may be replaced by the q-sequence $\varphi(q_n)$. Both sequences converge, according to the above remarks, and can be intended as different viewpoints on the approximation of Liouville numbers, according to the consequences from Khintchine's theorem at the end of section 2.3. Thus $\varphi_h(n) \simeq \varphi(q_n)$. Combining (3.1.4) with (3.1.5) for $\theta \in \mathbb{L}$, we obtain:

$$\frac{\epsilon}{q_n^2\varphi(q_n)} \leq \|q_n\theta\| < \frac{\epsilon}{q_n\varphi(q_n)}.$$

We know

$$\liminf_{i\to\infty} \|q_{n+i}\theta\| = 0 \tag{3.1.6}$$

for $\theta \in \mathbb{L}$: the sequence continues until $\epsilon = 0$ is reached. Theorem 3.1.3 is susceptible to recursive application: while q_n grows, one has

$$\frac{\epsilon}{q_{n+1}^2 \varphi(q_{n+1})} \leq \|q_{n+1}\theta\| \leq \frac{\epsilon}{q_{n+1}\varphi(q_{n+1})}.$$

The left-hand side can be used to define q_n *recursively* in the following manner

$$q_{n+1} = q_n^2 \quad \Rightarrow \quad q_n = q_1^{2^n} \quad \Leftarrow q_n = (((q_1^2)^2)^2)^{\cdots},$$

and we obtain this double inequality:

$$0 \leq \frac{\epsilon}{q_1^{2^{n+1}} \varphi(q_{n+1})} \leq \|q_{n+1}\theta\| \leq \frac{\epsilon}{q_1^{2^n} \varphi(q_{n+1})}. \tag{3.1.7}$$

The meaning of 3.1.7 is two-fold: first it shows that the left fractions shall vanish identically for p_n/q_n to become $\theta \in \mathbb{L}$. In addition, the *double nature* of the sequence $\{q_1^{2^{n+1}} \varphi(q_{n+1})\}$ implies two ways for $\|q_n\theta\| = 0$ to hold: one via the growing factors q^{2^n+1} which shall be *maximal*, or via the approximating function φ acting as *the fastest one*. We will later discuss the details of these two conclusions. According to the previous remarks on non-effectiveness (p. 32), it is also worth investigating on the fate of ϵ, which shrinks to 0 as q_n grow. In terms of indexed sequences,

Question 3.1.2. *Does it make sense to admit the existence of a maximal index* n *so that* (1) ≤ 0 *and the sequence stops,* (2) $\epsilon = 0$ *because of the magnitude of* q_n *and* (3) $\theta \in \mathbb{L}$ *is finally reached by* p_n/q_n *?*

Before attacking this question, we show that the proposition 2.5.1 at p. 33 holds: *there exist infinitely many solutions of* (2.5.1) *and which get arbitrarily close to a Liouville number* θ. The 'arbitrary closeness' property cannot apply to Diophantine irrationals, because of (2.1.1).

Given the index set I, let $\{S_i : i \in I\}$ be the collection of intervals such that the width of S_i is $\|q_n\theta\|$ and S_i includes finitely many solutions of $\|q\theta\|$. At the largest extent of their whole sequence, the approximants p_n/q_n shall run towards $\theta \in \mathbb{L}$, according to the *recursive inequality*[24] (3.1.7). The union

[24] Resulting from 3.1.3 and the Khintchine's convergence theorem 2.2.1.

$U_S = \bigcup_{i=1}^{\infty} S_i$ of infinitely many nested intervals includes infinitely many solutions of $\|q\theta\|$. For the following corollary, solutions spread arbitrarily close to θ ([58], p. 29):

Corollary 3.1.4. *Let $0 < a \leq 1$ and let $\rho(t) = at$. Then $\lambda(N)$ is the number of the ordered pair integers (p, q) satisfying*

$$0 < \|q\alpha\| < a, \qquad 1 \leq q < N,$$

where

$$\lambda(N) = \rho(N) + O(N).$$

As $n = 1, 2, \ldots,$ q_n grow and the equi-distribution of the solutions of (2.5.1) sharpens within the interval $[0, \theta] \subset [0, 1]$. Now we draw another straightforward consequence from the reading of (3.1.7):

Proposition 3.1.1. *The existence of sequences of approximants $\{p_n/q_n\}$, converging arbitrarily close to irrational numbers, can be effective. Liouville numbers θ can be reached when the sequence behavior is tuned to sufficiently fast approximating functions $\varphi(q_n)$ or sufficiently large q_n.*

Our further discussion will score two goals: *showing the type of Liouville numbers θ and determining the behavior of the sequence $\{p_n/q_n\}$*. The double inequality (3.1.7) gives the clue: the q_n need to be sequenced at an *increasing exponential rate*, faster than the speeds induced by the coefficients a_i of the polynomial in the form $\sum_{i=0}^{d<\infty} a_i z^i$.

3.2 Analogy with Topology

The construction of the regular sets model is independent[25] from the rationality or the irrationality of a and of θ ([11], p. 15). Both $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ are dense in $\mathbb{R}$: this opens to the plausibility of finding approximants at any arbitrarily small

[25]Borel also wrote ([11], p. 15): '*Therefore, in order to study the regular sets of zero measure, whose fundamental points are dense over one domain, one could, with no loss of generality, suppose that such points are, rational coordinates* [A/N: or, equivalently for our purposes, 'irrational']*; then the study is reduced by the applications of the properties related to continued fractions*'.

distance to the irrational value. This independence relates to the structure of the sets $\mathcal{D}(\kappa)$ of Diophantine irrationals, as shown in section 2.7.

For our purposes and because $\mathbb{Q}$ is the numerical environment for computers – whether rationals are good approximation of irrationals or not – we will apply this model through the rational approximations a of θ (see fig. 3.2.1). It

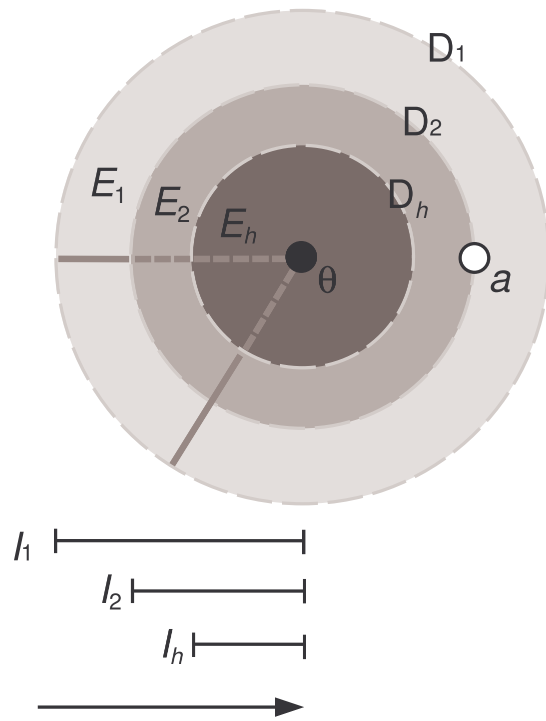


Figure 3.2.1: **Analogy with Liouville's definition.** Using (3.1.2), one model to approximate θ, assumed as the fundamental point attained via exclusion (E_h) and inclusion sets (I_h), represented as nested discs. As the external width (full grey line) grows, the internal width (dotted grey line) shrinks to zero.

is straightforward to move to the analogous classic definition of Diophantine irrationals by Liouville:

$$\liminf \|q\theta\| > \frac{\epsilon}{q^{\kappa}}, \quad p, q \in \mathbb{N} \qquad n \geq 1, \kappa \geq 2. \tag{3.2.1}$$

The width of each E_h can be represented by the distance in the left member of (3.1.2) or (3.2.1), and the equivalence is set. In light of corollary 3.1.2 and after assuming the right-hand expression of (2.5.1) in terms of (3.1.1), one can look at the operator '$\leq$' in (2.5.1) and deduce that

Proposition 3.2.1. *Given a decreasing sequence of values ϵ, a Liouville number admits the existence of a sequence of approximants p_n/q_n and which is faster than any sequence of rational approximants ϵ/q^κ converging to θ.*

If the lower bound on the right is > 0, (3.2.1) holds. If it decreases asymptotically to 0 or finally vanishes identically, (2.5.1) holds instead. The limit shall be considered from the opposite side as one looks at the sequence (refer to (3.1.2)) and at the nested circles D_n, from the shrinking perspective of the problem. Let the boundaries ∂D_n be located at a sequence of rational points $a_n = p_n/q_n$ (fig. 3.2.1). It is easy to combine (3.1.2) with (2.5.1) and obtain:

$$\|q\theta\| \leq \frac{\epsilon}{\varphi_h(n)} \simeq \frac{\epsilon}{q_n^\kappa}, \tag{3.2.2}$$

where $\epsilon/q_n^\kappa, \kappa = 1, 2, \ldots$ decreases with $\epsilon/\varphi_h(n)$, as $h = 1, 2, \ldots$ and in conformity to (3.1.5). Liouville's formulation applies when the left modulus (the distance between θ and p_n/q_n) is 0, enjoying again a straightforward similarity with the limit (3.1.6):

$$\liminf \|q\alpha\| = 0. \tag{3.2.3}$$

Elementary Calculus suggests that (3.2.3) vanishes identically as $\varphi_h(n) \equiv \infty$. Through a later refinement of concepts, we will come to a sharper and more useful result. *Both the approximation of Liouville numbers θ (in numerical terms) and the display of non-linearizable hedgehogs (in graphical terms) are equivalent problems relating to the speed and to the coefficients of $\varphi_h(n)$ speed, tied by the quest for the reaching of θ through arbitrarily large q.*

4 On the speed of approximants

4.1 Kicking to the limit

In his work [11] on regular sets, Émile Borel suggested to push the du-Bois Reymond theorem to the extreme consequences. Let F be the dense (limit) set of fundamental points. Borel stated that there exists no sufficiently fast sequence of $\varphi_h(n)$ for $E \equiv F \equiv X \backslash \bigcup E_h$ to hold. Borel observed that regular sets R are maximally dense in the sense of cardinality when F is also dense ([11], p. 15):

Proposition 4.1.1. *Every regular set of zero measure, whose fundamental points are dense in a domain, has the cardinality of the continuum. In other words, if the decrease rate of the exclusion squares* [[26]] *around the fundamental points is arbitrary, it is not possible for such decrease to be fast so that the fundamental points are the only points of the set.*

Refer to footnote 25. Because of the collection of fundamental points is countable, Borel's result implies that it is not possible for the decrease to be fast enough to leave only the fundamental points. The collection of non-fundamental points that are left will also have cardinality of the continuum in every open disk contained in the domain.[27] We can restate it in terms of rational approximants:

Proposition 4.1.2. *No decreasing sequence of regular sets that share the same set of fundamental points can have an intersection (i.e. the limit of the regular sets) that is equal to the set of fundamental points.*

These remarks on how to not reach ∞ and on the possibility of reaching $\theta \in \mathbb{L}$ echo Borel's words zero measure sets $E \equiv F$. After these two quite not-so-promising statements, Borel opened to a possibility: he barely suggested that this goal can be only achieved via a '*transfinite infinity*' (in his original terminology) of inclusion sets I_h. The next sections are devoted to the sense of this clue and to related developments, relying on remarks about approximating functions which are faster than polynomials, in particular about the *(multi-)exponential speed*, because of the double inequality (3.1.7).

4.2 Transfinite induction

Several remarkable results on *well-ordered sets* are collected into modern Set Theory. The advancements during late XIXth century, inspired George Cantor to look far beyond countability and finite ordinals. Finally he built up a robust corpus of rigorous concepts and contributed for Set Theory to be more coherent. Cantor needed to trespass the frontier of *finite* numbers and came to the concept of *transfiniteness*, as coined by him. The early application was

[26]For our purposes, let them be *circles*; read the caption of fig. 2.6.1.

[27]Today we understand it as the fact that every G_δ set that is dense in a domain has cardinality continuum in every open disk contained in that domain.

to numbers, giving rise to the concept of transfinite values, i.e. *cardinal or ordinal numbers which are larger than all finite numbers and not necessarily be absolutely infinite*: Cantor's goal was in fact to define the largest numerical nature which, unlike the infinity concept, could share some of the well-known computability properties of finite ordinals, in order to rely on a solid mathematical basis. transfiniteness later involved relations and Logic: it is the so-called procedure of *transfinite induction*, where the principle of induction extends to well-ordered sets including transfinite elements:

Proposition 4.2.1 (Principle of induction). *Let S be a well-ordered set, and $A \subseteq S$. If $y \in A$ for every $y < x \in S$, then $x \in A$.*

Let S be a well-ordered set with maximal element x, so $y < x$ for all $x, y \in S$. If the property $\mathcal{T}(y)$ (say, '*y belongs to the set A*') holds for all $b < x$, $\mathcal{T}(x)$ (say, '*x belongs to the set A*') holds too. Formally:

$$\forall x(\forall y(y < x \Rightarrow \mathcal{T}(y)) \Rightarrow \mathcal{T}(x)) \Rightarrow \forall x(\mathcal{T}(x)).$$

Extending a property $\mathcal{T}$, from a well-ordered subset A to the maximal element of the related superset S, is a procedure of the three cases:

1. *zero case*: proving that $\mathcal{T}(0)$ holds, where 0 is the minimal element of a well-ordered set.

2. *successor case*: proving that, given one ordinal y and its successor $y + 1$, where $0 < y < y + 1 < \ldots$, $\mathcal{T}(y)$ holds and $\mathcal{T}(y + 1)$, following from $\mathcal{T}(y)$, also holds. One notices the recursive character.

3. *limit case*: proving that for any limit ordinal γ, $\mathcal{T}(\gamma)$ holds and follows from $\mathcal{T}(y)$ for all $y < \gamma$.

One speaks of *transfinite induction* when, assuming that both zero and successor cases apply to finite ordinals, the nature of the limit ordinal is transfinite.

4.3 transfiniteness for Γ sequences

The relation (3.2.3) holds, depending on these special conditions:

1) if the index h is *larger than any finite ordinal* (in terms of amount of $\varphi_h(n)$ in the sequence);

2) or if the input variable n or the polynomial coefficients are *larger than any finite ordinal* (in terms of the speed rate for each function element $\varphi(n)$ in the sequence).

We are going to focus on 1) now. Exploring the sequence Γ of approximating functions involves a systematic study of the set of indexes h. As we look at this condition from the operative viewpoint, i.e. at how the sequence works, it is clear that we need to focus on the numerical nature of h, which affects the approximation performance.[28] On the other hand, 2) belongs to the nature of $\varphi_h(n)$, whose discussion will naturally follow after section 4.8.

Let $h \in \mathbb{N}$ be a finite and countable ordinal. We already know that (3.2.3) cannot hold according to (3.2.2) and to proposition 4.1.2: $\theta \in \mathbb{L}$ cannot be reached by Γ. Since Liouville numbers are accessible by the approximants p_n/q_n, one wants to understand what conditions have to be set up for $\|q\theta\| = 0$ to hold. We showed that it makes sense to discuss $\|q\theta\| = 0$ via $\liminf \|q\theta\| = 0$, which decreases infinitesimally but never vanishes identically if $h \in \mathbb{N}$. Inductively, h is required to assume larger values than any finite ordinal, so to push Γ far beyond the reach allowed by $h \in \mathbb{N}$. One such role is played by transfinite numbers – the extension of finite ordinals, as postulated by Cantor. In our environment, we will deal with either finite and transfinite ordinals, assuming they are both countable. Let $\mathbb{O}$ be the union set of finite ordinals with the least transfinite ordinal ω_0: $\mathbb{O} \equiv \mathbb{N} \cup \{\omega_0\}$. $\mathbb{O}$ is also countable.[29] Let the index $h \in \mathbb{O}$ be a *von Neumann integer.* [30]

Alike $\mathbb{N}$, let $\mathbb{O}$ be a *zero-start* set, i.e. where $\min(h) = 0$. The indexing role of h is an ordering relation for all elements of $\mathbb{O}$. When applied to $\varphi_h(n)$ for the whole Γ sequence inherits the zero-start property ($\varphi_0(n) = n$). The sequence

[28] This side of the question has been further investigated in the next section.

[29] Both components, $\mathbb{N}$ and ω_0, are countable.

[30] In this model, any well-ordered set is isomorphic to a von Neumann ordinal. One refers to the *von Neumann cardinal assignment* and for the zero-index property associated to cardinality of the subsets of $A_h \subseteq \mathbb{O}$: here for a well-ordered set U, we define its cardinal number to be the smallest ordinal number equinumerous to U. Von Neumann integers naturally apply the transfinite induction to generate a succession of supersets with growing size.

of h is ruled by this incremental relation:

$$h_{n+1} = h_n + 1, \tag{4.3.1}$$

where $h, n \geq 0$. From the hypotheses on the construction of Γ, (4.3.1) holds for any index $h \in \mathbb{O}$. The values h_n and h_{n+1} are said *predecessor* and the *successor* respectively ([35], p. 5), because of

$$h_n < h_{n+1}.$$

From (4.3.1), one finds $h_0 = 0, h_1 = 1, \dots$ and in general $h_k = k$, where k is the maximal element $M_h = k = h_k = \max(h)$. In this beginning, we just let A be a subset of finite ordinals $h \in A$, thus $A \subset \mathbb{O}$ and

$$\max(h) = h_k \equiv \mathrm{Card}(A) < \mathrm{Card}(\mathbb{O}).$$

Since $\min(h) = 0$ and A is *ordered* by the relation '$<$', A is a *well-ordered set*. We need to prove that this ordering relation extends *naturally* to '$\leq$' in $\mathbb{O}$. It is sufficient to refine the definition of *maximal element* M_A of a given well-ordered set A: if $M_A = \max(A)$ for any subset $A \subset \mathbb{O}$, then $\forall h \in A : h \leq M_A$. Given $M_{A_n} = \max(A_n)$, if we consider the sequence of nested subsets $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \dots$, then

$$M_{A_1} \leq M_{A_2} \leq \cdots \leq M_{A_n} \leq \dots \ . \tag{4.3.2}$$

One notices that all M_{A_n} are *relatively maximal* elements for the chain of nested subsets A_n. The existence of relatively maximal elements is equivalent to the possibility of nesting one set $A_{n>0}$ into another A_{n+1}. Now let this fundamental proposition in General Topology:

Proposition 4.3.1. *Every well-ordered set is order isomorphic to a unique ordinal number.*

Given a well-ordered set A with maximal element at M_A, there exists an ordinal number counting the cardinality of A; but A is also zero-start by hypothesis and $\mathrm{Card}(A) \equiv \max(h) \equiv h_k$ holds. Recall the Axiom of Choice (AC) ([35], p. 8):

Proposition 4.3.2 (The Axiom of Choice). *For every family $\{X_s\}_{s\in S}$ of non-empty sets, there exists a function f from S to $\bigcup_{s\in S} X_s$ such that $f(s) \in X$ for every $s \in S$.*

Here φ is f, $S \equiv \mathbb{O}$ and $X \equiv \Gamma$. Our construction requires (AC) to show that $\mathbb{O}$ is *linearly ordered* ([35], p. 4) and, more strongly, that $\mathbb{O}$ is *well-ordered set*. In fact, given any triplet of distinct elements h_{n-1}, h_n, h_{n+1}, one has $h_{n-1} < h_n < h_{n+1}$. Moreover, every non-empty subset of $\mathbb{O}$ has a smallest element ([35], p. 5). Because of $\mathbb{O} = \mathbb{N} \cup \{\omega_0\}$ and it is zero-start and well-ordered, $\mathbb{O}$ is an *infinite Dedekind set* and enjoys these interesting properties:

a) if $\eta \in \mathbb{O}$ is transfinite, $\text{Card}(\mathbb{O}) \equiv \eta$. According to the previous discussion, this property extends via transfinite induction, the concept of maximal element from finite to transfinite ordinals. While $\max(\mathbb{O}) = \omega_0$ in terms of ordinals, $\text{Card}(\mathbb{O}) = \aleph_0$ in terms of cardinals.

b) $\eta + 1 = \eta$. This relation is consistent with the construction of $\mathbb{O}$ and of Γ: the least transfinite ω_0 is either *the maximal element* for Γ and *the natural limit* for Γ. Transfinite ordinals give rise to a mismatch between the cardinality and the number of elements in Γ anyway, according to the Burali-Forti Paradox. As shown in the next section, the paradox cannot extend to the Γ-type sequence of $\varphi_h(n)$;

c) $\aleph_0 \leq \eta$: we showed that $\eta \equiv \omega_0$ and its cardinality is $\aleph_0$;

d) there exists a cardinal number t, so that $\aleph_0 + t = \eta$: this follows from joining *b*) to *c*).

Now let the Kuratowski-Zorn lemma ([35], p. 8):

Lemma 4.3.1 (Kuratowski-Zorn). *If for each linearly ordered subset A of a set X ordered by $\leq$ there exists an $x_0 \in X$ such that $x \leq x_0$ for every $x \in A$, then X has a maximal element.*

Since $\mathbb{O}$ is well-ordered and, for.the above lemma, there exists one maximal element in $\mathbb{O}$ so that the relation '$\leq$' applies to $\mathbb{O}$. The lemma can be restated as follows, after dropping the strict inclusion with no loss of generality:

Corollary 4.3.2. *If for each linearly ordered subset $A \subseteq X$, X has a maximal element for A, then*

- *if $A \subset X$, there exists an $x_0 \in X$ such that $x < x_0$ for every $x \in A \subset X$ and X is ordered by $<$.*
- *if $A \subseteq X$, there exists an $x_0 \in X$ such that $x \leq x_0$ for every $x \in A \subseteq X$ and X is ordered by $\leq$.*

Also this corollary applies to $\mathbb{O}$.[31] As we link the Kuratowski-Zorn lemma to properties $b), c), d)$, we can show that the maximal element in $\mathbb{O}$ is a countable transfinite ordinal, $M_{\mathbb{O}} = \max(\mathbb{O})$; in particular, according to Cantor's theory, $M_{\mathbb{O}}$ is the *least* transfinite countable value, conventionally defined as ω_0. According to the same properties, ω_0 is the maximal element of $\mathbb{O}$. Since $\mathbb{N} \subset \mathbb{O}$ and (AC) applies to $\mathbb{O}$, the pair $(\mathbb{O}, \leq)$ defines *a well-ordered set of ordinals where the maximal element is not a finite but a transfinite ordinal.* The well-orderedness of $\mathbb{O}$ allows to apply the transfinite induction to sets with cardinality $\text{Card}(A) \equiv h_k$ and to $\mathbb{O}$, where $\text{Card}(\mathbb{O}) \equiv \aleph_0$. Therefore $\mathbb{O}$ *is a countable set with transfinite maximal element.* Comparatively, the sequence Γ shall be a *countable transfinite sequence of approximating functions* too, and one obtains this remarkable equivalence:

$$\text{Card}(\mathbb{O}) \equiv \text{Card}(\mathbb{N}) \equiv \aleph_0 \equiv \text{ Card } (\mathbb{Q}). \tag{4.3.3}$$

Again, the transfinite induction of the property $\mathcal{T}(h) : h \leq M_{A_n}$ extends from subsets A_n of finite ordinals to $\mathbb{O}$, including both countable finite and transfinite ordinals. The same properties attest that $M_{A_n} \leq M_{\mathbb{O}}$. So the previous chain (4.3.2) definitely closes as follows

$$M_{A_1} \leq M_{A_2} \leq \cdots \leq M_{A_n} \leq \cdots \leq M_{\mathbb{O}}. \tag{4.3.4}$$

So $h_k \in A \subseteq \mathbb{O}$ for $h \leq \omega_0 \equiv M_{\mathbb{O}} \equiv \aleph_0 \equiv \text{Card}(\mathbb{O})$.

4.4 Remarks on applications and the Burali-Forti paradox

In the last section, we introduced the transfinite sequences of ordinals $h \in \mathbb{O}$ and their properties. Here we illustrate that their application to the topology

[31] This version of the lemma could help to figure out the induction from finite to transfinite ordinals, since it finds a topological analogy which splits into two cases, depending on whether the inclusion is strict or not.

induced by indexed and transfinite sequences of contracting maps φ_h cannot be affected by the Burali-Forti paradox.

The Burali-Forti paradox brings in the evidence of a contradiction in the construction of the set of all ordinal numbers. The latter are either used to describe the size of a sequence and to indicate the location of an element within. More rigorously, ordinal numbers represent equivalence classes of well-ordered sets with order-isomorphism being the equivalence relationship itself. Since the set Ω of all ordinals enjoys the properties of the ordinal number, Ω can be assumed as the ordinal number itself. (This is also the case for the set of ordinals indexing our sequence of contracting maps φ_h.) The successor element $\Omega + 1$ is strictly greater than Ω. But $\Omega + 1$ is again an element of Ω, because Ω includes all ordinals; stated in terms of von Neumann ordinals (see p. 50), Burali-Forti came to the contradiction:

$$\Omega < \Omega + 1 \leq \Omega. \tag{4.4.1}$$

The paradox originates from a property which applies *recursively* from the elements to the whole set itself. One such kind of recursion affects Γ-type sequences too. For example, the set of all functions φ_h is a new φ-function: $\varphi(\varphi(\varphi(\ldots\varphi(n)))) \equiv \varphi_{h>0}(n)$. The central question is to check if the paradox may hold for φ_h or not.

Let $h \in \Omega$ be the index indicating univocally each function φ_h of the well-ordered sequence. Any element of Γ indicated by the pair (φ, h). The original double inequality (4.4.1) turns into the double topological relation

$$\varphi_\Omega \supset \varphi_{\Omega+1} \subseteq \varphi_\Omega \tag{4.4.2}$$

for Γ-type sequences. The limit (2.5.1) for Liouville numbers, together with the regular sets model, satisfies the left-hand relation $\varphi_\Omega \supset \varphi_{\Omega+1}$ and the operator '$\subseteq$' in the right-hand side $\varphi_{\Omega+1} \subseteq \varphi_\Omega$. The limit prevents the paradox from holding, because of the following *fixed point property* for approximating functions: in the topological terms of the regular sets model applied to that limit, the Cremer points $\delta \in \mathbb{L}$ are fixed for the whole sequence of nested and shrinking inclusions set $I_{h+1} \subseteq I_h$ (see definition of fundamental points at p. 35 here), i.e. the intersection for all I_h where $h = 1, 2, 3, \ldots$. According to the Brouwer theorem, there exists one fixed point δ for $\bigcap_{h=0}^{\Omega} I_h \equiv \{\delta\}$. Because of δ is a fixed point with period 1 and accessed by Γ when h is transfinite ($\varphi_\Omega(n) \equiv \delta$), the formula (4.4.2) can be equivalently rewritten as follows:

$$\varphi_\Omega(n) \supset \varphi_{\Omega+1}(n) \subseteq \varphi_\Omega(n). \tag{4.4.3}$$

Therefore

$$\delta \supset \varphi[\varphi_\Omega(\delta)] \subseteq \delta.$$

Because of δ is a fixed point, one has

$$\delta \supset \varphi_1(\delta) \subseteq \delta;$$

or the tautology

$$\delta \supset \delta \subseteq \delta,$$

where no contradictions arise. In no strict inclusion terms, every set includes all elements of the set itself. Therefore the one-element set includes itself and it is either subset, superset and equivalent to itself (and it coincides with the universal covering). Although h is ordinal and Γ inherits the properties of Ω, the fixed point property of $\varphi_h(n)$ *naturally* resolves the paradox.[32] This same property does not hold for the set Ω of indexes: in fact, there is no fixed value for the relation (4.4.1).

One further application involves Number Theory, where ordinals $h \in \mathbb{O}$ are applied to the transfinite sequence of real exponential maps q^h, with $q \geq 2$. According to p. 44, on the approximation of Liouville numbers and the required speed for sequences of denominators $\{q^h \varphi_h(n)\}$, we will focus on the sequence $q^1, q^2, \ldots, q^h$ and the consequences drawn in the limit case of transfinite h.

4.5 A higher degree of freedom

We are going to refine our viewpoint on transfinite sequences of approximating functions in order to have a fuller control on these sequences. The following revisitation accomplishes in terms of indexes. As we showed at p. 35, Borel's and

[32] In the classic case of ordinal numbers, modern axiomatic Set Theory resolves the paradox by preventing sets from sharing the same properties as of their elements and does not allow the generation of sets with unrestricted comprehension terms. For example, general (and thus ambiguous) statements in the form "*all sets with the property P*", which involves each ordinal and their entire set, cannot hold.

Lebesgue's constructions are quite similar. They just differ because of Borel assumed that E_h are overlapping. Given a superset X, $X \setminus \bigcup E_h$ with pairwise disjoint sets E_h (Lebesgue), is equivalent to $\bigcap E_h$ with non-empty intersection: $E_h \cap E_{h+1} \not\equiv \emptyset$ and $E_{h+1} \subseteq E_h$(Borel). So theorem 2.6.1 turns into:

Theorem 4.5.1. *Every non-empty open set $E \subset \mathbb{R}$ can be uniquely expressed as a finite or countably transfinite and non-empty intersection of open subsets E_h, E_{h+1} where $E_{h+1} \subset E_h$.*

Both theorems 2.6.1 and 4.5.1 are *recursive*: the roles of E and of E_h *can be played by E_h and by one subsequence E_{h_i} respectively.* The nature of such subsequences will be introduced here, during the two stages graduation of the indexes domain from $\mathbb{N}$ to $\mathbb{R}$. First $h \in \mathbb{N}$ turns into $g \in \mathbb{Q}$, later g into $f \in \mathbb{R}$. We will pull out refined versions of the regular sets model, earning more freedom to walk around the reals.

We recall that Γ is a sequence of approximating functions with integer indexes $h \in \mathbb{O}$ (Γ-type), whereas Λ is an analogous sequence indexed by $g \in \mathbb{Q}$ (Λ-type); finally, Δ is a sequence indexed by $f \in \mathbb{R}$ (Δ-type). Given $h \in \mathbb{N}$, the sequence Γ can just enjoy one degree of freedom: either convergence (incrementing h by 1) or divergence (decrementing h by 1) along the prescribed shrinking [or expanding rate] of Γ [or of Γ^{-1}]. As the indexes domain first extends to $\mathbb{Q}$ (and later to $\mathbb{R}$), the chain of decimals turns the Γ-sequence into the Λ-type and finally into the Δ-type, enjoying the maximal degree of freedom: starting from any real point $\alpha_1 \in \mathbb{R}$, one can reach another real point $\alpha_2 \in \mathbb{R}$ by finely tuning the integer and decimal digits of the index. The shrinking [and the expanding rate] of the Λ-type sequence is arbitrary (see fig. 4.5.1). The sense of such extensions[33] is not to drop but to keep up and sharpen the same fundamental condition for all these sequences to work. One sees Λ-type or Δ-type sequences as disguised versions of the Γ-type: the (ir-)rational form of the indexes can be equivalently deconstructed into subsequences of arbitrarily low depth and through integer sub-indexes h.

Let I be the inclusion set (refer back to p. 40), where the index $h \in \mathbb{N}$.

[33]Similar studies have been developed in the field of Holomorphic Dynamics which relate to sequences of iterated maps with non-integer index: results are collected under the rubric of *Analytic Iteration* or of *Fractional Iteration*, where the sequence of iterations can be assumed to be continuous semi-groups.

Now let a Γ-type sequence. For example, as h increases from 3 to 4, we find the topological inclusion $\varphi_4(I) \equiv I_4 \subset I_3 \equiv \varphi_3(I)$. This generally holds from any h to $h+1$ or, reversely, from $h+1$ to h.

Analogously let $3 \leq g \leq 4, g \in \mathbb{Q}$. For one such Λ-type sequence of continuous φ-functions inside the difference set $I_{g=3} \backslash I_{g+1=4}$, it is easy to check that the index value $g = 3,8$ alludes to the intermediate subset $I_{3,8}$:

$$I_4 \subset I_{3,8} \subset I_3.$$

Because of $3,8 = \mathrm{Int}(3,8) + \mathrm{Frac}(3,8)$, one may conventionally assume the existence of one main sequence and one subsequence: here the integer part (from 3 to 4) points to the main sequence, while decimals $(0,8)$ to the nested subsequence of, say, depth -1. It is a recursive, top down generation process, where the terms '*main*' and '*sub*' shall not be intended in absolute terms. Rather they refer to a sub-dependency relation of arbitrary depth and between two sequences at consecutive depths. Hence we can generate subsequences *recursively* over and over again, downwards to arbitrary depth and decimal n-th place of the (ir-)rational index. For example, at a next stage, let $g = 3,81$. The index g has two decimal places. According to the above topological inclusion,

$$I_4 \subset I_{3,81} \subset I_{3,8} \subset I_3,$$

with two orders of sub-dependency: one main sequence and two subsequences. Again, let $g = 3,79$, then we find

$$I_4 \subset I_{3,81} \subset I_{3,8} \subset I_{3,79} \subset I_3.$$

Moreover we find

$$I_4 \subset I_{3,82} \subset I_{3,81} \subset I_{3,8} \subset I_{3,79} \subset I_{3,78} \subset I_3.$$

One such higher degree of freedom for Γ-sequences fits better and more intuitively (than the Γ-type) the topological properties of the regular sets model, where convergence of inclusion sets *was not supposed to be always uniform.*[34]

[34] Refer back to fig. 2.6.1 at p. 36, which is very close to the sense of fig. 4.5.1. The magnitude of the integer and decimal digits in g, from 0 to 9 at the digits n-th place in g, affects the convergence or divergence speed rate.

Proposition 4.5.1. *Every Λ-type sequence with rational indexes can be assumed as the composition of Γ-type subsequences, each one with integer indexes.*

It is easy to extend this discussion to irrational indexes, when the sequence may finally reach any real value. For (4.6.1), one can state:

Proposition 4.5.2. *Let $\alpha \in \mathbb{R}$ be a limit point. We can resume the types and behaviors of sequences as follows:*

1. *the limit α can be reached via finitely many steps by a Γ-type sequence;*
2. *if $\alpha \in \mathbb{L} \subset \mathbb{R}\backslash\mathbb{Q}$, the limit α can be reached via transfinitely many steps by a Δ-type sequence;*
3. *if $\alpha \in \mathcal{D}(\kappa) \subset \mathbb{R}\backslash\mathbb{Q}, \kappa \geq 2$, the limit α can be infinitesimally approximated by a Δ-type sequence.*

We are going to show the possibility of matching the numerical natures of the indexes with the real limits. The limits for sequences with with rational indexes can be exclusively and sequences with irrational indexes follow the above points 2 and 3.

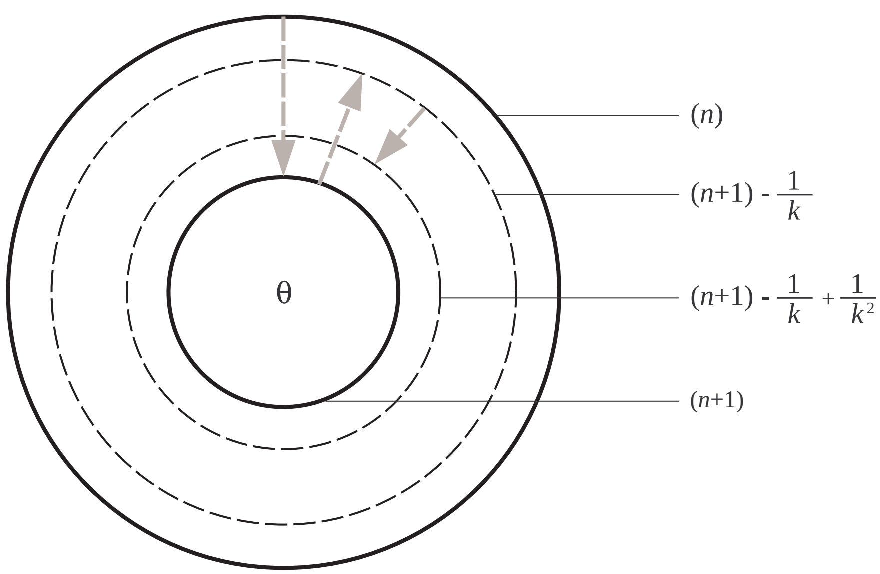


Figure 4.5.1: **Nested domains for sequences with non-integer indexes.**

Any rational (index g) can be computed by this bounded sum:

$$g = \frac{p}{q} \equiv \sum_{i=-c}^{d} \frac{n}{10^i}, \qquad c, d < +\infty, \quad n \in \mathbb{N}, 1 \leq n \leq 9, \quad g \in [0, 1]. \qquad (4.5.1)$$

Given any $i \in \mathbb{Z}$, the summands $n/10^i$ compute the integer and the decimal part of p/q. Since each summand $n/10^i, i > 0$ is again rational and because of $\mathbb{N}$ and $\mathbb{Q}$ are equipotent, we can find an integer (sub-)index h_1 for another Γ-type subsequence. Each level associates to the interval width $L_g = |\varphi_{g_0} - \varphi_{g_1}|$. The speed of one such function is measured by the length L_g covered at each step, i.e. at each variation of the index h or g: for example L_g. The convergence speed rate of subsequences cannot exceed the given L_n. See for example fig. 4.5.1, where we represent two orders of nested subsequences through the black bold and the dashed circles: the indexes, associated to each nested domain, are listed on the right.

It is straightforward that *the smaller* $n/10^i$ *is, the slower* $\varphi_{h_{p_{o_{q\dots}}}}(n)$ (where $h, m, n, o \in \mathbb{N}$) *is.*[35] The nesting rule $I_{h+1} \subset I_h$, for integer indexes, extends to rationals and inductively leads to

$$I_{g+1} \subset I_{g \equiv h + \frac{p_i}{q_i}} \subset I_g, \qquad 0 \leq \frac{p_i}{q_i} \leq 1. \qquad (4.5.2)$$

Let p_i/q_i be a sequence. The domain of the subsequence of inclusion sets $I_{g+\frac{p_i}{q_i}}$ is smaller than the domain of the main sequence. In fact, as it is bounded by two rational values which are topologically represented by the subsets $I_g, I_{g+\frac{p_i}{q_i}}$, according to (4.5.2) and to section 3.

A longer chain of decimals in the index equals to a slower approximating function. Likewise, a longer chain of decimals for a Λ-type sequence to a given limit point can be turned into a Γ-type subsequence with increasing integer indexes and accumulating to the same limit point. Moreover, given a different approximating function, we can replace the Λ-type with only one Γ-type sequence. The speed of $\varphi_{\frac{p_i}{q_i}}$ relates to the amount of decimal places in g. One

[35] In addition, the sign $\pm$ may be applied to denote moving forwards or backwards via Γ, Λ or Δ-type sequences.

also observes that

$$\liminf_{\frac{p_i}{q_i}\to 0} |\varphi_n - \varphi_{n+\frac{p_i}{q_i}}| = 0, \qquad 0 \le \frac{p_i}{q_i} \le 1. \tag{4.5.3}$$

applies recursively for all indexes and sub-indexes $h_{p_{o\ldots q}}$, i.e. to any (sub-)sequence with (sub-)indexes $h_{p_{o\ldots q}}$ and its immediate subsequence $n+\frac{p_i}{q_i}$. Thus

Proposition 4.5.3. *The increasing and decreasing nature of the magnitude of each sub-index relates to the faster or slower speed rate of the related subsequence of approximating functions.*

The final extension applies inductively to Δ-type sequences with real indexes. The possibility for the index g to be *irrational* certainly raises the question on *how slow one subsequence is required to run.* Recall that if a sequence of any type Γ, Λ or Δ, say for example $\Gamma : \{\varphi_h(n)\}$, does reach the given limit, say λ, then the following two conditions are equivalent:

1. in terms of values, $\varphi_h(n) \equiv \lambda$ for a given h;
2. in terms of indexes, the convergence $\varphi_k(n) \to \varphi_h(n) \equiv \lambda$ stops as $\lim h = k$ holds after the sequence of indexes $h = 1, 2, \ldots, k$.

We remark that the indexes sequence is not the Γ-type or the Λ-type sequence; while the former involves indexes exclusively, the latter deals with the values returned by the approximating functions.[36] The reaching or even the approximation of limit values by sequences follow an analogous discussion about the approximation of indexes. The response may optionally rely on the theory of Diophantine Approximations. Here one differs Diophantine from Liouville irrationals.

In this direction, the limit index $s \in \mathbb{R}\backslash\mathbb{Q}$ and Diophantine, the behaviour of Δ-type sequence is ruled by the same conditions, as stated in the theory of Diophantine Approximations. In fact, $f \equiv \frac{p_i}{q_i} \to s$ and $\liminf |s - \frac{p_i}{q_i}| > 0$: given a Diophantine α and $\varphi_s(n) \equiv \theta$, we find again (4.6.1). On the other hand, if

[36] This consideration is not pointless, because indexes and functions may match. For example, in the case $\varphi_h(1) \equiv 1 \cdot h \equiv \varphi(h)$. Otherwise, they may follow the same trend: as the chain of decimals elongates. The approximating functions are applied to smaller and smaller domains.

the limit index $s \in \mathbb{L}$, Δ-type sequence can reach Liouville numbers because of $g \equiv \frac{p_i}{q_i} \to s$ and $\liminf |s - \frac{p_i}{q_i}| = 0$. This is one further evidence of the intimate relation between the numerical nature of the limit and of the dynamical nature of the sequence.[37]

Proposition 4.5.4. *The possibility of reaching to $\theta \in \mathbb{L}$ associates instead to the existence of a subsequence of approximating functions whose speed rate decreases infinitesimally or it is 'absolutely'* [38] *slower than any time-constructable function.*[39]

Roughly speaking, reaching to $\theta \in \mathbb{L}$ requires a transfinite number of approximating functions to compensate the decreasing speed rate; otherwise, finitely many such functions would not be enough.

An approximating function $\varphi_f(h)$ of Δ-type sequence, with $f \in \mathbb{R}$, can reach the limit $\theta \in \mathbb{L}$ when $f \in \mathbb{R}\backslash\mathbb{Q}$. Hence its speed rate decreases infinitesimally to zero as the chain of decimals develops. One would like to distinguish two cases: when the limit index $g \in \mathbb{L}$ and when $s \in \mathcal{D}(\kappa), \kappa \geq 2$, i.e. when the speed rate can decrease to zero or just slow infinitesimally to zero. In both cases, we notice that the limit value is irrational and accessible. The above discussion followed according to the theory of Diophantine approximations, which collects the study of the best conditions for approximating irrationals via rational convergents p_i/q_i among the goals. But, since $\mathbb{Q}$ is dense in $\mathbb{R}$, one can choose convergents which do not satisfy these rules and which are arbitrarily close to the given irrational limit. The condition $f \in \mathbb{R}$ shows up organically as the extremal completion of the regular set construction, because it is natural for Δ-type sequences to reach any real number, in conformity to the relation (4.6.1), when rational or irrational limits are reached after finitely many or, if required, transfinitely many steps respectively. We need now to restate propo-

[37] One can add something more along these lines. Dynamically speaking, slower speed rates are both shown during the convergence of iterated points to a Cremer point as well as inside any petal of the Leau-Fatou flower, while reaching to the fixed saddle point. See section 5.1.

[38] After sections 4.8 and 4.9, focusing on metrics, we will understand how the expression *in absolute* refers to the (approximating) function speed.

[39] In computational complexity theory, it is a function χ from naturals to naturals (thus we can find a map $\sigma(n), n \in \mathbb{N}$ so that $\sigma \circ \chi(n) : \mathbb{N} \to \mathbb{Q}$) and enjoying the property that $\chi(n), n \in \mathbb{N}$ can be constructed from n by a deterministic Turing machine in the time of order $\chi(n)$. For example, polynomials with positive non-integral coefficients are time-constructable, as well as exponentials in the form q^n where $q, n \in \mathbb{N}$.

sition 4.5.4 in terms of sequences:

Proposition 4.5.5. *The possibility of reaching $\theta \in \mathbb{L}$ depends on the existence of a sequence (either of Γ or of Δ-type) whose set of approximating functions (or the set of indexes) has cardinality $\aleph_0$, or equivalently where the maximal index is ω_0.*

4.6 Again on non-integer indexes

The equipotency between $\mathbb{N}$ and $\mathbb{Q}$ motivates the assumption of the injection $\tau : \mathbb{N} \to \mathbb{Q}$ and of sequences with non-integer indexes h. This affects the behavior of the Γ sequence. Regarding rational approximation of irrationals, a special attention will also be paid to the limit case of $h \in \mathbb{R} \backslash \mathbb{Q}$, after drawing the following, opportune conclusions.

According to the previous properties b), c) and d) of transfinite numbers, a first result is as follows:

Proposition 4.6.1. *The least transfinite ordinal ω_0 is the natural limit for the set of finite ordinals $h < \omega_0$ so that, given $h \to \omega_0$:*

- *the Γ-type sequence reaches the limit θ when h is transfinite;*
- *the sequence $\varphi_h(n)$ ends with the transfinite index $h = \omega_0$;*
- *$\varphi_{\omega_0}(n)$ is the limit case, i.e. it is the limit function at the end of the sequence $\varphi_h(n)$.*

From the transfinite induction, it turns out that the question about the limit, as reconsidered in this new and wider transfinite environment, drops the assumption that the fate of the Γ-type sequence of $\varphi_h(n)$ is to decrease infinitesimally as the index h grows, that is, always running but never reaching to its natural stop, the limit value. Transfinite indexes push the Γ-type sequence $\varphi_h(n)$ to the limit function $\varphi_{\omega_0}(n)$.

Since $\theta \in \mathbb{L}$ is accessible, we showed here that Γ can reach θ when Card(Γ) is a transfinite value exclusively. The accessibility property of θ and transfinite sequence condition are intimately related. Otherwise, if transfinite sequences are not sufficient to reach the lower bound in the Liouville inequality (2.5.1) and θ is not be accessible, like it happens for Diophantine irrationals: in fact, transfinite integers are maximal in absolute and there are no ordinals with greater magnitude. The accessibility property, if enjoyed but not satisfied through finite ordinals, it shall be through transfinite integers.

Countable transfinitely many approximating functions $\varphi_h(n)$ are required to reach θ. These last remarks help us to anticipate what follows:

Proposition 4.6.2. *Let Γ be a sequence, indexed by ordinals $h \in \mathbb{O}$ and which reaches $\theta \in \mathbb{L}$. There exists an equipotent sequence Λ, indexed by transfinitely many indexes g and reaching to θ.*

This wider opening to rational indexes will bring much more insight on the nature and on the dynamics of Γ and of Λ, yet unclear when h was just an integer. More benefits will follow, like in the final proof here. Let $h \in \mathbb{Q}$. One likes to push Λ-type sequences to the extremely efficient performance. Questions naturally arise on the sense of a sequence with irrational indexes, which results from their approximation via rationals. The case of $h \in \mathbb{R}\backslash\mathbb{Q}$ may seem to represent an exception for the dynamics of Γ-type or of Λ-type sequences of approximating functions, unless we look at things as follows.

Recall that the original index was $h \in \mathbb{N}$. Let Γ be a function space $\mathcal{W}$, generated by the sequence of all functions $\varphi_h(n)$. It is well known in Set Theory and Functional Analysis that $\mathcal{W}$ is a *sequence space*, that is, the set of all functions with $n \in \mathbb{N}$. Since $h \in \mathbb{N}$, then

$$\varphi : \mathbb{N} \to \mathcal{I} \equiv [0,1]\backslash\mathcal{D}(\infty). \tag{4.6.1}$$

(In the next section, we will show the reason why we excluded the set $\mathcal{D}(2+)$.) The equivalence between Γ and the regular sets model for Liouville numbers, shown in section 3, allows to drop $\mathbb{N}$ and work with the larger set $\mathbb{O}$ instead, $\mathbb{O} \supset \mathbb{N}$. If we assume that the image domain of φ extends to $\mathbb{R}$, (4.6.1) forks into these two cases:

1. $\varphi_{\mathbb{Q}} : \mathbb{O} \to L_{\mathbb{Q}} \subset \mathcal{I}$, where $h < \omega_0$ and $L_{\mathbb{Q}} \subset \mathbb{Q}$ is the limit set of rational points. Γ is a countable sequence of *finitely many* approximating functions;

2. $\varphi_{\mathbb{L}} : \mathbb{O} \to L_{\mathbb{L}} \subset \mathcal{I}$, where $h = \omega_0$, $L_{\mathbb{L}} \subset \mathbb{R}\backslash(\mathbb{Q} \cup \mathcal{D}(+\infty))$ and $L_{\mathbb{L}}$ is the limit set of Liouville numbers. Γ is a countable sequence of *transfinitely many* approximating functions.

4.7 Equivalences

The equivalence between the Diophantine Approximations of Liouville irrationals and the regular sets model allows to review the latter in terms of sequences of indexes. Let G and H be two functions, so that

$$\begin{array}{ccc} \mathbb{Q} & \underset{\rightarrow}{G} & \mathbb{L} \\ \psi\ \downarrow\uparrow & & \downarrow\uparrow\ \psi \\ \mathbb{N} & \underset{\rightarrow}{H} & \mathbb{O} \end{array} \qquad \begin{array}{l} \text{for real indexes } f_{n+1} = G(f_n) \\ \\ \text{for integer indexes } h_{n+1} = H(h_n) \end{array}$$

where $\psi : \mathbb{N} \to \mathbb{Q}$ (or, more extensively, $\psi : \mathbb{O} \to \mathbb{R}$) is an invertible map.[40] Then $G \circ \psi = \psi \circ H$ holds.

We address one open question to the reader. With regard to the property *c*) at p. 52, the value we found is enough for our further purposes, but we wonder whether such estimation could be improved by showing that the cardinality could be even larger than $\aleph_0$ or not. [41]

We showed that moving from $\mathbb{Q}$ to $\mathbb{R}$ is *theoretically possible* via sequences of transfinitely many steps, provided that the functions

$$\varphi_h(n) \to \psi(n) \Leftrightarrow \mathbb{Q} \to \mathbb{R}$$

are sufficiently fast. $\mathbb{L} \subset \mathbb{R}\backslash\mathbb{Q} \subset \mathbb{R}$ is everywhere dense and includes uncountably many points. This amounts to look for a function ψ from a set of cardinality $\aleph_0$ (say $\mathbb{N}$ or $\mathbb{Q}$) to a set of cardinality $\aleph_1$ (say $\mathbb{R}$ or $\mathbb{L}$). In the topological terms of inclusion sets I_h for the regular sets model, the possibility to reach the given limit equals to $E \equiv F \equiv \bigcap E_h$: this equivalence is out of reach if $h \in \mathbb{N}$ or if $g \in \mathbb{Q}$. If $E_h \not\equiv \emptyset$ and $E_{h+1} \subset E_h$ is a nesting sequence inside $\mathcal{I} \equiv [0, 1]$, there is no loss of generality if $\psi(n) \equiv 1/\varphi_h(n)$, termed the

[40] In the sense that one such function ψ maps natural numbers to rationals and, equivalently, a transfinite value to a real number.

[41] If the cardinality of such sequences should be larger than $\aleph_0$, (AC) and the Continuum Hypothesis would lead to $\aleph_1$. The maximal index would be the first uncountable transfinite value ω_1.

choice function, with regard to (AC).[42] Therefore, if $\epsilon/\varphi_h(n)$ with $h \in \mathbb{O}$ (or another map,[43] say $\chi_f(n) : \mathbb{N} \to \mathbb{R}$ with $f \in \mathbb{R}$), the topological equivalences (in Lebesgue's terms)

$$E \equiv F \equiv X \setminus \bigcup E_h$$

and (in Borel's terms)

$$E \equiv F \equiv \bigcap E_h$$

hold.[44] We have been working with approximating functions $\varphi_h(n)$ of abstract type up to now, i.e. with no regard of their nature. Rather we focused on the properties of the sequences Γ, Λ and Δ for (3.2.3) to hold. We will check later whether the *standard polynomial type* may work or not. If not, we will look for another, performing type.

Finite	**Transfinite**
Sequenced indexes $h, h+1, \ldots$	*Limit ordinal* η
Γ *sequence* $\varphi_h(n), \varphi_{h+1}(n), \ldots$	*Limit function* $\varphi_\eta(n)$
Rational approximants $\frac{p_n}{q_n}, \frac{p_{n+1}}{q_{n+1}}$	*Liouville number* θ
Leau-Fatou flower	*Non-linearizable hedgehogs*

4.8 Transfinite fractions

We need to upgrade our tools and move from the '*ordinal fractions*', where p and q are finite cardinal numbers, to the analogous '*transfinite fractions*', the

[42] We will come back to these remarks during the final proof, especially during the construction in section 6.6, p. 84.

[43] In general, the approximation function reshapes when the index turns from naturals to rationals, covering ranges of values not usually retrieved if indexed by natural numbers.

[44] The questions discussed in these last two section will be the topic of a further work.

quotient between two integers where at least one is transfinite. This extension is plausible in terms of formal algebraic systems of symbols, according to Gleyzal [44] and to Artin and Schreier [2]. Gleyzal formulated the existence of a parametrizable and ordered field O, which represents the maximal generalization of $\mathbb{R}$. When such parameter is 0, Gleyzal showed that O turns into $\mathbb{R}$. Otherwise, O turn into distinct algebraic systems of prime numbers, or of integers, rationals, of reals or of complex numbers of transfinite order ([44], p. 586). As the existence of transfinite fraction is formally acknowledged, we call upon the theory of Grzegorczyk hierarchy [39] and in particular *Ackermann functions* [20, 31, 55], in order to generate the transfinite fractions.

Ackermann functions are computable, but not primitively recursive; their growth is much faster than single exponential functions a^x and even than the multiple version $a^{a^{\cdots^{a^x}}}$. One defines the Ackermann function $\oplus$ as follows:

$$m \oplus^1 n = m$$
$$m \oplus^k n = m \oplus^k [m \oplus^k [\dots [m \oplus^k m]]],$$

where m is exponentiated to the $n^h, h = k - 1$ power:

$$m \oplus^k n \Rightarrow m^{n^{k-1}}. \tag{4.8.1}$$

For example,

$$3 \oplus^2 4 \Rightarrow 3^{4^1} = 81, \qquad m = 3, n = 2, k = 4;$$
$$10 \oplus^3 4 \Rightarrow 10^{(4^2)} = 10^{16}, \qquad m = 10, n = 3, k = 4.$$

It is evident that they are as incredibly fast as they can reach large numbers through relatively small values of the parameters. From the completely parametric formulation of Ackermann functions, one realizes that their speed rate is larger than standard polynomials and that the fine tuning of m and k can open the reach to cardinals with arbitrarily large magnitude. Since primes are subset of integers, we can generate transfinite integers or transfinite primes via Ackermann functions.

4.9 On the Inverse Ackermann functions

In this section we will focus on deductions we have already drawn. According to Ackermann's developments, it is also worthwhile to extend the Ackermann functions to negative exponents k, they are analogously termed as *Inverse Ackermann functions* $\ominus$. Their formal definition requires only a slight twist of the sense of (4.8.1):

$$\epsilon \cdot m \ominus^k n \Rightarrow \epsilon \cdot m^{n^{|k-1|}} = \frac{\epsilon}{m^{n^{|k-1|}}} \qquad m \neq 0, k > 0, n > 0. \tag{4.9.1}$$

It is clear that

$$\liminf \; \epsilon \cdot m \ominus^k n = 0. \tag{4.9.2}$$

This condition satisfies the definition of zero Lebesgue measure sets, as we compare it to corollary (3.1.2). Let m again be a transfinite value. Therefore (4.9.1) turns into a transfinite fraction and the lower bound (4.9.2) finally vanishes identically:

$$\liminf \frac{\epsilon}{m^{n^{|k-1|}}} = 0. \tag{4.9.3}$$

Otherwise, the zero limit would not be accessible to such fractions.

With regards to the analogy between the regular sets model and Liouville's limit (discussed in section 2.1), we would be remiss not to point out the implications of (4.9.3) viewed from these new perspective. In particular, (4.9.3) can be revisited in light of (3.2.2), which leads to the Liouville limit. So the Inverse Ackermann function plays the same role as of approximating function

$$\liminf \|q\theta\| \leq \frac{\epsilon}{\varphi_h(n)} \quad \overset{?}{\rightarrow} \quad \liminf \|q\theta\| = \frac{\epsilon}{m^{n^{|k-1|}}} = 0. \tag{4.9.4}$$

This last passage is critical: although it follows a flow of inductions, it gives rise to important questions on the possibility of moving from left to right of (4.9.4); in fact, some exceptions may come up. Both the approximation function $\varphi(n)$ and the parameters need to be carefully examined. Let $\theta \in \mathbb{L}$. We show that the limit equality

$$\liminf \|q\theta\| = \frac{\epsilon}{q^{\kappa}} = 0$$

holds when, equivalently, q_n is transfinite or p_n/q_n is a transfinite fraction. More interestingly, we need to link the above lower bounds to the double inequality

(3.1.7) at p. 44, which we restate again here, for lessening the connection:

$$\frac{\epsilon}{q_{n+1}^{i+1}\varphi(q_{n+1})} \leq \|q_{n+i}\theta\| \leq \frac{\epsilon}{q_n^i\varphi(q_n)}. \tag{4.9.5}$$

Once again, the sequence of denominators $\{q_n^i\varphi(q_{n+1})\}$ shall enjoy the same properties as by Inverse Ackermann functions in the form (4.9.4). How? The double nature of the expression $\{q_n^i\varphi(q_{n+1})\}$ states that this goal can be accomplished:

- either via *one transfinite sequence*, i.e. a sequence with at least one element endowed with a transfinite index, say i here;
- or via *an approximating function with (multi-)exponential shrinking speed*, i.e. whose coefficients show up in the same form of Inverse Ackermann function as above.

4.10 Categories of transfinite fractions

In both cases, the denominators of (4.9.5) shall include at least one factor behaving such as (4.9.4). From a wider viewpoint, the right-hand side fraction in (4.9.1), can be listed into one among the five categories below for fractions with transfinite elements:

1. m is transfinite and n, k are non-transfinite;
2. m, k are non-transfinite and n is transfinite;
3. m, n are non-transfinite and k is transfinite;
4. m, n are transfinite and k is non-transfinite;
5. m, n, k are transfinite.

One can also state an alternative definition of the transfinite extension in (4.9.1), as one argues that the limit (4.9.3) is always satisfied by all these categories.

Our final purposes dictate to pay a closer attention to transfinite fractions, but in a more general flavor, where the fraction is endowed with both Ackermann and Inverse Ackermann functions in the numerator and in the denominator respectively. The main restriction in the previous form ϵ/q^n was that transfiniteness can be called in the denominator exclusively, so that transfinite sequences do always decrease to zero: this may help to understand the fate of the sequence of approximants to Liouville numbers in metric terms, but not the nature of the fraction p_n/q_n. Hence we will consider this fraction in lowest terms:

$$\frac{p^{n^{|k-1|}}}{q^{n^{|k-1|}}}. \tag{4.10.1}$$

For sake of simplicity, we can melt exponents into one, n for example, and assume that the sequence of numerators and of denominators grows together with n. We find this more comfortable version, with no loss of generality for the further conclusions:

$$\frac{p_n^n}{q_n^n}. \tag{4.10.2}$$

Now we can resume the previous list into this shorter version including rational non-rational fractions:

a) p^n is non-transfinite and q^n is transfinite;

b) p^n is transfinite and q^n is non-transfinite;

c) p^n and q^n are non-transfinite;

d) p^n and q^n are transfinite;

Question 4.10.1. *What is the range of values assumed by these four categories? Again, which values can be accessed by sequences of fractions belonging to each of these four categories?*

Category *a*) gives rise to fractions tending to vanish identically, while category *b*), being the inverse of *a*) shall necessarily tend to transfinite rationals. Category *c*) gives rise to rational numbers, as known. Since p and q are coprime, then p^n and q^n are coprime too: no integer can be generated by (4.10.2). Therefore category *d*) gives rise to none of the above categories (: no integers, no rationals), thus to irrational numbers. When the question is explored in terms of accessibility property, we refer to irrationals which are accessible to rational fractions. By definition, these are Liouville numbers.[45] One straightforward example of category d) is the well-known *Liouville constant*:

$$c = \sum_{j=1}^{\aleph_0} 10^{-j!} = 0.110001000000000000000001000\ldots,$$

which is approximated by

$$|c - p_n/q_n| = \sum_{j=n+1}^{\infty} 10^{-j!} = 10^{-(n+1)!} + 10^{-(n+2)!+\cdots<10^{-(n!n)}} = 1/q_n^n. \quad (4.10.3)$$

where[46]

$$p_n = \sum_{j=1}^{n} 10^{(n!-j!)}, \quad q_n = 10^{n!}.$$

The factorials appearing in both numerator p_n and denominator q_n attest that they are of multi-exponential kind, as we can check for the simpler form of q_n:

$$q_n = 10^{n!} = 10^{2^{3^{\cdots^{n}}}}.$$

Since $n!$ grows faster than $(n! - j!)$, then $p_n < q_n$ and $0 < c < 1$. The relation (4.10.3) brings back again to light the 'Inverse Ackermann nature' of category a) for functions governing the metrics between the sequence of rational approximants p_n/q_n and the Liouville number. As we refer back to the definition of Diophantine irrationals (3.2.1) at p. 46, it is straightforward that κ in (3.2.1) plays the same role in the ordinary fraction ϵ/q^κ as the exponent $n^{|h-1|}$ does in the transfinite fraction $\epsilon/m^{n^{|h-1|}}$ (both fractions belong to the unit interval) like for (4.9.4), and release this new definition tying to corollary 6.3.2 at p. 81:

Definition 4.10.1. *According to Liouville's approximation formula, Liouville irrationals are the only Diophantine irrationals of transfinite order. The converse holds as well: the only Diophantine irrationals of transfinite order are Liouville.*

[45] Refer back to our remarks on the accessibility to irrationals via transfinite sequences in section 2.5 and ff.

[46] One notices a close analogy to the sequences of approximating functions with rational indexes (4.5.1), whose speed can slow infinitesimally to down to zero at Liouville numbers.

One can add the following corollary:

Corollary 4.10.1. *If $e^{2\pi i\theta} - 1 = 0$ holds, then $\theta \in \mathbb{Q}$ or $\theta \in \mathbb{L}$. In the latter case, θ can be expressed in terms of a transfinite fraction exclusively.*

In light of transfinite sequences, there exists no drastic separation between Diophantine and Liouville irrationals: both can be viewed under the same coherent approach, determined by the speed of convergents and growth of denominators q_n. As we will understand below, the effect of this numerical definition is analogous to the gluing power of hedgehogs in summarizing all invariant sets arising in indifferent dynamics. In theoretical terms, we refer back to the possibility, given by the regular sets model, to obtain a dual viewpoint on the question of Diophantine Approximations which we will address at the end of section 5.1. In practical terms, one easily understands that, after walking along a different path, we claimed the impossibility of algorithmically computing Liouville numbers and also of electronically drawing hedgehogs.

Last but not least we will give a response to the question mark in (4.9.4), raising the possibility of moving from the left to the right side. It is not so obvious that we can do so. First the deductions from the last two sections call for revisiting the du-Bois Reymond's theorem 3.1.1 at p. 41. This statement holds when $\varphi_h(n)$ is of polynomial type, thus it is not the case for Ackermann functions check in, being evidently exponential. The condition $\|q_n\theta\| = 0$ then holds under the latter situation exclusively. Polynomial speed is much slower than that of exponentials because of the degree of freedom in the input of the parameter values (base and exponent) inside the Ackermann function formula: the largely parametrizable nature together with the exponential form of such functions allow the latter to run transfinitely faster than polynomials. The lower bound cannot be zero. Moving from the left side to the right of (4.9.4) is impossible when $\varphi_h(n)$ is a polynomial. Otherwise, when the approximation function is multi-exponential, this extension (4.9.4) shall take place. One can give the following complement to du-Bois Reymond's theorem:

Proposition 4.10.1. *Positive and monotonically increasing Ackermann functions with transfinite input values are absolutely the fastest ones.*

It is clear that, in these new circumstances, the corollary 3.1.2 holds no more. The studies of exponential maps in the form $a^{a^{\cdots^{a}}}, a \in \mathbb{R}$ go long back in time and they are collected nowadays under the definitions of *infinitely iterated exponentials*, *chains* or *towers* of exponentials, or of *hyper-powers*. See [26, 43, 45, 46, 56, 60, 66] among the very wide literature on this last topic.

5 Forward to applications

5.1 Forcing through the folds of Reals

Moving from the previous discussion, we settle the equivalence between the regular sets model and the Liouville inequality. We will later apply it to questions of different nature. The construction of the regular sets model with Liouville numbers as fundamental points is *artificial*, not peculiar to such numbers. One can freely apply it when the limit is Diophantine and draw the same consequences like in section 2.7.

Given $\theta \in \mathbb{R}\backslash\mathbb{Q}$ and an infinitesimal inclusion set $I_h, \theta \in I_h$, we know that θ cannot be the only Liouville number inside I_h. There exist uncountably many Diophantine and Liouville irrationals, and θ cannot be regarded as the *natural limit* of a regular set. The previous degree of freedom suggests to run towards Liouville numbers from two distinct paths of approximation: one is *external*, along rationals (handling p_n and q_n), the other is *internal* along Diophantine irrationals (through growing sequences of order κ).[47]

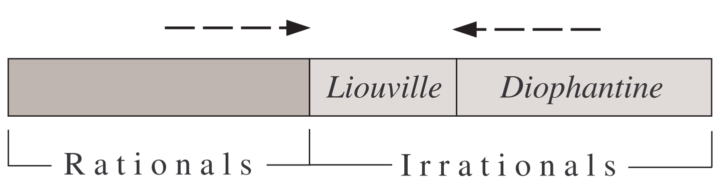


Figure 5.1.1: **Paths of approximation.**

The former path is much more convenient to us, since we can count on

[47] If Liouville numbers are approximated by Diophantine irrationals of arbitrarily large order κ, Siegel compacta progressively shrink to the Cremer point.

the results for continued fractions as well as on Set Theory. More promising deductions will be drawn later, after focusing on the topological affinity between hedgehogs and Leau-Fatou flowers, namely the empty linearization domains that occur with non-linearizable hedgehogs and the rationally indifferent fixed point δ. Both belong to the Julia set J_θ. The definition of the Leau-Fatou flower is among the basics in Holomorphic Dynamics and can be found inside the textbooks [25, 64].

5.2 Cremer values and algorithms

According to du-Bois Reymond's theorem 3.1.1 (see p. 41), a portion of statement of 4.1.2 can be restated as follows:

Proposition 5.2.1. $\mathbb{L}$ *is fundamental for the regular sets model and it cannot be attained via one approximation process of countably many steps*

What is the connection to hedgehogs?

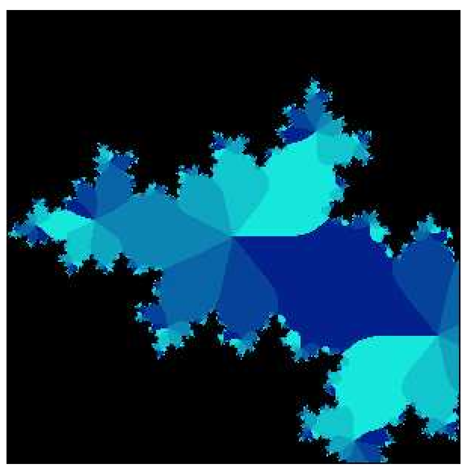


Figure 5.2.1: The 7-petal Leau-Fatou flower for the quadratic type germ with rational value $\theta = 3/7 = 0,\overline{428571}$.

The fundamental points of a zero measure set enjoy the same topological distribution as of values θ in the germ (1.1.5). It is clear that the shrinking process of E_h relates to the growth of h and one can associate each rational approximant to the width of each E_h. The degree of freedom for this process allows to apply the model to Diophantine numbers, where the lower bound of the widths ties to the maximal order κ. We can thus assert:

a) that *numerically* the existence of Diophantine irrationals of arbitrarily high but finite $\kappa \to \omega_0$ ties to the existence of inclusions intervals $I_{h+1} \subset I_h$ with positive width;

b) that *topologically* the shrinking process $I_{h+1} \subset I_h$ is of countable nature, i.e. it relies on (in-)finitely many steps, and we can approximate hedgehogs with Siegel compacta $\mathcal{S}$ of positive area;

c) that the growth rate of the convergents p_n/q_n can give rise to differently sized compacta[48] according to the dynamics of the quadratic type germ (1.1.5), which relate to the wedging action of the unbounded Fatou component (whose attracting point is at ∞) into the bounded basin;

d) that irrationals θ are not computable via finitely many steps.

According to the algorithm definition,[49] the entry *d*), as well as 5.2.1, can be revised for non-linearizable hedgehogs:

Proposition 5.2.2. *Reaching Liouville numbers via a sequenced process is not algorithmically feasible. Non-linearizable hedgehogs with Cremer points are not computable.*

6 Results in Holomorphic Dynamics

Computing the Hausdorff dimension of non-linearizable hedgehogs involves a strategy of several stages, where in essence we convert the results already obtained for the regular sets model into equivalent statements for Holomorphic Dynamics. A summary appears in the table below, mirroring the concepts in table 2.8 at p. 39.

[48] Related speed rates are expressed in logarithmic terms, See [73], p. 249. $\mathcal{S}$ is maximal for θ of constant type ($\kappa = 2$, see [64], p. 119), whereas, we find hedgehogs with Siegel compacta for $\kappa > 2$, with a sufficiently small neighborhood of θ, isomorphic to a rotational disc centered at δ.

[49] Any procedure (a finite set of well-defined instructions) to accomplish a task which, given an initial state, will terminate into one defined end-state.

Number Theory		**Holomorphic dynamics**
Decreasing modulus	$\leftrightarrow$	Petals metrics
Rational approximants	$\leftrightarrow$	Number of petals
Liouville number	$\leftrightarrow$	The hedgehog

Posthumously to the compilation of the results in this article, the author became acquainted with two works based upon a same heuristic approach as in our further discussion.

First we mention [47], where Heinemann and Strattman investigated on the Hausdorff dimension $H^s(J_\alpha)$ of Julia sets J_α for the iterates of the quadratic germ $f_\alpha : e^{2\pi i\alpha}z+\mathcal{O}(z^2)$, where $\alpha \in \mathbb{R}$. Even if their attack consists in increasing the Leau-Fatou petals number ad infinitum, id did not rely on Diophantine Approximations. Their goal was to show that $\limsup H^s(J_\alpha) = 2$, while we are going to prove that the case of $H^s(J_\alpha) = 2$ holds, provided that $\alpha \in \mathbb{L} \subset \mathbb{R}\backslash\mathbb{Q}$.

Secondly, one short passage in the celebrated work [36] (bottom of p. 247), where Fatou introduced a rough approach by incrementing the petals ad infinitum, i.e. via $q_n \to \infty$ in p_n/q_n of $e^{2\pi i \frac{p_n}{q_n}}$, so that $p_n/q_n \to \theta \in \mathbb{R}\backslash\mathbb{Q}$. Fatou believed to have brought more evidences to his (erroneous) opinion that the functional equation (1.1.6) fails inside any small neighborhood of δ. The passage also attests that Fatou *pioneered* the attack via rational approximants.[50] The lack of the theoretical background attests that Fatou's heuristic approach was sketched but not deepened. Despite the results, Fatou's strategy represented an important advancement and his mistakes, although improperly, invite to check if such method could represent an opening to further developments.

6.1 1st Part: the rational path to Liouville numbers

We start from the conversion formula for periodic rationals $x \in [0, 1]$ into the fractional notation:

$$x = \sup_n x_n = \frac{p_n}{q_n} = x_s + \frac{c_{s+1}10^{t-1} + c_{s+2}10^{t-2} + \ldots + c_{s+t}}{(10^t - 1)10^s}, \tag{6.1.1}$$

[50] About twenty four years later, Siegel showed that Diophantine Approximations were exactly the striking tool to solve Fatou's concerns.

where $s \geq 0$ is the *anti-period* length, x_s is the anti-period value, t is the period length, and $c_{s+1}, c_{s+2}, \ldots, c_{s+t}$ are the periodic digits. For example, let $0.1\overline{245}$; then $x_s = 0.1, s = 1, t = 3$ and $c_2 = 2, c_3 = 4, c_4 = 5$.

This formula extends to rationals with anti-period only through slight modifications. We cancel out the anti-period x_s. Non-periodic rationals may be reconsidered as periodic with zero digit period. Let $s > 0$ the number of decimals and n the digits index. One has $1 \leq n \leq s$ and c_n are the anti-period digits; n increases from 1 to s, while $n = 1$ holds for the most significant digit and $n = s$ for the least significant. Then

$$x = \sup_n x_n = \frac{p_n}{q_n} = \frac{c_1 10^{s-1} + c_2 10^{s-2} + \ldots + c_n 10^{s-n}}{10^s}. \tag{6.1.2}$$

The rational approximants p_n/q_n in the Liouville limit formula can be re-viewed as non-periodic rationals whose anti-period grows ad infinitum, as illustrated in the sequence

$$\begin{array}{llll}
0.\mathbf{8}\overline{0}, & s = 1, & t = 1, & c_1 = 8 \\
0.\mathbf{82}\overline{0}, & s = 2, & t = 1, & c_1 = 8, c_2 = 2 \\
0.\mathbf{827}\overline{0}, & s = 3, & t = 1, & c_1 = 8, c_2 = 2, c_3 = 7 \\
0.\mathbf{8271}\overline{0}, & s = 4, & t = 1, & c_1 = 8, c_2 = 2, c_3 = 7, c_4 = 1 \\
\ldots & & & \\
0.\mathbf{8271}\ldots\overline{0}, & s \to \infty, & t = 1, & c_1 = 8, c_2 = 2, c_3 = 7, c_4 = 1, \ldots
\end{array}$$

Equivalently this can be interpreted as the sequence of exclusion intervals $\bigcup E_n$ in the regular sets model. In the fraction form of (6.1.2), or even when reduced to the lowest terms, both the numerator p_n and the denominator q_n grow to infinity (each one according to a given speed rate). Since $0 \leq x_n < 1$, the growth for sequence of denominators q_n will be faster.

6.2 2nd Part: petal metrics and approximation conditions

Let the family of polynomials $f_\lambda = e^{2\pi i\lambda}z + \mathcal{O}(z^k), k \geq 2$, holomorphic at the origin and with one super-attracting fixed point at ∞. We are assume there exist no other (essential, pole-like) singularities all over $\mathbb{C}_\infty$.

Let $\mathcal{P}_{1\leq n\leq p}$ be one petal of any Leau-Fatou flower with p petals. Let $h_{\mathcal{P}_n}$ be the *petal height*, i.e. the measure of the longest straight-line segment emanating from the rationally indifferent fixed point and included inside $\mathcal{P}_n$. Both ends belong to the Julia set. Let $w_{\mathcal{P}_n}$ be the *petal width*, i.e. the longest straight-line segment, perpendicular to $h_{\mathcal{P}_n}$ and inside the given petal $\mathcal{P}_n$, whose both ends belong to the Julia set. See fig. 6.2.2.

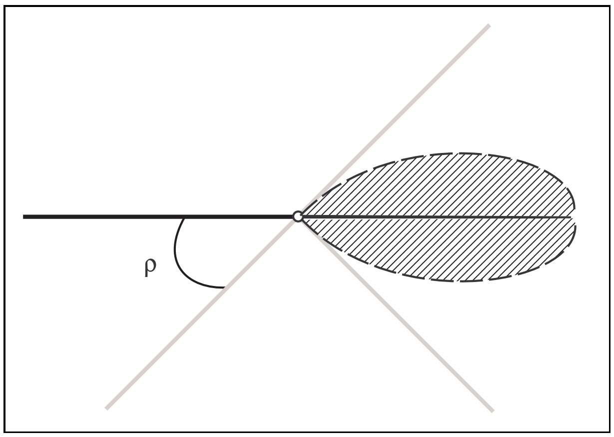


Figure 6.2.1: **Zooming into petals.** In a neighborhood of a rationally indifferent fixed point, attracting and repelling directions alternate. As the number of petals grows, the angle ω tends to 0 and petals shrink.

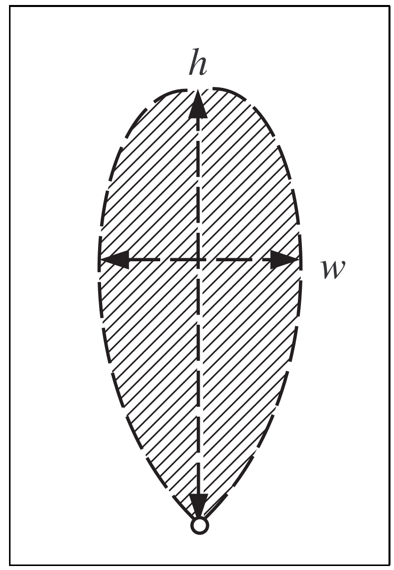


Figure 6.2.2: **Petal metrics.**

Let the Leau-Fatou flower be *regular* if both $h_{\mathcal{P}}$ and $w_{\mathcal{P}}$ are the same for all petals; otherwise the flower is *irregular*. The petal's area $A_{\mathcal{P}}$ is positive, as well as the *petal angle* $\omega_{\mathcal{P}}$ so that $\omega_{\mathcal{P}}$ decreases to 0 (see fig. 6.2.1) together with $A_{\mathcal{P}}$. According to proposition 2.5.1 at p. 33, we know that Diophantine irrationals cannot be reached by rational approximants. But Liouville numbers can. Proposition 3.1.1 in section 3.1 grants the accessibility for sequences of rational approximants $\{p_n/q_n\}$ to Liouville numbers. At this point, we summarize all the previous results into this list of conditions: namely ...

1. ... that the destination exists (*Liouville's inequality and the metrics for the set* $\mathbb{L}$ *of Liouville numbers*);
2. ... that the destination is accessible and how to get to it (*transfinite induction and sequences of rational approximants*);

3. … how many steps are required to get to destination (*transfinitely many*).
4. … the required speed to get to destination (*faster than polynomials; in fact, at least multi-exponential*).

Now we can safely run from rationals to Liouville numbers. According to (3.1.7), one considers the sequence of fractions

$$\frac{p_n^n}{q_n^n},$$

where the exponent n governs the growth of both numerators p_n^n and denominators q_n^n. We assume that both p_n and q_n belong to $\mathbb{O}$ (refer back to section 4.3). These fractions can represent Liouville numbers, according to category 5 of the list at p. 68. It is easy to find close analogies with (3.1.7) at p. 44 and to show that we can set

$$\varphi_n(q) = q_n^n \quad \Rightarrow \quad \frac{p_n^n}{\varphi_n(q)}.$$ [51]

With no loss of generality, we replace the general form $\frac{p_n^n}{q_n^n \varphi_n(q)}$ with the simpler $\frac{p_n^n}{q_n^n}$, whose speed is quite slower but still enough for our purposes.

6.3 Transfinite Induction and Topology: the inflation

We apply the transfinite induction to $\{p_n^n/q_n^n\}$. In section 4.2 we remarked that one such process affects ordinal numbers (the indexes). After it has been exported to iterates, the topology of the Leau-Fatou flower around δ too. Thus we need to understand how one could apply the transfinite induction while 'travelling from flowers to non-linearizable hedgehogs'. The *accomplishing of this goal* requires the introduction of one topological property $\mathcal{T}$, which holds for the iterates f_θ^n of (1.1.5), where $n \leq \omega_0$. Equivalently, from complex functions $f_\alpha^n : e^{2\pi i \alpha_n} z + \mathcal{O}(z^k)$ with $\alpha_n \equiv p_n/q_n \in \mathbb{Q}$ to same functions but with

[51] Assuming that p^n grows slowlier than $\varphi_n(q)$, the maps on the left and the right are Ackermann and Inverse Ackermann functions respectively.

$\alpha_n \equiv \theta \in \mathbb{L}$. The property $\mathcal{T}$ is: '*the existence of a Julia set curve, locally pathwise connected and intersecting the fixed point δ at the origin*'. The further construction shows that the boundary topology of flowers (when $\theta \in \mathbb{Q}$) and of non-linearizable hedgehogs (when $\theta \in \mathbb{L}$) look like similar: in both cases, the Julia set is a continuous curve intersecting the indifferent fixed point δ. Now it is useful to assume the existence of one transfinite sequence of fractions with increasing denominators $q_n^n \geq 1$ (with integer exponents n), running from rational numbers to $\theta \in \mathbb{L}$:

$$1 = \frac{1}{1} \equiv \frac{p_0^0}{q_0^0} \to \frac{p_1^1}{q_1^1} \to \frac{p_2^2}{q_2^2} \to \frac{p_3^3}{q_3^3} \to \cdots \to \frac{p_{\aleph_0}^{\aleph_0}}{q_{\aleph_0}^{\aleph_0}} \equiv \theta. \tag{6.3.1}$$

Question 6.3.1. *Why did we set up this particular sequence?*

Answer: Recall the reader that the construction of transfinite sequences approximating the Liouville irrationals through rational numbers was illustrated in section 4.3. We pointed out to the double inequality (3.1.7), whose formulation asserts that the required speed rate of the approximants *shall not be slower* than the (multi)-exponential rate 2^{n+1} in $q_n = q^{2^{n+1}}$. We showed that the pair $(\mathbb{O}, \leq)$ defines the zero-start union set of all finite with the least transfinite ordinals ω_0: thus $\mathbb{O}$ has an absolute maximum element $M_{\mathbb{O}}$ of transfinite kind. Then $\text{Card}(\mathbb{O}) \equiv \aleph_0$ and $M_{\mathbb{O}} = \aleph_0$. Let $p_n, q_n \in \mathbb{O}$ from now on. The maximal index $M_{\mathbb{O}} = \max(\mathbb{O})$ shall be the least countable transfinite ordinal ω_0. Equivalently, $\text{Card}(\Gamma) \equiv \aleph_0$ if we assume that n starts from 0 (given a transfinite set, we can assume the converse according to conditions b) and d) at p. 52). For zero-start sets (see p. 51), the double role of ordinals allows to assume that the cardinality $\aleph_0$ of $\mathbb{O}$ is equal to the magnitude of the maximal index of the sequence. It is sufficient to push[52] the exponent 2^{n+1} up to $\aleph_0$ and to rewrite the right-hand side of (3.2.2), according to the terms of theorem 2.5.4, into this lowest term fraction:

$$\text{Card}(\Gamma) = \aleph_0 \quad \Rightarrow \quad \frac{p_{\aleph_0}^{\aleph_0}}{q_{\aleph_0}^{\aleph_0}} \tag{6.3.2}$$

[52] We may be contented of this estimation here. But we can show a stronger result. Because of the index $n \in \mathbb{O}$, we can push it to transfinite values and find that the double exponent form in the denominator q^{2^n+1} rewrites as follows:

$$2^{\aleph_0+1} \equiv 2^{\aleph_0} \equiv \aleph_1.$$

This generate fractions in the form $p_{\aleph_1}^{\aleph_1}/q_{\aleph_1}^{\aleph_1}$: but this goes far beyond what we need to apply the lemma 6.3.1.

After the developments at p. 69, the resulting fraction $\frac{p_{\aleph_0}^{\aleph_0}}{q_{\aleph_0}^{\aleph_0}}$ belongs to category d): the only one being able representing the Liouville numbers. Now we invoke this lemma about Leau-Fatou flowers ([64], p. 104):

Lemma 6.3.1. *If λ from* (1.1.5) *is a primitive q-th root of unity, the number n of attracting directions at the fixed point δ is a multiple of q. In other words, the multiplicity $n+1$ of δ must be congruent to 1 modulo q.*

We are not interested here in keeping the whole fraction p_n^n/q_n^n. This lemma allows to drop numerators and keep the denominators exclusively. We will revisit the sequence (6.3.1) in these latter terms. By transfinite induction, (6.3.1) involves an increasing amount of q^n attracting and repelling directions until $q^{\aleph_0}$, when $\theta \in \mathbb{L}$ is reached.[53] After transfinitely many steps, assume we finally get to the fraction at the right-hand side of (6.3.2). Thanks to Cantor, we know the power-sets formula $2^{\aleph_0} = \aleph_1$ and in general that $q^{\aleph_0} = \aleph_1$ for $q \geq 2$. We inductively draw this (still raw) conclusion: *given* (1.1.5) *and* $\theta \in \mathbb{L}$, *there exists an invariant set such as Leau-Fatou flower with uncountably many directions emanating radially everywhere from the fixed point δ.* The approximation of Liouville irrationals requires that the speed of convergents q_n shall not be slower than the (multi)-exponential rate, with positive and integer coefficients.

Question 6.3.2. *Why can we not set up a slower sequence than* (6.3.1) *?*

Answer: Suppose we do not, despite the Liouville approximation formula. For example, the sequence consists of denominators q_n, whose exponents growth rate is 1: $q_1^1, q_2^1, q_3^1, \ldots, q_n^1$. The largest $q_n^1 \in \mathbb{O}$ is $\aleph_0$ and no slower speed can guarantee the approximation of Liouville numbers as well as the previous lemma would retrieve a number of $\aleph_0$ directions. For the sequences of q_n, the (multi)-exponential growth rate with magnitude ≥ 2 is required: slower speeds cannot fulfill the prerequisites to get to Liouville numbers. We will go over this conclusion, thereby realizing what topological contradictions arise and discussing the topology of the Julia set when $\theta \in \mathbb{L}$ in (1.1.5). We can state the following corollary which extends the lemma 6.3.1 to transfinite fractions:

[53] According to (3.1.7) at p. 44, the application of multi-exponential operators (or inverse Ackermann functions for transfinitely many convergent functions to approximate the Liouville number) can help to get one such denominator with transfinite exponent. Refer to the discussion after Khintchine's theorem 2.2.1.

Corollary 6.3.2. *Given $\theta \in \mathbb{L} \subset [0,1]$, the following conditions are equivalent:*

1. *θ is exclusively representable by a transfinite fraction p/q, where p, q are transfinite;*

2. *according to the construction in lemma* 6.3.1*, the set of attracting and repelling directions for a Leau-Fatou flower, continuously impresses a circle C centered at δ and with positive radius;*

3. *the petal angle ω vanishes identically.*

While rational approximants $a_n = p_n/q_n$ get closer and closer to $\theta \in \mathbb{L}$, the denominators q_n grow together with the number of alternating (repelling and attracting) directions in the Leau-Fatou flowers. See the simulation along the slides sequence in fig. 6.3.1, where a regular flower was assumed for sake of simplicity. This lemma also implies that, as the denominators q_n grow in mag-

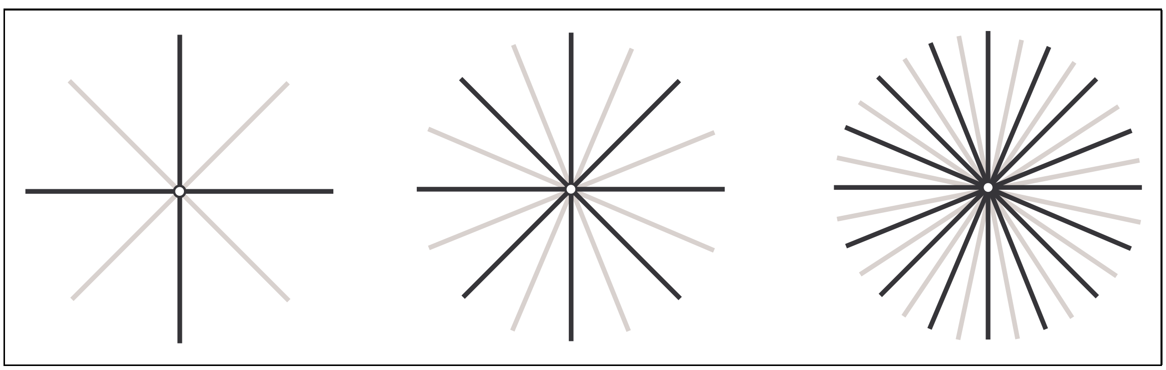


Figure 6.3.1: **The rational path.** As p_n and q_n in (6.1.2) grow, the number of repelling (in grey) and attracting directions (in black) grows. Directions need not to be uniformly distributed around the fixed points for the construction to work, as it is the case here.

nitude, the petals' width infinitesimally shrink and the attracting (or repelling) directions progressively inflate a disk-shaped neighborhood of the fixed point δ, i.e. they spread radially everywhere. By making use of the hypotheses in the beginning of section 6.2, we can state:

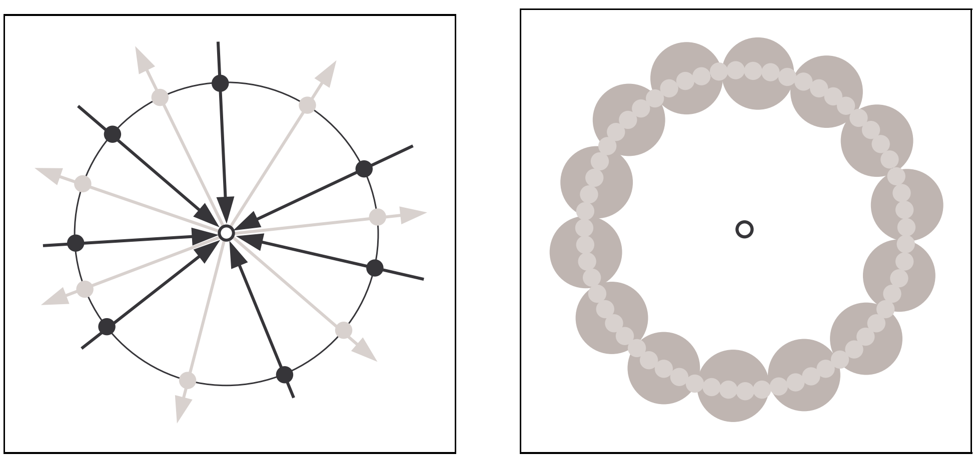
Proposition 6.3.1. *Let $f_\lambda : e^{2\pi i\lambda}z + \mathcal{O}(z^k), k \geq 2$ so that the (essential, pole-like) singularities of f_λ do not belong to any repelling cycle. The related Leau-Fatou flower enjoys the following properties:*

a) *the petal width* $w_{\mathcal{P}}$ *is positive;*

b) *the petal height* $h_{\mathcal{P}}$ *is upper bounded.*

c) *the argument* $\lambda \in \mathbb{Q}, \lambda = p/q \in [0, 1)$ *in* $e^{2\pi i\lambda}$*, where* p *and* q *are finite ordinals.*

One understands that a) is straightforward because of the condition about the petal angle $\omega > 0$ assures that $A_{\mathcal{P}} > 0$ too. Then b) follows because of $h_{\mathcal{P}} < \infty$. Since f_λ is a polynomial, there exists necessarily one super-attracting fixed point at infinity and the basin $\mathcal{B}_\infty \not\equiv \emptyset$. The basin $\bigcup_n^{q<+\infty} \mathcal{P}_n \equiv \mathcal{B}_0$ cannot include the point at infinity and the elongation of $\mathcal{P}_n$ is necessarily upper bounded[54] and positive, $0 < h_{\mathcal{P}} < \infty$, for allowing the flower existence. While a) follows from c), the metrics in b) just follow from the polynomial f_λ.[55]



[A] [B]

Figure 6.3.2: **Progressive inflation.** As q_n grows, attracting/repelling directions emanate from the fixed point δ and intersect the circle C. On the right, a later stage. The grey shaded disks illustrate the forward process. In any case, $U_{\mathcal{A}} \cup U_{\mathcal{R}} \equiv U \subset C$.

The remarks in sections 4.1 and 4.3, on the necessary cardinality of the Γ-type sequence for reaching $\theta \in \mathbb{L}$, finally find here their full import. The countable sequence of rational approximants p_n/q_n relates to a discrete-type inflation as the decimals chain grows longer and the denominators q_n grow in

[54] A deeper study on the flower metrics of $f_\lambda : e^{2\pi i\alpha}z + \mathcal{O}(z^k), k \geq 2$ would be helpful to understand whether ∞ is the correct upper bound or it could be sharpened to an optimal value. One such approach is not known to the author.

[55] The restriction of proposition 6.3.1 to f_λ will be further discussed at the end of section 6.8.

magnitude. From Liouville's limit and proposition 5.2.1, we recall that it is not possible to reach $\theta \in \mathbb{L}$ after finitely many steps and or not sufficiently fast.

This discussion is still unsatisfactory. Let's visualize it. We associate the sequence of fractions p_n/q_n to the possibility of determining the number $k \leq \aleph_1$ of attracting and repelling directions, the sets $\mathcal{A}_k$ and $\mathcal{R}_k$ respectively. According to lemma 6.3.1, we know that k is a multiple of q. Let C be a circle centered at the fixed point δ. Let $U \subseteq C$, where U is the set of intersection points u_n between C and the attracting/repelling directions emanating from δ. One sees that they 'impress' the circle C (see fig. 6.6.2/A). We are led to consider the following two subsets of U:

$$\begin{aligned} U_{\mathcal{A}} &= \{u_{\mathcal{A}_k} : \mathcal{A}_k \cap C\} \\ U_{\mathcal{R}} &= \{u_{\mathcal{R}_k} : \mathcal{R}_k \cap C\} \end{aligned} \tag{6.3.3}$$

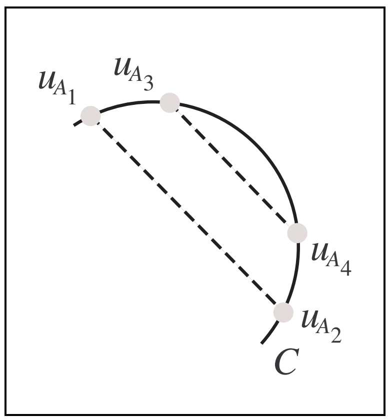


Figure 6.3.3: **Zooming into the inflation.** If the process is countable, the distance between any pair of points u is strictly positive.

And $U \equiv U_{\mathcal{A}} \cup U_{\mathcal{R}}$. According to the finite sequence of p_n/q_n, we have $U \subset C$, where $\mathrm{Card}(U) < \aleph_1$ and the distance between any pair of intersection points $u_{1_n}, u_{1_{n+1}}$ is always positive, $|u_{1_n} - u_{1_{n+1}}| > 0$.

6.4 3rd part: Hunting with Liouville and Cantor

This construction relies on the full and continuous inflation of the circle C and enjoys a precise correspondence to the classical *Cantor's diagonalization argument* on the impossibility of a one-to-one map between the natural numbers and a continuous line; indeed, one can use the diagonalization process to draw almost identical consequences here. Since C is a continuum, we need uncountably many intersection points u to cover it. The cardinalities of both $U_{\mathcal{A}}$ and $U_{\mathcal{R}}$ need to be 'upgraded' to $2^{\aleph_0}$, for U to cover completely C; the cardinality of the union set U will be $2^{\aleph_0}$ too. This was possible by merging the extension of theorem 2.5.4 to transfinite sequences of rational approximants with lemma 6.3.1. This translates the conclusions from such theorem into the geometries of complex iterated maps. The terms of Cantor's diagonalization and the chance of its accomplishment shall be revisited in light of transfinite sequences or of the speed rate of the one-to-one map.

6.5 Heuristics

We complete this discussion by looking at the question from the opposite viewpoint of non-transfinite sequences, because our construction develops some restrictions. It is helpful to recall this basic theorem from Set Theory:

Theorem 6.5.1. *The set $\mathbb{Q}$ of rational numbers, consisting of positive and negative fractions, can be put into one-to-one correspondence with $\mathbb{N}^+$.*

This theorem leads to the unfortunate empirical implication that, starting from $\mathbb{Q}$, one cannot reach $\theta \in \mathbb{L}$ and have uncountably many attracting and repelling directions which cover *continuously* the circle C around δ. It looks like impossible to generate transfinite fractions from fractions whose terms are finite ordinals. Fractions with finite ordinals as terms generate finitely many attracting and repelling directions impressing on C *discretely*: $|u_1 - u_2| > 0$ holds for any given pair of intersection points u_n, u_{n+1} on C. The cardinality of $\mathbb{Q}$ is $\aleph_0 < \aleph_1$. The 'jump' from finite to transfinite ordinals is *purely theoretical*. Non-linearizable hedgehogs are definitely out of reach in computational terms. The next results shall be interpreted under this remark.

6.6 4th Part: The basin annihilation

We move to the geometry of iterates near a Cremer point δ and we discuss the topology of the related invariant set.

Question 6.6.1. *Could a Leau-Fatou flower with uncountably many petals exist? Does it make sense?*

Given $\theta \in \mathbb{R}$, the run from $\mathbb{Q}$ to $\mathbb{L}$ stops when $\omega_{\mathcal{P}} = 0$: it is the extremal stage of our construction. It is a unique and new situation, where the flower topology around δ *is not kept up but annihilated.* One such mismatch may come up as the transfinite induction applies to the numerical terms of rational approximants p_n/q_n and to the geometrical terms of Leau-Fatou flowers with increasing petals number. One could expect, like for numbers, that any property, holding in *all successor cases* (with finite ordinals), would also hold in the *limit transfinite case.* This new situation alerts to take care of the topological implications, *as well as to focus on the Julia set instead of the basin component in the Leau-Fatou flower, during the transition from finite to transfinite ordinals*, in order to cross-check all statements and to prevent mistakes and contradictions. We study the shrinking width of petals and the consequences in topological terms. From the contraction process one expects that each single petal finally turns into a uni-dimensional curve. The construction itself can be compared to a race splitting into two stages, the *run* and the *arrival*:

1. the *run* consists of a *countable union* of attracting and of repelling directions while $\theta \in \mathbb{L}$ is *approximated* through rational convergents p_n/q_n; petals still have positive area.

2. the *arrival* is the *uncountable union set* of such directions when $\theta \in \mathbb{L}$ is *reached* by rational convergents p_n/q_n of transfinite kind. Inductively, the petal angle ω vanishes identically.

Question 6.6.2. *Could petals have zero area and be uni-dimensional curves?*

For sake of simplicity, we will *consider petals as open subsets* of the basin $\mathcal{B}_0$ of attraction (to the fixed point $\delta = 0$) *exclusively here.* The consequences on the related Julia set will be drawn at last. We show that petals do not degenerate into curves when the angle $\omega = 0$ or when the repelling and attracting petals (according to the classic Leau-Fatou flower construction) disappear. It is not so premature then to infer that Fatou components cannot be uni-dimensional by invoking one fundamental theorem in classic holomorphic dynamics, in the same original version as stated by Fatou and by Julia for meromorphic maps:

Figure 6.6.1: **Area and segments.** The rectangle area can be considered as the union of uncountably many segments $\overline{AB}$.

Theorem 6.6.1. *The Julia set J is totally disconnected or a continuous line, unless it covers the entire Riemann sphere: $J \equiv \mathbb{C}_\infty$.*

Given a two-dimensional set (either $\mathbb{C}$ or $\mathbb{C}_\infty$) and according to the topological definition of Fatou set ($\mathcal{F} \equiv \mathbb{C}_\infty \backslash J$), one understands that $\mathcal{F}$ shall have positive area or be empty. Hence:

Proposition 6.6.1. *$\mathcal{F}$ cannot consist of isolated points.*

Nonetheless these are still weak assumptions. Stronger evidences will be brought in. First, ad absurdum, let each petal be a curve by now.

Let $\theta \in \mathbb{L}$. According to our deductions from the application of Leau-Fatou flower theorem to transfinite fractions, there exists uncountably many attracting/repelling directions around the Cremer point δ. By the combinatorial viewpoint, it is easy to check that there are only four distributions for the two sets $U_{\mathcal{A}_k}$ and $U_{\mathcal{R}_k}$ in the construction (6.3.3):

A)	$U_{\mathcal{A}_k}$	
B)	$U_{\mathcal{R}_k}$	
C)	$U_{\mathcal{A}_k}$ $U_{\mathcal{R}_k}$	$U_{\mathcal{R}_k}$ $U_{\mathcal{A}_k}$
D)	$U_{\mathcal{A}_k} \equiv \emptyset$	$U_{\mathcal{R}_k} \equiv \emptyset$

Since $U_{\mathcal{A}_k}$ and $U_{\mathcal{R}_k}$ are sets of attracting and of repelling directions, distributions are permutable: so cases 2 and 3 are the same. The case 4 assumes that both sets are empty, *locally around the fixed point.* Since attracting and repelling directions point out to the neighboring regions of the basins of attraction to δ and to ∞ respectively, that previous table offers the resume of the 4 distribution cases for the basins $\mathcal{B}_0$ and $\mathcal{B}_\infty$ around δ. We will check which one holds or not. With no loss of generality, these four distributions radially spread inside regular sectors with amplitude $\omega, 0 \leq \omega_i \leq 2\pi$, as it customarily holds for Leau-Fatou flowers. We will dismiss all cases but D) because of A), B) and C) lead to contradictions (refer to figs. 6.6.2):

A) the *full distribution* of $\mathcal{R}_0$ over 2π. There is one only sector with two angles $\omega_0 = 2\pi$ and $\omega_\infty = 0$. The basin $\mathcal{B}_0$ surrounds the Cremer point δ, which by definition belongs to J_θ. Thus δ is isolated. Contradiction: Julia sets for polynomials (1.1.5), have no isolated points and are not of (multi-)exponential kind ([64], p. 47).

B) *full distribution* of $\mathcal{R}_\infty$ over 2π. Again, one only sector where amplitudes are $\omega_\infty = 2\pi$ and $\omega_0 = 0$. Here $\mathcal{B}_\infty$ surrounds the Cremer point. Thus δ is isolated and the contradiction holds in the same terms as at A).

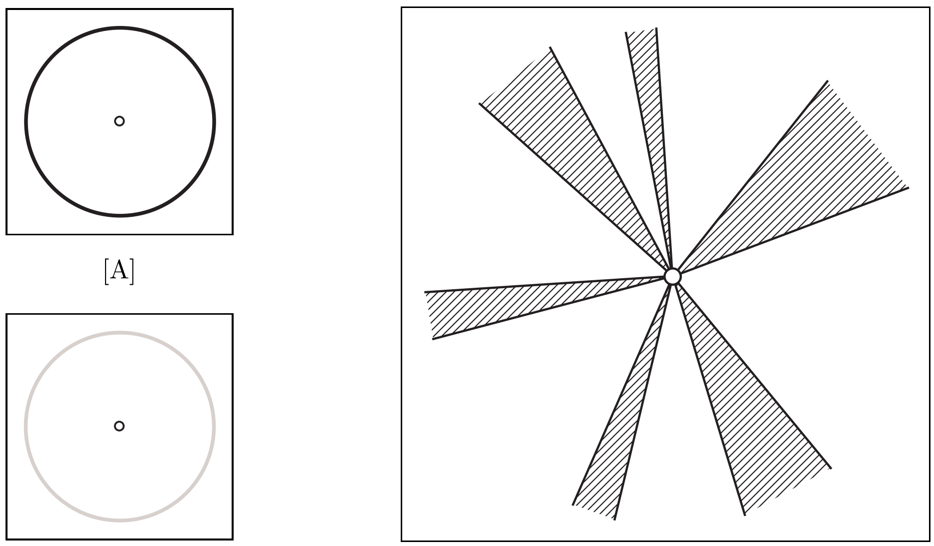
C) *sectorial (or interlacing) distribution* of $\mathcal{R}_0$ and of $\mathcal{R}_\infty$ over 2π. Because of entry 2 at p. 85 holds for $\theta \in \mathbb{L}$, one assumes the existence of more unions of uncountably many uni-dimensional petals. Around δ, the basins $\mathcal{B}_0$ and $\mathcal{B}_\infty$ form $2n$ alternating sectors, with strictly positive angles ω_{0_i} and ω_{∞_i} respectively, so that

$$\sum_{i=1}^{n} \omega_{0_i} < 2\pi, \quad \sum_{i=1}^{n} \omega_{\infty_i} < 2\pi \Rightarrow \sum_{i=1}^{n} \omega_{0_i} + \sum_{i=1}^{n} \omega_{\infty_i} = 2\pi.$$

See fig. 6.6.1. We want to discuss on the number of such sectors. This situation splits into 2 sub-cases with interlacing distributions:

C.a) *if alternating and repelling sectors are uncountably many*, A) and B) hold simultaneously, as if alternating directions superimpose on repelling ones. Besides the odd implication that both basins would fill-in the same neighborhood around δ, we would draw again the contradiction: the Cremer point δ is isolated.

C.b) *if alternating and repelling sectors are countably many*, the same distribution like for Leau-Fatou flowers, when $\theta \in \mathbb{Q}$, holds. But (3.2.1) holds by hypothesis here; equivalently, if $\omega > 0$ we find countably many elements $u_{\mathcal{A}_k}, u_{\mathcal{R}_k}$ of the construction (6.3.3). Thus both sets $U_\mathcal{A}, U_\mathcal{R}$ are countable. But $\theta \in \mathbb{L}$: the number of elements $u_{\mathcal{A}_k}, u_{\mathcal{R}_k}$ shall be necessarily uncountable and yielding $\omega = 0$. Contradiction.



[A] [B] [C]

Figure 6.6.2: **Contradictions arise.** These drawings show the three hypothetical distributions of petals when $\omega = 0$. They all lead to contradictions. In [A], one shows the distribution when $\mathcal{B}_0$ would be a continuum which completely surrounds δ. We find the same configuration in [B] but for $\mathcal{B}_\infty$. In [C], the sectorial distribution where continua of $\mathcal{B}_0$ and $\mathcal{B}_\infty$ alternate.

We know that the point at infinity is super-attracting and fixed for (1.1.5): the basin $\mathcal{B}_\infty$ to infinity is non-empty. The dichotomy between Fatou sets F and Julia sets J shall necessarily hold for such polynomials in $\mathbb{C}_\infty$: $\mathbb{C}_\infty \equiv F \cup J$. We have that this fourth and last case shall be true:

D) *empty distribution* of $\mathcal{B}_0$ and of $\mathcal{B}_\infty$ all over 2π.

We term the situation in D) as the *basin annihilation.* Therefore this local statement follows:

Proposition 6.6.2. *Given a Cremer fixed point* δ *and the holomorphic germ* (1.1.5), *there exists a bounded neighborhood* $\mathcal{H} \subset \mathbb{C}$ *of* δ, *so that* $\dim(\mathcal{B}_0) \not\equiv 1, 2$ *and* $\dim(\mathcal{B}_\infty) \not\equiv 1, 2$. *In addition,* $\dim(\mathcal{B}_0) \not\equiv 0$ *and* $\dim(\mathcal{B}_\infty) \not\equiv 0$ *because of proposition 6.6.1. Thus* $\mathcal{B}_0 \equiv \emptyset$ *and* $\mathcal{H} \cap \mathcal{B}_\infty \equiv \emptyset$*: finally,* $\mathcal{H} \equiv J$.

$\omega > 0$ holds when $\theta \in \mathbb{Q}$. We deduce the following corollary:

Corollary 6.6.2. *Given $f_n(\delta) \equiv \delta, f'(\delta) \equiv e^{2\pi i\theta}, \theta \in \mathbb{Q}$, the petals area is positive.*

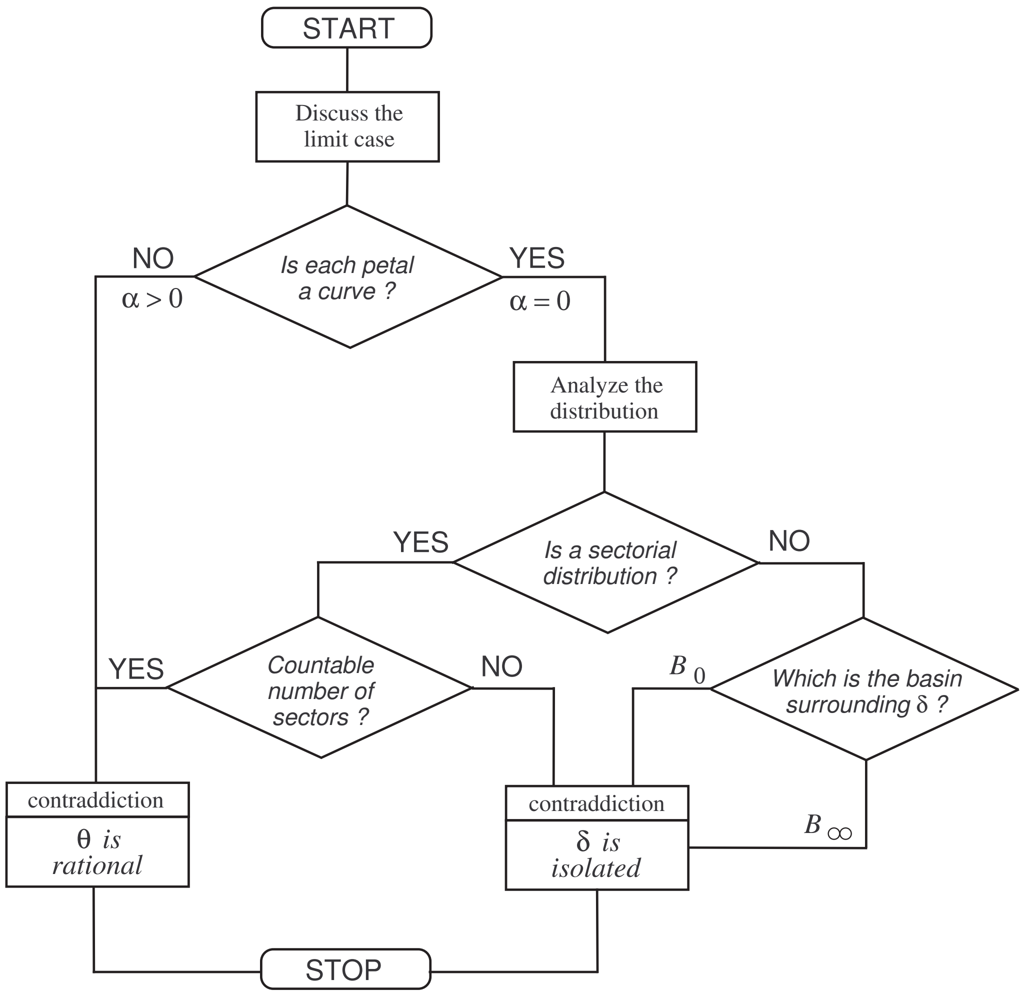


Figure 6.6.3: **Flow diagram of the proof.**

The proposition 6.6.2, although labelled as '*pathological*' in respect of the usual Julia set topologies for the iterates of rational maps, does not give rise to contradictions. The annihilation of the bounded basin $\mathcal{B}_0$ follows the accomplishment of the transfinite induction (discussed in section 6.3), where the property $\mathcal{T}$, namely the existence of a Julia set curve intersecting the finite fixed point, always holds while the argument θ runs from $\mathbb{Q}$ to $\mathbb{L}$. The bounded basin of attraction cannot grant one such property, because it disappears when $\theta \in \mathbb{L}$. For the family of polynomials f_α, when $\alpha \in \mathbb{Q}$ or $\alpha \in \mathbb{L}$, the Julia set always intersects $\delta = 0$. One minor conclusion from the previous proof is that *the existence of the Leau-Fatou flower*, assumed as a neighborhood U of δ, where $\delta \in U$, $U \cap \mathcal{B}_0 \not\equiv \emptyset$ and $f_\alpha^n \to \delta$, *may not be meant as the dominant property for the transfinite induction to work.* One understands the existence of a (bounded) region $\mathcal{H}$ about δ, whose multiplier is $e^{2\pi i\theta}$ and for $\theta \in \mathbb{L}$, so that $\mathcal{B}_0 \cap \mathcal{H} \equiv \emptyset$ and $\mathcal{B}_\infty \cap \mathcal{H} \equiv \emptyset$. After the propositions 6.6.1 and 6.6.2, we argue that $\dim(\mathcal{B}_0) \not\equiv 0, 1, 2$ over $\mathbb{C}_\infty$, when $\theta \in \mathbb{L}$. We can finally state:

Theorem 6.6.3. *Let $f_\theta = e^{2\pi i\theta} z + \mathcal{O}(z^2)$ and $\theta \in [0, 1]$ be Liouville. There exists a bounded, non-empty neighborhood of δ which intersects no Fatou components.*

This can be equivalently stated in terms of Julia sets:

Theorem 6.6.4. *Let $f_\theta = e^{2\pi i\theta} z + \mathcal{O}(z^2)$ and $\theta \in [0, 1]$ be Liouville. Let J_θ be the related Julia set. There exists a bounded, non-empty area $\mathcal{H} \supset \delta$ so that $\mathcal{H} \equiv J_\theta$. The Julia set fills-in $\mathcal{H}$ like a plane-filling curve.*

It would be interesting to investigate on the metrics. In particular, on the petal height $h_{\mathcal{P}}$ when p_n/q_n approximate $\theta \in \mathbb{L}$, in order to determine the size of $\mathcal{H}$.

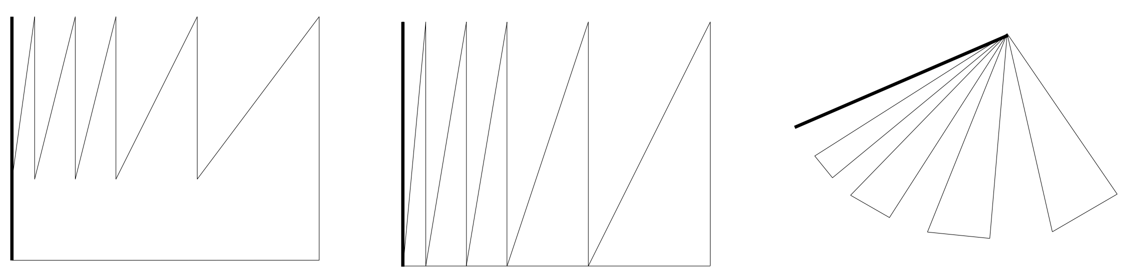
6.7 The infinite broom and local connectivity

The previous construction splits into the non-limit and the limit stage, where there are countably and uncountably many attracting/repelling directions respectively. The investigation of the topological properties can be made through the revisitation into a parameterized version resulting from the modifications applied to the *infinite broom space B*, shown in fig. 6.7.1/*a*. We introduce this metric[56] definition of *local connectivity* ([64], p. 169):

Definition 6.7.1 (Local connectivity). *Let $X \subset \mathbb{C}_\infty$ be a compact metric space. X is locally connected if, for every $\epsilon > 0$, there exists $\delta > 0$ so that any two points of X of distance $< \delta$ are contained in a connected subset of X of diameter $\leq \epsilon$.*

[56] There exist more equivalent versions, but not involving metrics. We like this one because of matching our metric environment related to the basin annihilation. Observe that this definition is *recursive*: at the successive stage, the role of X can be played by the 'connected subset of X' and so on. Thus ϵ and δ can be arbitrarily decreased. Moreover, if the operator '$< \delta$' is replaced with '$\leq \delta$', this definition switches the local connectivity from the *open* to the *closed* type (which does not affect the possibility for X to be open or closed). We are not interested anyway in this further distinction, keeping up the above version.

The broom spaces are generated by a sequential construction where successive subset combs C_n infinitesimally squeeze. B is not locally connected because of the positive distance between any pair of C_n. This lets the above connectivity test fail at any pair of points belonging to two distinct combs. Our modifications strengthens the reasons for this failure by first (6.7.1/b) letting $C_n \cap C_{n+1} = a_n, n \to \infty$, where a_n is a sequence of points; finally (6.7.1/c) all resulting combs intersect at one point A, $\forall C_n : \bigcap C_n = A$. The sequence C_n tends to shrink up to a limit line, thus following the same fate as petals do in our construction. Equivalently, flowers (intended the union of disjoint petals) are compact, connected but not locally connected, according to definition 6.7.1: for example, consider two points of two distinct petals respectively. These conclusions extend to their boundaries: either ∂B and J_θ are not locally connected for all non-limit stages of the construction. Anyway one notices that both curves are *locally pathwise connected*.[57]



(a) (b) (c)

Figure 6.7.1: **Broom space and petals.** As the middle vertexes of fig. a are taken to the bottom baseline (b) and top vertexes are glued into the same point (c), the baseline breaks into a number of segments, playing as the bases of the resulting triangles. The original broom space on the left (a) turns into another equivalent model on the right (c), being petals-like: the latter does not match the flower configuration, but it plays as a version parametrized by one petal which, under the action of q^κ in the lemma 6.3.1, squeezes to a line (marked in bold), being the limit to this construction.

At the limit stage, combs shrink to a line. The parametrized construction of fig. 6.7.1 just offers a restricted view, playing only for the broom space version and equivalently for just one petal. But the full envision of Julia sets J_θ around the origin opens the broom parametrization to an arbitrary number of petals: here the limit stage corresponds to uncountably many such lines belonging to

[57]In particular, the broom boundary ∂B is a simple curve and can be also *locally arcwise connected*, while the Julia set J_θ cannot because it may include double points.

J_θ. It is easy to assume the existence of a planar continuum filled in by the boundary J_θ at the non-linearizable case. It is straightforward that J_θ is also locally pathwise connected. But, in addition, J_θ is a continuum now and spreads radially everywhere: therefore J_θ is *locally connected*. One sees that the origin is *inaccessible* from any point $z \notin \mathbb{C}_\infty$. This consequence finds one analogy to the following theorem, stated by Pérez-Marco in 1994 ([72], p. 4):

Theorem 6.7.1. *If a rational function has a Cremer point, its Julia set contains a dense set of non-accessible points.*

The persistence of the Julia set curve (along our whole construction and always enjoying the local pathwise connectivity property) shows that the transfinite induction shall apply to J_θ exclusively and to its topological properties $\mathcal{T}$, like we remarked before in the beginning of section 6.3.

6.8 5th Part: Locally connected non-linearizable Hedgehogs

We explain what happens to non-linearizable hedgehogs in local terms, that is, inside a sufficiently small and bounded neighborhood $\mathcal{H}$ of the fixed point $\delta = 0$, where the previous construction took place. This motivates our conclusions through complementary and known results about Diophantine irrationals. If $\theta \in \mathbb{L}$, the previous construction leads us to assume that

$$\mathcal{B}_0 \cap \mathcal{H} = \emptyset \qquad \textit{and} \qquad \mathcal{B}_\infty \cap \mathcal{H} = \emptyset.$$

It is straightforward that consequences shall be drawn in terms of the local action induced by our construction, which just works locally, hence the results shall be also locally evaluated. We already showed that the petal height is necessarily bounded in proposition 6.3.1: so hedgehogs cannot extend up to the point at infinity but they just fill in a bounded neighborhood of δ – in particular the Leau-Fatou flower. Petals shrink to zero area and they finally disappear. One is able to show that $\mathcal{B}_0 \equiv \emptyset$ for $\theta \in \mathbb{L}$. Let the quadratic polynomial $P_\theta : e^{2\pi i\theta}z + z^2$ where the argument $\theta \in \mathbb{L}$ and the related Julia set J_θ.

As we remarked before, J_θ is a (1°) *continuum*. The transfinite induction from $\mathbb{Q}$ to $\mathbb{L}$ also assures that J_θ is a (2°) *curve* too. Thus J_θ *is locally con-*

nected[58] *and non-linearizable Julia set* spreading radially everywhere around the Cremer point δ. For sake of clarity, we liked to follow the terminology in the recent preprint [9], where such J_θ is classified as of *solar* type:[59] in light of these results, as recently stated into the theorem 1.1 at [9], p. 3, the local connectivity property can be re-framed as follows:

Proposition 6.8.1. *J_θ is connected im-kleinen.*

When $\theta \in \mathbb{L}$, it is like there would be transfinitely many attracting/repelling directions, or equivalently the Julia set J_θ emanates from δ so that, in a sufficiently small disk C centered at δ, one has $J_\theta \cap C \equiv C$. By a proper homeomorphism, there exists a full Lebesgue measure set of angles over S^1, where *degenerate* impressions (i.e., consisting of one point) are given. Resuming, given $\theta \in \mathbb{L}$, the Julia set for (1.1.5) is (1°) a locally connected curve, (2°) spreads radially everywhere about the origin, (3°) fills in a bounded and connected region (theorem 6.6.4). If $\theta \in \mathbb{L}$, J_θ is a plane-filling curve inside a bounded neighborhood of δ and we can state:

Theorem 6.8.1. *Given $\theta \in \mathbb{L}$, the Hausdorff dimension of the Julia set J_θ is 2.*

On the other hand, if $\theta \notin \mathbb{L}$ but $\theta \in \mathcal{D}(2+)$, one refers back to a complementary result by McMullen ([63], p. 1):

Theorem 6.8.2 (McMullen). *The Hausdorff dimension for Julia sets of polynomials f_θ, where θ is Diophantine, is strictly less than 2.*

Let $f_\theta : e^{2\pi i\theta} z + \mathcal{O}(z^2)$, where $\theta \in \mathbb{L} \subset [0, 1]$. We summarize all the topological situation around the Cremer point into these equivalent conditions:

1. The Julia set J_θ of f_θ has Hausdorff dimension 2, and $\delta \in J_\theta$;

[58] This property allows to apply the so-called Carathéodory's extension theorem ([64], p. 169): *a conformal isomorphism $\psi : \mathbb{D} \overset{\cong}{\to} U \subset \mathbb{C}_\infty$ extends to a continuous map from the closed disk* $\overline{\mathbb{D}}$ onto $\overline{U}$ *if and only if the boundary ∂U is locally connected, or if and only if the complement $\mathbb{C}_\infty \backslash U$ is locally connected.* Here we can set $\overline{U} \equiv \partial U \equiv \mathcal{H} \equiv J_\theta$. See also [73], pp. 246 and ff., where the correspondence between the two models was shown in order to earn one more degree of freedom and study iterates via analytic circle diffeomorphism.

[59] This work lists a second configuration, termed *red dwarf* and featuring no local connectivity around δ. The existence of a similar case was shown by Douady and Sullivan [92] for the quadratic family $P_c(z) : z^2 + c$, with Cremer points and where $z, c \in \mathbb{C}$. See also [91] for a study on the mono and bi-accessibility to Cremer points for polynomials $P_c(z)$.

2. J_θ is a plane-filling curve inside a bounded and connected, planar neighborhood with strictly positive area around δ;

3. $\mathcal{B}_\infty$ is the only Fatou component, so $\mathcal{B}_\infty \equiv \mathbb{C}_\infty \backslash J_\theta$.

The non-linearizable hedgehog is one such Julia set J_θ. This corollary follows:

Corollary 6.8.3. *Let $\theta \in \mathbb{L} \subset [0,1]$. The dynamical system generated by the iterates of $f(z) : e^{2\pi i\theta}z + \mathcal{O}(z^2)$ consists of the basin $\mathcal{B}_\infty$ of attraction to ∞ and of the Julia set J_θ, whose Hausdorff dimension is 2 and fills in a bounded, connected, two-dimensional neighborhood around the Cremer point $\delta = 0$.*

It would be interesting to study how the Hausdorff dimension increases together with the order κ of the Diophantine value θ and how this affects both the topology and geometry of hedgehogs. Along this direction, we mention the following theorem by Heinemann and Stratmann ([47], p. 572):

Theorem 6.8.4. *Let $f_t : \mathbb{C} \to \mathbb{C}$ be given by $z \mapsto e^{2\pi i\theta}z + z^2$. There exists a sequence $\{\theta_n\}$ such that, where $\dim_H(J(f_{\theta_n}))$ denotes the Hausdorff dimension of the Julia set $J(f_{\theta_n})$, the following formula holds:*

$$\limsup_{n\to\infty} \dim_H(J(f_{\theta_n})) = 2.$$

In our notation, since the Julia set J_θ of theorem 6.8.1 covers a non-empty planar region $\mathcal{H}$ and spreads radially around δ with no provileged directions, one argues that J_θ shall include non-accessible points from $\mathbb{C}\backslash\mathcal{H}$.

7 Remarks and speculations

7.1 Flowers and singularities distribution

In conclusion, our hypotheses and results about non-linearizable hedgehogs depend on the polynomial nature of $P_\theta : e^{2\pi i\theta}z + \mathcal{O}(z^k), k \geq 2$ and to $\theta \in \mathbb{L}$. Therefore we cannot miss to remark the following restrictions.

First, our construction depends on the geometry of the Leau-Fatou flower: as the sequence $\{p_n/q_n\}$ approximates $\theta \in \mathbb{L}$, the shape of J_θ is deduced by that of the flower, while the number of petals grows. Given a rational angle, this flower has no privileged direction(s) to spread its petals. Via transfinite induction, this same property is inherited by J_θ. We found out the so-called solar hedgehog J_θ, i.e. a Julia set curve with (1°) Hausdorff dimension 2, (2°) filling in a non-empty region, (3°) including the irrationally indifferent fixed point δ, (4°) spreading radially everywhere around it. In second instance, we assumed a very simple singularities distribution for P_θ: lacking of poles at finite distance and of essential points, a relatively easy local environment was to be investigated about the origin. Here nothing affects the radially everywhere distribution of J_θ because, as one guesses, the distance from the origin to the closest singularity of P_θ at ∞ would be shorter than the maximal modulus of any point $z \in J_\theta$ (i.e.: the singularity seems to be too far to not affect the radially everywhere extension of J_θ). Hence the 'solar' term follows.

At this stage, we might want to get rid of these restrictions, looking at the wider realm of complex rational functions $R(z)$. The investigation on the iterates of $R(z)$ was also the starting point for Fatou and Julia, back in late 1910s. Regarding the questions about local behavior, these two French mathematicians, and later Cremer, Siegel, Cherry et Alia, were concerned of determining the possibility of the linearization $R^n(z) \mapsto e^{2\pi i \alpha n}, \alpha \in \mathbb{R}\backslash\mathbb{Q}$ in a sufficiently small neighborhood of the indifferent fixed point δ. After the outstanding contributions by Cremer and Siegel begun to settle the question definitely, it was natural to deepen several aspects by focusing on the iterations of polynomials P_α, working out the linearization when α is algebraic of finite order κ. It turned out that this trend shifted the original question from rational maps to polynomials P_α or in the form $z^2 + c$ where $c \in \mathbb{C}$, producing a wide literature.

Although the knowledge of hedgehogs topology for a general polynomial is still far from the full accomplishment, the question on linearization has been completely settled for the quadratic type of P_α. Besides higher orders, one would also like to pay the attention again to rational maps,[60] where the investigation of hedgehogs might open to newer perspectives, as we try to illustrate further.

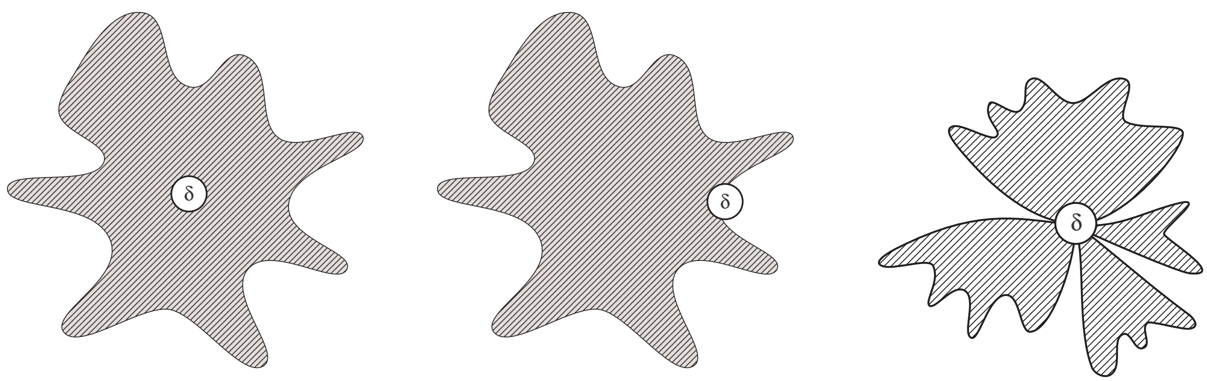
Given an hedgehog $\mathcal{H}$, its geometry (as well as topology) could be shaped,

[60] Indifferent dynamics for rational maps have been mostly investigated in terms of Blaschke's products, introduced in the next section. Refer to [48, 49, 69, 74, 88]. See also [75, 80, 81] for general rational maps.

as we conjecture, by the singularities distribution around $\mathcal{H}$; for example, by singularities at finite distance and of different order. No doubt that another remarkable question is about the accessibility to $\mathcal{H}$. In this direction, the reader can find some related results scattered in a number of works [8, 9, 54, 85, 91, 97].

Because every polynomial is a rational map, this latter family shows up as the best place to ground the general theory on hedgehogs:

Question 7.1.1. *Given a complex rational map* $R(z) \equiv \frac{p(z)}{q(z)}$ *with Cremer point* δ*, how can the distribution of singularities of* $R(z)$ *affect the shape and the extension of the non-linearizable hedgehog* $\mathcal{H}$ *?*



(*a*) (*b*) (*c*)

Figure 7.1.1: **May it be a full resume about the hedgehog topologies for rational maps?** Besides the solar type, which we showed to appear for iterates of polynomial P_θ and drawn in (a), we guessed two more configurations in (b) and in (c), possibly occurring for rational maps R. Cremer point δ is the white disc, located in the interior (thus inaccessible from $\mathcal{B}_0$) or on the boundary (with one or more accessibility directions respectively). The number of components in (c) is guessed to be $n \geq 2$ in general.

It seems likely that an answer to this question can be presumably given by looking at the algebraic or transcendental polynomial nature of $p(z)$ and of $q(z)$. In this direction, as even shown in [9], hedgehogs are not only of *solar type*, thus the Leau-Fatou flower cannot be the only model to shape $\mathcal{H}$ for a rational map with Cremer points. One might want to deepen the conditions determining what are the shapes of $\mathcal{H}$. One hypothetical scenario was sketched out in figures 7.1. Drawings are compatible to $\mathbb{C}_\infty = J_\theta \cup \mathcal{B}_\infty$, when $\theta \in \mathbb{L}$. One would like checking if different situations occur: for example *if the Hausdorff* *dimension of non-linearizable hedgehogs is lesser than* 2, or *if the complement* $\mathbb{C}\backslash\mathcal{H}$ *includes more than one basin of attraction.* One notices that $\mathcal{B}_0 \equiv \emptyset$ holds in all the three suggested configurations; the Julia set curve has still Hausdorff dimension 2 and fills in the dashed regions. With regard to figs. 7.1, we wonder whether (b) or (c) might hold and at what conditions: if just numerical because of the given rotational angle or they depend on the existence of neighboring singularities which are sufficiently close to J_{R_θ}. It would be also interesting to determine theorems ruling these possibilities and, if so, to give examples of rational maps related to such two cases. Since the Julia set structure is homogeneous, the situations where non-linearizable hedgehogs can be met seem to restrict to rational maps with two basins of attraction.

7.2 On the doubly connected case

The construction we illustrated in section 6.1 intimately relates to the polynomial (1.1.5) and to the Leau-Fatou flower, when $|e^{2\pi i\theta}| = 1$ and $\theta \in \mathbb{Q}$. By looking at hedgehogs as degenerate Siegel compacta, one understands that the previous proof is peculiar to the simple connectivity of such compacta. This represent a restriction because it cannot be exported to doubly connected hedgehogs, when the rotation domain, termed a 'Herman ring', is isomorphic to an annulus ([73], p. 281). One also knows that Herman rings cannot arise for polynomials according to Sullivan's classification (theorem 9.2, [7], p. 122), but they can for Blaschke products

$$f(z) : e^{2\pi i\theta} z^2 \frac{z-a}{1-\overline{a}z}, \tag{7.2.1}$$

i.e. for the group of conformal automorphisms of the unit disc, with degree $d \geq 3$. The Diophantine-Liouville dichotomy holds here as well ([64], pp. 148–151), like in the simple connected case of Siegel discs [88]. One wonders if a modification of our previous construction may help. If so, it would be interesting to check first the local invariant set for (7.2.1), when $\theta \in \mathbb{Q}$ and then let the construction run, with proper adaptations. Given $\theta \in \mathbb{L}$, one expects a plane-filling curve inside a doubly connected neighborhood here.

7.3 On computer simulations

Computer simulations, in the two cases where the Siegel compactum does not degenerate into the Cremer point, are very close to their actual shape. Although computers can handle rational numbers, we do not see Leau-Fatou flowers when we input a sufficiently sharp rational approximation of an irrational number μ, but quasi-rotational dynamics, i.e. fairly close to what the full Siegel compactum related to μ would look like if the endless chain of decimals could actually be input.

'Irrationality' does not only involve the existence of the endless chain of decimals, but also the non-periodicity of the sequence. In addition, we do not feel the separation between rationals and irrationals as drastic as the fractional representation might suggest. The existence of such a continuity can be further strengthened by noticing the almost-reproducibility of some computer simulations mentioned at the top, which depict hedgehogs with non-empty Siegel compacta. The classic computer experiments on Siegel disks showed that we deal with numbers which are not properly rationals, in the sense that, as we input them into the value θ of $e^{2\pi i\theta}z + \mathcal{O}(z^2)$, we do not see Leau-Fatou flowers; on the contrary, they cannot be assumed as properly irrationals, because of decimals truncation. We know that the former values, say ξ, are approximations of irrationals; the term 'approximation' is just a numerical convention about ξ being arbitrarily close to another value θ, anyway this term does not retrieve any information on the dynamical properties we are interested in here. Therefore, from the dynamical viewpoint, one may speak of 'almost-irrational' numbers, mid-way between rational and irrationals: again, rationals showing dynamical properties very close to those for irrationals. A first clue can be watched into the decimals chain. Being periodic for rationals and aperiodic for irrationals, is it then possible to classify all reals by a parameter which serves as a *disorder indicator*, telling how irregular the decimals chain is?

If so, we guess that one such indicator should take on the lowest value for rationals whose period starts with the first decimal place (no disorder), then grows with the place of the period, finally becoming maximal for irrationals. One might try to check whether there are irrationals whose decimal chain is 'wilder' than others. One such disorder parameter could even help to realize

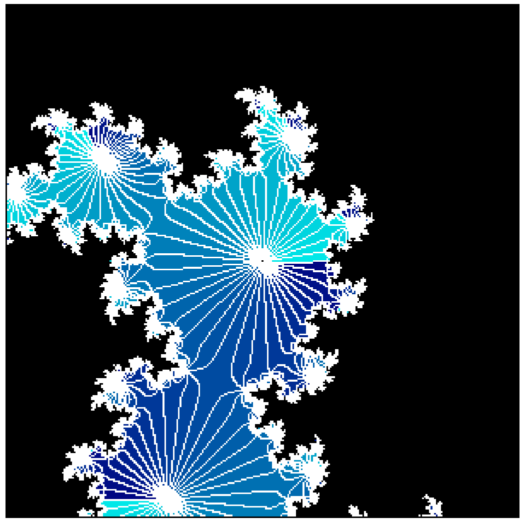


Figure 7.3.1: The 37-petals Leau-Fatou flower for the quadratic germ with $\theta = 7/37 = 0.\overline{189}$. Petals number is as larger as they are tinier. They have been contoured in white to be highlighted.

what we termed as the 'ideal journey' through the indifferent dynamics[61] and sketched out in the previous proof. Although irrational values are beyond the scope of algorithms, one can classify their resistance to rational approximation, according to the known limit (3.2.1), which states that Diophantine irrationals may stop access to rational approximants, while Liouville numbers do not, at least theoretically through a suitable sequence.

7.4 On the computational complexity of Julia sets

According to proposition 5.2.2 (p. 74) and to the previous discussion, one can easily argue that *it makes no sense to look for features of dynamical systems when θ is Liouville via rational approximants.* As approximation is the art of getting arbitrarily close to a given value, we do not want to approximate Liouville numbers because we find rationals (from both theoretical and empirical viewpoint) being "arbitrarily close" to Diophantine numbers of large order κ. Thus they cannot help to retrieve the expected properties (either in dynamical or numerical terms) of Liouville numbers. Approximation still works fine for Diophantine irrationals, although their computation complexity grows with

[61]From rationals to Liouville numbers and from Diophantine to Liouville irrationals (see [83]).

their order. We come finally to the same conclusions as stated in theorem 2.5 by Braverman and Yampolsky (see [5], p. 5).

In any event, we do not agree with this assertion ([5], p. 4): '[...] (1°) *all Cremer quadratic Julia sets are computable - this despite the fact that* (2°) *no informative high resolution images of such sets have ever been produced*'. We refute (1°) this claim both in theoretical (because of the regular sets model) and in empirical terms (providing digital images in [83]): non-linearizable hedgehogs cannot be displayed electronically. In addition, one should remark that (2°) (see [15], section 3, p. 4) relies on the empirical investigation of the classic graphical methods (working through an exclusively iterative approach) for Julia sets and therefore cannot attest if the problem is solvable or not. Such flaws have been discussed in details by Milnor [64], again by Binder, Braverman and Yampolsky [5, 6, 13, 14, 15, 98] and finally by the author in terms of hedgehogs exclusively [83], where a time saving approach was given (see examples in figs. 1.1.1/A-B-C here at page 16). See also [10] for a general overview.

As we remarked in section 1.3, global graphical methods in Holomorphic Dynamics cannot be customized to fit the local configurations. Our approach, extensively discussed in [83], joins iteration to the imitative strategy and covers linearizable hedgehogs exclusively.

7.5 Trends are eloquent

The last remarks on the computability of Julia sets imply that, although algorithms cannot visual the dynamics attached to Liouville numbers, the sequence of nested sets E_h (approaching θ) yields, an endlessly shrinking sequence as the width of E_h tends infinitesimally to 0, as $h \to \aleph_0$:

$$\liminf_{h=1,2,\ldots,\aleph_0} \|q_h\theta\| = 0,$$

without taking on the limit itself. Given $\theta \in \mathbb{L}$, one cannot obtain its value but can at least indicate the trend of what is happening inside the sequence of nested, arbitrarily small neighborhoods of θ itself. The results we stated before offer a response to the problem 11-b on generic angles, posed by Milnor in [64], p. 130: the width of E_h (the countable intersection of open sets) is always positive, so there exists at least one generic value $\xi \in \mathbb{R}$ inside each E_h, because $\mathbb{R}$ is dense in itself. In addition, the monotone and strictly decreasing trend ties to the growth terms of Cremer's Non-linearization theorem[62] ([64], p. 117):

Theorem 7.5.1 (Cremer Non-linearization theorem). *Given $\lambda = e^{2\pi i\theta}$ on the unit circle and given $d \geq 2$, if the d^k-th root of $1/|\lambda^k - 1|$ is unbounded as $k \to \aleph_0$, then no rational function of degree d with a fixed point of multiplier λ is locally linearizable.*

In fact

$$\limsup \frac{1}{\sqrt[d^k]{|e^{2\pi i\theta k} - 1|}} = \infty$$

either as d^k increases by $k \to \aleph_0$ or when $\liminf |e^{2\pi i\theta k} - 1| = 0$, amounting to the shrinking widths of E_h in the multiplicative model or as the limit (3.2.3) in the the additive model over $\mathbb{R}\backslash\mathbb{Z}$ respectively.

7.6 Cremer values from a different viewpoint

Recalling the proposition 4.1.1, we claim that given a dense set E of fundamental points (for example, $\mathbb{L}$), the zero measure set E_h cannot include the fundamental points exclusively and it is dense too: hence $E \subset E_h$. Suppose it does not. Because of the nesting relation, there are two possible cases. The first is $E \supseteq E_h$. But $E \supset E_h$ cannot hold because of the shrinking behavior of the sequence E_h. Let $E \equiv E_h$ and E_h^θ be a sequence of sets so that E_h shrinks to θ (see fig. 3.2.1). We can find either an arbitrarily large index h and a many-to-one formula f which transforms E_h^θ into θ, that is, $f(E_h) = \theta$. Thus the inverse f^{-1} is a multi-valued function mapping θ back to the dense set E_h of points a or, equivalently, the roots of

$$F(\theta) : f^{-1}(\theta) - a = 0$$

are transfinitely many. Evidently, f^{-1} cannot be a polynomial of finite degree. In other words, one such formula $F : U^{k+1} \to V^k$ should map a continuum U

[62]George Adam Pfeiffer was the first to attest the possibility that our quadratic polynomials with an irrationally indifferent fixed point could even not be linearizable inside a sufficiently small neighborhood of the finite fixed point δ at 0.

of Euclidean dimension $k + 1$ to a continuum V of dimension k, for example. The condition $k = 0$ holds for the case of Liouville numbers/Cremer points, but the same discussion could extend to Euclidean spaces of arbitrarily higher dimensions. As we showed in section 4.9, Inverse Ackermann functions just offer a theoretical way to make it happen although, according to our remarks on Ackermann functions in section 6.4, we are inclined to believe that it is improbable that one such function f^{-1} is writable. The examples we gave belong to formal systems only, whose existence is theoretical. We would crash onto the same problems that arose from the Burali-Forti paradox, when we dealt with the forward sequence. The relation between backward indexes would be now restated as

$$\Omega - 1 < \Omega \leq \Omega.$$

Again, the solution to move backwards from transfinite to finite ordinals is theoretical exclusively.

When dealing with polynomials, the discussion requires us to be practical, in which case the Ackermann functions work as accelerators for the speed of coefficients a_n inside the polynomials. But the response seems to be negative again. Otherwise one such function would offer a counter-example to the concept of *topological invariance*, unless we do not change the terms of the input function. With regard to our context of regular sets, the discs (E_n) and the limit value θ are not equivalent, as was also proved during the early XXth century, most notably in the celebrated works by Brouwer in 1910 and 1911 [17, 18].

7.7 Speed affects the Julia sets topology

Following the application of the regular sets model and of the previous considerations on (inverse) Ackermann functions, we found that, given a sequence of nested sets I_n generated by the application of a family of functions φ_h, there exists one fixed point $\delta \in \bigcap_{n=1}^{\omega_0} I_n$ for such sequence and it is *accessible*

1. either by a sequence of transfinitely many φ_h;
2. or by countably many φ_h whose speed rate is as fast as (multi-) exponential functions;

3. or jointly by transfinitely many φ_h of (multi-) exponential kind.

One notices that the graphical representation of regular sets model fits the dynamics of approximating Γ-sequences, as for example in the Diophantine-Liouville dichotomy of irrational numbers. In this sense, there exists an interesting similarity with the sequence of nested sets induced by the application of Montel's normal families of iterates $f^{n\geq 0}$ near a (super-)attracting fixed point $\delta, f'(\delta) < 1$. On the other hand, if δ is repelling ($\delta \in J$), δ could be assumed as the limit for the family of inverse maps $f^{n<0}$ converging to δ.

Perfectness and *closedness* are two among the classic properties of Julia sets for iterates of polynomials in the form

$$\sum_{i=0}^{d<+\infty} a_i z^i. \tag{7.7.1}$$

With regard to the degree of coefficients exclusively, i.e. with no allusion to the polynomial terms they belong to, we say that all a_i are '*linear*' in (7.7.2), because of $\deg(a_i) = 1$. If at least one point of J belongs to a given backward orbit $\mathcal{O}$, then $\mathcal{O} \subset J$. Given a transfinite sequence of iterates or a function with transfinite speed, the points of J would be accessed by the n^{th}-iterate of any seed point $z \in \mathbb{C}\setminus \bigcup \mathcal{B}_b$ where $\mathcal{B}_b$ are the basins of attraction and $\mathcal{F}$ is the set of points belonging to non-repelling and periodic cycles, so that $\mathcal{F} \subset \bigcup \mathcal{B}_b$. If $\mathcal{O} \subset J$, most complex points would be mapped to a repelling cycle.[63] Thus $J \equiv \mathbb{C}_\infty$ also depends on $\mathcal{F} \neq \emptyset$. We sketch out some examples here below: they show that *the possibility of accomplishing $\mathcal{O} \subset J$ could offer a fuller overview on Julia sets topologies than the approach through standard polynomials.*

First, in section 6 we proved that, given the polynomial (1.1.5) with $\theta \in \mathbb{L}$, the Julia set J_θ is a plane-filling curve with Hausdorff dimension 2 and extending over a bounded and two-dimensional region: in fact, ∞ is a super-attracting fixed point and cannot belong to J_θ. Then $\infty \in \mathcal{F} \neq \emptyset$ and there exists at least one non-empty basin of attraction $\mathcal{B}_\infty$.

The second example is the iteration of the complex exponential function e^z. The Julia set of e^z was conjectured to satisfy $J \equiv \mathbb{C}_\infty$ by Fatou in 1926 [37].

[63]One also notices here that definitions of Julia and Fatou sets swap for the families of the iterates of inverse maps, so that the Julia set of one such family has finitely many points.

This was proved by Misiurewicz in 1981 [65] via one approach not involving function speed. The iterates $e^{e^{\cdots^{e^z}}}$ are complex Ackermann functions of (multi-)exponential kind. There is no room here to try out a new proof of Fatou's conjecture of 1926, but it would be interesting to check if this approach really works. A different strategy may stand behind the corner: since one can show that $\mathcal{F} \equiv \emptyset$ holds in this case and that the iterates of e^z run as fast as Γ-sequences in the above point 2, the repelling cycles are accessible from any $z \in \mathbb{C}_\infty$ via sequences of nested subsets and it turns out that $J \equiv \mathbb{C}_\infty$.

A third interesting example appears in [19], p. 206, where the authors investigated the Julia sets J of the complex iterates:

$$f_n(z) = a_n z^2, \qquad a_n = \frac{1}{n^{1+2^n}}. \tag{7.7.2}$$

Brück and Büger showed that J consists of the point at infinity exclusively ($J \equiv \{\infty\}$): J is a singleton, isolated, not perfect, not enjoying the two classic properties of closedness and perfectness as by the Julia sets of iterated polynomials (7.7.1) or of rational maps $R(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are coprime polynomials. The coefficients a_n of (7.7.2) are Inverse Ackermann functions: as $n \to \infty, a_n \to 0$ but now they decrease with exponential speed, i.e. faster than the coefficients a_i of (7.7.1).

The first and the third example add two new entries to the standard classification of Julia sets topologies. About the first one, the only Julia sets with Hausdorff dimension 2 were known to satisfy $J \equiv \mathbb{C}_\infty$. After our proof on the geometry of the non-linearizable hedgehogs, the related casuistry forks into two sub-cases, which share a transfinite sequence: the approximants $p_n/q_n \to \theta \in \mathbb{L}$ or the iterates of e^z.

Again, coefficients or exponents – whose growth speeds are expressed by Ackermann functions – may generate Julia sets whose topological properties could offer counter-examples to the classic properties enjoyed by the iterates of rational maps $R(z)$, according to Fatou's and Julia's classic environment. For example, the speed rates of the iterated polynomial maps in the form (7.7.1) are evidently too slow to reach the Julia set points. This situation cannot be obviously belong to the cases of non-linearizable hedgehogs or of Julia sets $J \equiv \mathbb{C}_\infty$. So $H^s(J) < 2$: J can be a connected line or totally disconnected.

It is clear that the features of the cases proposed do not rely on the polynomial sum formula – for example, (7.7.1) –, but both on the *magnitude* and on the *degree* of their coefficients a_n. One can generate polynomials whose coefficients are Ackermann functions $a_n^{a_n^{\cdot^{\cdot^{\cdot^{a_n}}}}}$, so that the required speeds can be obtained after a lesser amount of iterates.

The full overview of Julia sets topologies for rational and transcendental maps seems then to involve the investigation on the degree of coefficients and on the related speed rates of iterated maps. At this point, we can sketch out this tentative table:[64]

Hausdorff dimension $H^s(J)$	linear coefficients	multi-exponential coefficients
$H^s(J) = 0$	*Totally disconnected set*	*Isolated point*
$H^s(J) = 1$	*Connected line*	?
$H^s(J) = 2$	$J \equiv \mathbb{C}_\infty$	*Plane-filling curve*, $J \equiv \mathbb{C}_\infty$

Hopefully, in light of these results and of other ones scattered among several publications [19, 77, 78, 79], the corpus of the Holomorphic Dynamics could be rebuilt by improving the original Fatou-Julia approach through a more systematic attention to questions inherent to the growth speed of coefficients. These new cases could be no longer listed as exceptions, as counter-examples or as ‘pathological’.

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[64]This overview was filtered by coefficients. The case $J \equiv \mathbb{C}_\infty$ in the second column from the left relates to the iteration of so-called ‘Lattès’ maps’.

9 Conclusions

From [83], we understand that, from both the theoretical and the empirical viewpoint, only approximations of Siegel discs or of Hedgehogs with Siegel compacta, endowed with positive area exclusively, are currently possible. Our investigations on regular sets, together with the above discussion on topological invariance, suggest that non-linearizable hedgehog visualization is impossible on a practical basis, even if we are able to learn their behavior over $\mathbb{C}_\infty$.

In conclusion, we can state general considerations on the features enjoyed by the construction in the final proof and by the method developed to display hedgehogs on computers [83]: the approach via rational approximation can effectively lead to relevant achievements in both topological and visual terms, as our graphical method discussed in [83] runs analogously to rational convergents or to regular sets.

Our remarks about Ackermann functions suggest that polynomial functions f_θ in the form (1.1.5) might be just the 'narrow door' to hedgehogs. Such invariant sets could be more easily tracked, and possibly occur more often for the family of non-polynomial functions, especially exponentials.

Liouville numbers and Hedgehogs enjoy properties beyond the computational possibilities of classic mathematical approaches, just as God transcends the cogitation by the human mind and senses. We can postulate their existence through the clues we find, on one hand, from the existence of limits, on the other, from the infinite beauty and complexity of Nature. To a watchful eye, they are impossible not to notice. But the values are too little to pick up, the speeds are too fast to follow and the entities are too big to be wrapped up: we cannot watch hedgehogs directly (limits of graphical methods), nor we can hope to compute Liouville numbers (and irrationals, in general) via any process developable by the finiteness of human thinking (limit for processes of countably many steps, limits in the algorithmic approach, limit in the approximation through rational convergents p_n/q_n): it is right the marginal error between our approximations and the effective value to decree us as losers in the war against infinity. Non-linearizable hedgehogs represent a little battle. Reason should call fantasy to help seeing what is otherwise veiled to senses and to calculus.

God, infinity and hedgehogs, so to come all the way down from transcendental to concrete numerical and geometrical entities, they all belong to an outer reality which we can weakly understand, necessarily requiring us to drop everyday standards, to stop ambling and to jump into new thought patterns, as offered by transfinite sequences and faith respectively.

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