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Ordinary differential equations

ON THE STABILITY OF SOLUTIONS FOR NON-AUTONOMOUS DELAY DIFFERENTIAL EQUATIONS OF THIRD-ORDER

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Abstract

In this paper we give sufficient conditions for the asymptotic stability and uniform boundedness for certain third order delay differential equation by constructing Lyapounov functionnals. This result generalizes some asymptotic stability and uniform boundedness for certain third order delay differential equation.

Keywords: Stability; Boundedness; Lyapunov functional; Delay differential equations; Third-order differential equations.
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1 Introduction

In this paper we investigate the asymptotic stability of the zero solution of the delays differential equations

$$(q(t)(p(t)x'(t))')' + a(t)x''(t) + b(t)x'(t) + c(t)f(x(t-r)) = 0, \qquad (1.1)$$

and

$$(q(t)(p(t)x'(t))')' + a(t)x''(t) + b(t)x'(t) + c(t)f(x(t-r)) = R(t). \tag{1.2}$$

Where $a(t), b(t), c(t)$, $p(t), q(t), R(t)$, and $f(x)$ are real valued functions continuous in their respective argument, $f(0) = 0$.

In recent years, the asymptotic stability and boundedness of solutions of non-autonomous delay differential equation of the third order have been studied by a variety of authors, and we mention only a sampling of such papers [1–13] and other references therein.

Omeike, in 2009 [4], considered the following nonlinear differentiable of third order, with a constant deviating argument r ensure the stability and the boundedness of system

$$x''' + a(t)x'' + b(t)x' + c(t)f(x(t-r)) = R(t). \tag{1.3}$$

He discussed the stability and boundedness of solutions of this equation when $R(t) = 0$ and $R(t) \neq 0$.

Our objective in this paper is to show that Omeike results obtained in [3, 4] do hold equally well in the case of the more general third order nonlinear delay differential equation (1.1). Thus, our theorems contain the results of Omeike [3, 4] as special case ($p(t) = q(t) = 1$).

We shall use appropriate Lyapounov function and impose suitable conditions on the functions p, q and f.

2 Preliminaries

First, we will give the preliminary definitions and the stability criteria for the general non-autonomous delay differential system. We consider

$$\dot{x} = f(t, x_t), \quad x_t = x(t+\theta)\ , \quad -r \leq \theta \leq 0, \quad t \geq 0, \tag{2.1}$$

Where $f : I \times C_H \to \mathbb{R}^n$ is a continuous mapping, $f(t, 0) = 0$, and we suppose that f takes closed bounded sets into bounded sets of $\mathbb{R}^n$. Here $(C, \|.\|)$ is the Banach space of continuous function $\phi : [-r, 0] \to \mathbb{R}^n$ with supremum norm, $r > 0$; C_H is the open H-ball in C; $C_H := \{\phi \in (C[-r, 0],\ \mathbb{R}^n) : \|\phi\| \leq H\}$.

Definition 2.1 *[12] A function $x(t_0, \phi)$ is said to be a solution of the system (2.1) with the initial condition $\phi \in C_H$ at $t = t_0, t_0 \geq 0$, if there is a constant $A > 0$ such that $x(t_0, \phi)$ is a function from $[t_0 - r, t_0 + A]$ into $\mathbb{R}^n$ with the properties:*

1. *$x(t, t_0, \phi) \in C_H$ for $t_0 \leq t \leq t_0 + A$,*
2. *$x_t(t, \phi) = \phi$,*
3. *$x_t(t, \phi)$ satisfies (2.1) for $t_0 \leq t \leq t_0 + A$.*

Standard existence theory, see Burton [1], shows that if $\phi \in C_H$ and $t \geq 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ such that on $[t_0, t_0 + \alpha)$ satisfying Equation (2.1) for $t > t_0$, $x_t(t, \phi) = \phi$ and α is a positive constant.If there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. Further, the symbol $|.|$ will denote the norm in $\mathbb{R}^n$ with $|x| = \max_{1 \leq i \leq n} |x|$.

Definition 2.2 *A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of (2.1), $x(t, 0, \phi)$, is defined on $[0, \infty)$ and $x_t(\phi) \in Q$ for $t \in [0, \infty)$.*

Lemma 2.3 *If $\phi \in C_H$is such that the solution $x_t(\phi)$ of (1,3) with $x_0(\phi) = \phi$ is defined on $[0, \infty)$ and $\|x_t(\phi)\| \leq H_1 < H$ for $t \in [0, \infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and*

$$dist(x_t(\phi), \Omega(\phi)) \to 0 \quad as \quad t \to \infty.$$

Lemma 2.4 *let $V(t, \phi) : I \times C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition.$V(t, 0) = 0$, and such that:*
(i) *$W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$ where $W_1(r)$, $W_2(r)$ are wedges.*
(ii) *$\dot{V}_{(2,1)}(t, \phi) \leq 0$, for $\phi \leq C_H$.*
Then the zero solution of (2.1) is uniformly stable.
If $Z = \{\phi \in C_H : \dot{V}_{(2,1)}(t, \phi) = 0\}$, then the zero solution of (2.1) is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

3 Assumptions and main results

We shall state here some assumptions which will be used on the functions that appeared in equation (1.1):

i) $p(t)$ and $q(t)$ are positives and continuously differentiable functions on $[0,+\infty[$, and $f(x)$ is continuously differentiable for $x \in \mathbb{R}$,

ii) $0 < m \le p(t) \le M,\ 0 < m \le q(t) \le M$,

iii) $-L \le p'(t) \le 0$, $-L \le q'(t) \le 0$, and $p''(t) \ge 0,\ \ t \ge 0$,

iv) $f(0) = 0,\ 0 < \delta_0 \le \dfrac{f(x)}{x}$ with $x \ne 0$ and $|f'(x)| \le \delta_1$,

v) $0 < a_0 \le a(t) \le a_1,\ \ t \ge 0$,

vi) $0 < n \le c(t) \le b(t) \le N$, $-N \le b'(t) \le c'(t) \le 0,\ \ t \ge 0$,

vii) $(p(t)c(t))' \le (q(t)c(t))' \le 0$, $t \ge 0$.

To simplify the notation in what follows, we let

$$A(t) = \frac{a(t)}{p(t)q(t)}, \qquad B(t) = \frac{b(t)p(t) - a(t)p'(t)}{p^2(t)},$$

$$D(t) = \frac{\alpha M}{2}\left[\frac{2a_1p'^2(t) - a_2p(t)p'(t)}{p^3(t)} - a_3c'(t)\right] \ge 0,$$

where $a_2 = \dfrac{1}{\alpha M}a_1 + 2n(1-\alpha M\delta_1) + N$, and $a_3 = \dfrac{N}{nm} + \dfrac{a_1L}{nm^2}$.

Theorem 3.1 *Suppose that the assumptions (i)-(vii) hold. Then every solution of (1.1) is uniformly asymptotically stable, provided that there exists* α *satisfying* $\dfrac{M}{a_0} < \alpha < \dfrac{1}{M\delta_1}$ *such that*

$$\frac{1}{2}a'(t) \le d_0 < n(1-\alpha M\delta_1),$$

and

$$r < \min\left(\frac{2c_2m^2}{N\delta_1(1+\alpha+m^2)}, \frac{2c_3}{\alpha\delta_1 N}\right),$$

where $c_2 = \dfrac{1}{M}\left[n(1-\alpha M\delta_1) - d_0\right] > 0,$ *and* $c_3 = \dfrac{1}{M}\left(\dfrac{\alpha a_0}{M} - 1\right) > 0.$

Proof: We write the equation (1.1) as the following equivalent system:

$$\begin{aligned} x' &= \frac{1}{p(t)}y, \\ y' &= \frac{1}{q(t)}z, \\ z' &= -A(t)z - B(t)y - c(t)f(x) + c(t)\int_{t-r}^{t}\frac{y(s)}{p(s)}f'(x(s))ds, \end{aligned} \tag{3.1}$$

and denote $\theta(t) = \int_0^t D(s)ds$. We can see that since $p(t)$ and $q(t)$ are continuous bounded functions, then $\theta(t) = \int_0^t D(s)ds < \infty$; for all $t \geq 0$. Indeed,

$$\begin{aligned} \int_0^t D(s)ds &= \alpha M \int_0^t \left[\frac{a_1 p'^2(s)}{p^3(s)} - \frac{a_2 p'(s)}{2p^2(s)} - \frac{a_3}{2}c'(s)\right] ds \\ &\leq a_1\alpha M \int_0^t (\frac{-p'(s)}{p^2(s)})(-\frac{p'(s)}{p(s)})ds + \frac{\alpha a_2 M}{2m} + \frac{\alpha a_3 MN}{2} \\ &\leq \frac{a_1\alpha ML}{m^2} + \frac{\alpha a_2 M}{2m} + \frac{\alpha a_3 MN}{2} \leq \omega < \infty. \end{aligned}$$

We define the Lyapounov functional $W(t,x,y,z)$ as

$$W(t,x_t,y_t,z_t) = \exp(-\frac{\theta(t)}{\mu})V(t,x_t,y_t,z_t), \tag{3.2}$$

where

$$\begin{aligned} V(t,x_t,y_t,z_t) &= p(t)c(t)F(x) + \alpha q(t)B(t)\frac{y^2}{2} + \alpha q(t)c(t)f(x)y \\ &+ \frac{1}{2}\left\{\frac{a(t)}{p(t)}y^2 + \alpha z^2 + 2yz\right\} + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds, \end{aligned} \tag{3.3}$$

such that $F(x) = \int_0^x f(u)du$, μ and λ are positives constants which will be determined later. From the definition of V in (3.3), we observe that the above Lyapounov functional can be rewritten as follows

$$V = V_1 + V_2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds,$$

where

$$V_1 = p(t)c(t)F(x) + \alpha q(t)B(t)\frac{y^2}{2} + \alpha q(t)c(t)f(x)y,$$

and

$$V_2 = \frac{1}{2}\left\{\frac{a(t)}{p(t)}y^2 + \alpha z^2 + 2yz\right\}.$$

First consider

$$\begin{aligned} V_2 &= \frac{1}{2}\left\{\frac{a(t)}{p(t)}y^2+\alpha z^2+2yz\right\} \\ &= \frac{\alpha}{2}(z+\frac{y}{\alpha})^2+\frac{1}{2}y^2(\frac{a(t)}{p(t)}-\frac{1}{\alpha}). \end{aligned}$$

By (ii), (v) and since $\alpha > \dfrac{M}{a_0}$ we have

$$\frac{a(t)}{p(t)}-\frac{1}{\alpha} \geq \frac{a_0}{M}-\frac{1}{\alpha} > 0.$$

Thus there exist positives constants such that

$$V_2 \geq \delta_2 y^2+\delta_3 z^2. \tag{3.4}$$

On the other hand, using the assumptions (i)-(vi), and since $n \leq b(t) \leq p(t)B(t)$, after some rearrangements we obtain

$$\begin{aligned} V_1 &= p(t)c(t)F(x)+\frac{\alpha}{2}q(t)B(t)\left\{y+\frac{c(t)f(x)}{B(t)}\right\}^2-\frac{\alpha q(t)f^2(x)c^2(t)}{2B(t)} \\ &\geq p(t)c(t)\int_0^x\left[1-\frac{\alpha q(t)c(t)}{p(t)B(t)}f'(u)\right]f(u)du \\ &\geq p(t)c(t)\int_0^x(1-\alpha M\delta_1)f(u)du \\ &\geq \delta_4 F(x), \end{aligned}$$

where

$$\delta_4 = nm(1-\alpha M\delta_1) > nm(1-\frac{1}{M\delta_1}M\delta_1)=0.$$

Thus from (iv) we obtain,

$$V_1 \geq \frac{\delta_4\delta_0}{2}x^2. \tag{3.5}$$

Clearly, from (3.5),(3.4) and (3.3), we have

$$V(t,x_t,y_t,z_t) \geq \delta_2 y^2+\delta_3 z^2+\frac{\delta_4\delta_0}{2}x^2+\lambda\int_{-r}^0\int_{t+s}^t y^2(\xi)d\xi ds.$$

Hence, it is evident, from the terms contained in the last inequality, that there exist sufficiently small positive constant k, such that

$$V(t,x_t,y_t,z_t) \geq k(x^2+y^2+z^2), \tag{3.6}$$

since the integral $\int_{t+s}^{t} y^2(\xi)d\xi$ is nonnegative, where $k = min(\delta_2; \delta_3; \frac{\delta_4\delta_0}{2})$. Therefore we can find a continuous function $W_1(|\phi(0)|)$ with

$$W_1(|\phi(0)|) \geq 0 \quad \text{and} \quad W_1(|\phi(0)|) \leq W(t,\phi).$$

The existence of a continuous function $W_2(\|\phi\|)$ which satisfies the inequality $W(t,\phi) \leq W_2(\|\phi\|)$, is easily verified.
For the time derivative of the Lyapunov functional $V(t, x_t, y_t, z_t)$, along the trajectories of the system (3.1), we have

$$\begin{aligned}
\frac{d}{dt}V(t,x_t,y_t,z_t) &= (p(t)c(t))'F(x) + \frac{\alpha}{2}q'(t)B(t)y^2 + \alpha(q(t)c(t))'f(x)y \\
&+ \left[\frac{\alpha}{2}q(t)B'(t) - \frac{a(t)p'(t)}{2p^2(t)} + G(t) + \lambda r\right] y^2 \\
&+ \left[\frac{1}{q(t)} - \alpha A(t)\right] z^2 \\
&+ c(t)(y+\alpha z)\int_{t-r}^{t} \frac{y(s)}{p(s)} f'(x(s))ds - \lambda \int_{t-r}^{t} y^2(\xi)d\xi.
\end{aligned}$$

Where

$$G(t) = \frac{a'(t)}{2p(t)} + \alpha c(t)\frac{q(t)}{p(t)}f'(x) - B(t).$$

Since $q'c = (qc)' - qc'$, we obtain the following:

$$\begin{aligned}
\frac{\alpha}{2}q'(t)B(t)y^2 &= \frac{\alpha}{2}\frac{q'(t)c(t)}{c(t)}B(t)y^2 \\
&= \frac{\alpha}{2c(t)}(q(t)c(t))'B(t)y^2 - \frac{\alpha}{2c(t)}q(t)c'(t)B(t)y^2,
\end{aligned}$$

consequently, we have

$$\begin{aligned}
\frac{d}{dt}V &= (p(t)c(t))'F(x) + \frac{\alpha}{2c(t)}(q(t)c(t))'B(t)y^2 + \alpha(q(t)c(t))'f(x)y \\
&+ \left[\frac{\alpha q(t)B'(t)}{2} - \frac{\alpha q(t)c'(t)B(t)}{2c(t)} - \frac{a(t)p'(t)}{2p^2(t)} + G(t) + \lambda r\right] y^2 \\
&+ \left[\frac{1}{q(t)} - \alpha A(t)\right] z^2 \qquad\qquad (3.7) \\
&+ c(t)(y+\alpha z)\int_{t-r}^{t} \frac{y(s)}{p(s)} f'(x(s))ds - \lambda \int_{t-r}^{t} y^2(\xi)d\xi.
\end{aligned}$$

Now, we verify

$$H(t,x,y) = (p(t)c(t))'F(x) + \frac{\alpha}{2c(t)}(q(t)c(t))'B(t)y^2 + \alpha(q(t)c(t))'f(x)y \leq 0,$$

for all x, y and $t \geq 0$. If $(q(t)c(t))' = 0$, then

$$H(t,x,y) = (p(t)c(t))' F(x) \leq 0.$$

If $(q(t)c(t))' < 0$, the quantity in the brackets above can be written as,

$$\begin{aligned} H(t,x,y) &= (q(t)c(t))' \left[\frac{(p(t)c(t))'}{(q(t)c(t))'} F(x) + \frac{\alpha}{2c(t)} B(t) y^2 + \alpha f(x) y \right] \\ &= (q(t)c(t))' \left[\frac{(p(t)c(t))'}{(q(t)c(t))'} F(x) + \frac{\alpha B(t)}{2c(t)} \left\{ y + \frac{c(t)f(x)}{B(t)} \right\}^2 - \frac{\alpha c(t) f^2(x)}{2B(t)} \right] \end{aligned}$$

also by assumption (vii) we have $\dfrac{(p(t)c(t))'}{(q(t)c(t))'} \geq 1$ this implies

$$H(t,x,y) \leq (q(t)c(t))' \int_0^x \left[1 - \frac{\alpha c(t)}{B(t)} f'(u) \right] f(u) du.$$

From (ii) and (vi) we get $c(t) \leq MB(t)$, thus

$$\begin{aligned} H(t,x,y) &\leq (q(t)c(t))' \int_0^x (1 - \alpha M \delta_1) f(u) du \\ &\leq (q(t)c(t))' \frac{\delta_4}{nm} F(x) \leq 0. \end{aligned}$$

Thus, on combining the two cases, we have $H(t,x,y) \leq 0$ for all $t \geq 0$, x and y. Using the assumptions of theorem, we get

$$B(t) \leq \frac{N}{m} + \frac{a_1 L}{m^2},$$

and

$$\begin{aligned} B'(t) &= \frac{b'(t)p^2(t) - (b(t) + a'(t))p(t)p'(t) - a(t)p(t)p''(t) + 2a(t)p'^2(t)}{p^3(t)} \\ &\leq \frac{2a_1 p'^2(t) - [2n(1 - \alpha M \delta_1) + N] p(t) p'(t)}{p^3(t)}, \end{aligned}$$

hence, it is easy to see that

$$\begin{aligned} &\frac{\alpha q(t) B'(t)}{2} - \frac{\alpha q(t) c'(t) B(t)}{2c(t)} - \frac{a(t)p'(t)}{2p^2(t)} \\ \leq\ &\frac{\alpha M}{2} \left[\frac{2a_1 p'^2(t) - a_2 p(t) p'(t)}{p^3(t)} - a_3 c'(t) \right] = D(t), \end{aligned}$$

and

$$\begin{aligned} G(t) &\le \frac{1}{p(t)}\left[d_0 + b(t)(\alpha\frac{c(t)}{b(t)}q(t)\delta_1 - 1)\right] \\ &\le \frac{1}{M}\left[d_0 + n(\alpha M\delta_1 - 1)\right] = -c_2 < 0, \end{aligned}$$

we have also,

$$\begin{aligned} \frac{1}{q(t)} - \alpha A(t) &= \frac{1}{q(t)}(1 - \frac{\alpha a(t)}{p(t)}) \\ &\le \frac{1}{M}\left[1 - \frac{\alpha a_0}{M}\right] = -c_3 < 0. \end{aligned}$$

Therefore (3.7) becomes

$$\begin{aligned} \frac{d}{dt}V(t,x_t,y_t,z_t) &\le [D(t) - c_2 + \lambda r]\, y^2 - c_3 z^2 - \lambda\int_{t-r}^{t} y^2(\xi)d\xi \\ &+ c(t)(y+\alpha z)\int_{t-r}^{t}\frac{y(s)}{p(s)}f'(x(s))ds. \end{aligned}$$

Using the Schwartz inequality $|uv| \le \frac{1}{2}(u^2+v^2)$ and since $|f'(x)| \le \delta_1$, we obtain

$$c(t)y\int_{t-r}^{t}\frac{y(s)}{p(s)}f'(x(s))ds \le \frac{\delta_1 N r}{2}y^2 + \frac{\delta_1 N}{2m^2}\int_{t-r}^{t} y^2(\xi)d\xi,$$

and

$$\alpha c(t)z\int_{t-r}^{t}\frac{y(s)}{p(s)}f'(x(s))ds \le \alpha N\frac{\delta_1 r}{2}z^2 + \frac{\alpha N\delta_1}{2m^2}\int_{t-r}^{t} y^2(\xi)d\xi.$$

We rearrange

$$\begin{aligned} \frac{d}{dt}V(t,x_t,y_t,z_t) \le &- \left[c_2 - D(t) - r(\lambda + \frac{\delta_1 N}{2})\right] y^2 - \left[c_3 - \alpha\frac{\delta_1 N r}{2}\right] z^2 \\ &+ \left[\frac{\delta_1 N}{2m^2}(1+\alpha) - \lambda\right]\int_{t-r}^{t} y^2(\xi)d\xi. \end{aligned}$$

If we take $\dfrac{\delta_1 N}{2m^2}(1+\alpha) = \lambda$ the last inequality becomes

$$\begin{aligned} \frac{d}{dt}V(t,x_t,y_t,z_t) \le &- \left[c_2 - D(t) - \frac{\delta_1 N}{2}(\frac{1+\alpha}{m^2} + 1)r\right] y^2 \\ &- \left[c_3 - \alpha\frac{N\delta_1 r}{2}\right] z^2. \end{aligned}$$

Using (3.6), (3.2) and taking $\mu = k$ we obtain:

$$\begin{aligned}\frac{d}{dt}W(t,x_t,y_t,z_t) &= \exp(-\frac{\theta(t)}{k})(\frac{d}{dt}V(t,x_t,y_t,z_t) - \frac{D(t)}{k}V(t,x_t,y_t,z_t)) \\ &\le \exp(-\frac{\theta(t)}{k})[-(c_2 - \frac{\delta_1 N}{2}(\frac{1+\alpha}{m^2}+1)r)y^2 \\ &\qquad -(c_3 - \alpha\frac{\delta_1 N r}{2})z^2].\end{aligned}$$

Therefore, if

$$r < \min(\frac{2c_2m^2}{\delta_1 N(1+\alpha+m^2)}, \frac{2c_3}{\alpha\delta_1 N}),$$

we have

$$\frac{d}{dt}W(t,x_t,y_t,z_t) \le -\beta\exp(-\frac{\omega}{k})(y^2+z^2), \quad \text{for some} \quad \beta > 0.$$

It is clear that the largest invariant set in Z is $Q = \{0\}$, where

$$Z = \{\phi \in C_H : \frac{d}{dt}W(\phi) = 0\}.$$

Namely, the only solution of system (3.1) for which $\frac{d}{dt}W(t,x_t,y_t,z_t) = 0$ is the solution $x = y = z = 0$. Thus, under the above discussion, we conclude that the trivial solution of equation (1.1) is uniformly asymptotically stable. This fact completes the proof.

In the case $R(t) \neq 0$ we establish the following result:

Theorem 3.2 *In addition to the assumptions of Theorem 3.1, If we assume that $R(t)$ is continuous in $\mathbb{R}$ and*

$$\int_0^t R(s)ds < \infty \quad \textit{for all } t \ge 0,$$

then all solutions of the perturbed equation (1.2) are bounded.

Proof: The proof of this theorem is similar to that of the proof of Theorem 2 in [4] and hence it is omitted.

We give an example to illustrate our main results:

Example: We consider the following third order non-autonomous delay differential equation

$$((\frac{1}{1+t^2}+\frac{1}{2})((\frac{2}{1+t^2}+\frac{1}{2})x'(t))')'+(\frac{1}{4}\sin t+10)x''(t) \tag{3.8}$$
$$+(\frac{1}{1+t}+\frac{1}{2})x'(t)+(\frac{1}{4(1+t)}+\frac{1}{4})(x(t-r)+\frac{x(t-r)}{1+x^2(t-r)})=e^{-t}.$$

Now, it is easy to see that

$$\frac{1}{2}\le p(t)=\frac{2}{1+t^2}+\frac{1}{2}\le\frac{3}{2},\ \frac{1}{2}\le q(t)=\frac{1}{1+t^2}+\frac{1}{2}\le\frac{3}{2},$$

$$-1\le p'(t)\le 0,\ -1\le q'(t)\le 0,\ \text{and } p''(t)\ge 0 \text{ for all } t\in[1,+\infty[.$$

$$1\le\frac{f(x)}{x}=1+\frac{1}{1+x^2} \text{ with } x\ne 0,\ \text{and } |f'(x)|\le 2=\delta_1.$$

$$\frac{1}{4}\le c(t)=\frac{1}{4(1+t)}+\frac{1}{4}\le b(t)=\frac{1}{1+t}+\frac{1}{2}\le 1,$$

$$-1\le b'(t)\le c'(t)\le 0;\ \text{for all } t\in[1,+\infty[.$$

$$(p(t)c(t))'\le(q(t)c(t))'\le 0 \text{ for all } t\in[1,+\infty[.$$

$$\frac{39}{4}\le a(t)=\frac{1}{4}\sin t+10\le\frac{41}{4},\ t\in[1,+\infty[,$$
$$\frac{M}{a_0}=\frac{2}{13}<\alpha<\frac{1}{3}=\frac{1}{M\delta_1}.$$
$$\frac{1}{2}a'(t)=\frac{1}{8}\cos t<n(1-\alpha M\delta_1)<\frac{7}{52},$$

and

$$R(t)=e^{-t},$$

hence

$$\int_1^{\infty}e^{-t}<\infty.$$

All the assumptions (i) through (vii) are satisfied, we can conclude using Theorem 3.2 that every solution of (3.8) is bounded.

Remark 3.3 *Equation(1.1) can be rewritten as*

$$x'''(t)+\alpha(t)x''(t)+\beta(t)x'(t)+\gamma(t)f(x(t-r))=0, \tag{3.9}$$

where
$\alpha(t) = \dfrac{p(t)q'(t) + 2q(t)p'(t) + a(t)}{p(t)q(t)}; \beta(t) = \dfrac{q'(t)p'(t) + q(t)p''(t) + b(t)}{p(t)q(t)}$, *and*
$\gamma(t) = \dfrac{c(t)}{p(t)q(t)}$.
If we apply Omeike theorem [3] to show that every solution x(t) of (3.9) is uniform-bounded and satisfies $x(t) \to 0, x'(t) \to 0$ *and* $x''(t) \to 0$ *as* $t \to \infty$, *then the differentiability of* α *and* β *is needed, which implies the use of the second derivative of* q *and the third derivative of* p. *However in our theorem this latter conditions are not required since we just need to deal with* p', p'' *and* q'.

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