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# FREQUENCY DOMAIN CONDITIONS FOR THE EXISTENCE OF FINITE-DIMENSIONAL PROJECTORS AND DETERMINING OBSERVATIONS OF ATTRACTORS

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**Abstract.**

Frequency domain conditions for the existence of finite-dimensional projectors and determining observations for attractors of semi-dynamical systems in Hilbert spaces are derived. Evolutionary variational equations are considered as control systems in a rigged Hilbert space structure. As an example we investigate a coupled system of Maxwell's equations and the heat equation in one-space dimension. We show the controllability of the linear part and the frequency domain conditions for this example.

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# 1 Introduction

We investigate a class of evolutionary variational equations in general Hilbert spaces. The variational equations are considered as general control or feedback systems consisting of a linear part and a nonlinear part. A powerful method for the qualitative investigation of such systems is the frequency theorem for equations of evolutionary type ([4, 14, 16]). Using some properties of the transfer operator of the linear part of the given control system, the frequency theorem gives sufficient conditions for the existence of Lyapunov functionals for dissipativity, global stability and instability of nonlinear systems ([15, 17]). In this paper we extend the frequency domain approach to the investigation of global attractors generated by variational equations.

The article is divided as follows. In Sect. 2 we describe the rigged Hilbert space structure which is used for variational systems. General monotonicity conditions for the existence of global solutions to evolutionary problems are introduced. The frequency domain method is described in Sect. 3. We define here the transfer operator for the linear part of the system and quadratic forms for the characterization of the linear part of the equation. A general theorem for the existence of quadratic Lyapunov functionals is proved. In Sect. 4 such quadratic Lyapunov functionals are used for the construction of bijective continuous maps from an attractor onto a finite-dimensional projection. The realization of this approach for systems in finite-dimensional spaces is shown in Sect. 5. In Sect. 6 we derive frequency domain conditions for a class of evolutionary equations. In Sect. 7 we consider a coupled system of Maxwell's equations and the heat equation ([9, 10, 11, 18]). We give a characterization of this microwave heating problem as control system in certain Hilbert spaces. The controllability of the linear part of the system is shown and a frequency domain condition is verified. The nonlinearities of the heating problem are described by quadratic forms.

# 2 Feedback control systems

Suppose that $Y_0$ is a real Hilbert space with $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ as scalar product and a norm, respectively. Suppose also that on $Y_0$ there is a unbounded self-adjoint operator $\Lambda$ with dense domain $\mathcal{D}(\Lambda)$ such that

$$(\Lambda y, y)_0 \geq \|y\|_0^2, \quad \forall y \in \mathcal{D}(\Lambda).$$

Consider in $Y_0$ the new scalar product

$$(y,\eta)_{-1} := (\Lambda^{-1}y, \Lambda^{-1}\eta)_0, \quad \forall y,\eta \in Y_0$$

and let $Y_{-1}$ be the completion of $Y_0$ with respect to this scalar product. It is clear that $Y_{-1}$ is a Hilbert space. We denote the scalar product and norm of $Y_{-1}$ by $(\cdot,\cdot)_{-1}$ and $\|\cdot\|_{-1}$, respectively. Suppose that $Y_1 \subset Y_0$ is a Hilbert space which is dense and continuously embedded into $Y_0$. Thus we have the dense and continuous embedding $Y_1 \subset Y_0 \subset Y_{-1}$ which is called rigged Hilbert space structure ([1]). It follows from above that for $y \in Y_1$ and $\eta \in Y_0$ we have

$$|(\eta, y)_0| = |(\Lambda^{-1}\eta, \Lambda y)_0| \le \|\Lambda^{-1}\eta\|_0 \|\Lambda y\|_0 = \|\eta\|_{-1}\|y\|_1.$$

Extending by continuity the functionals $(\cdot, y)_0$ onto $Y_{-1}$ we obtain the bilinear form $(\cdot,\cdot)_{-1,1}$ ("scalar product") on $Y_{-1} \times Y_1$ coinciding with $(\cdot,\cdot)_0$ on $Y_0 \times Y_1$ and satisfying $|(\eta,y)_{-1,1}| \le \|\eta\|_{-1}\|y\|_1$, $\forall \eta \in Y_{-1}, y \in Y_1$.

If $-\infty \le T_1 < T_2 \le +\infty$ are two arbitrary numbers, we define the norm for Bochner measurable functions in $L^2(T_1,T_2;Y_j)$, $j = 1, 0, -1$ by

$$\|y\|_{2,j} := \left( \int\limits_{T_1}^{T_2} \|y(t)\|_j^2 dt \right)^{1/2}. \tag{1}$$

Let $W(T_1,T_2)$ denote the space of functions $y$ such that $y \in L^2(T_1,T_2;Y_1)$ and $\dot{y} \in L^2(T_1,T_2;Y_{-1})$ equipped with the norm

$$\|y\|_{W(T_1,T_2)} := (\|y\|_{2,1}^2 + \|\dot{y}\|_{2,-1}^2)^{1/2}. \tag{2}$$

Let us denote by $C(T_1,T_2;Y_0)$ the Banach space of continuous functions $y : [T_1,T_2] \to Y_0$ provided with the norm

$$\|y(\cdot)\|_{C(T_1,T_2;Y_0)} = \sup_{t\in[T_1,T_2]} \|y(t)\|_0.$$

By Sobolev's embedding theorem we can assume that any function from $W(T_1,T_2)$ belongs to $C(T_1,T_2;Y_0)$. Assume also that $\Xi$ and $W$ are two other real Hilbert spaces with scalar products $(\cdot,\cdot)_\Xi$, $(\cdot,\cdot)_W$ and norms $\|\cdot\|_\Xi$, $\|\cdot\|_W$, respectively, and

$$A : Y_1 \to Y_{-1}, \quad B : \Xi \to Y_{-1}, \quad C : Y_{-1} \to W \tag{3}$$

are linear continuous operators and define the nonlinearity $\phi : W \to \Xi$.

Consider the Cauchy problem for the evolutionary variational equation ([6])

$$\begin{aligned} &\dot{y}(t) = Ay(t) + B\phi(Cy(t)),\\ &y(0) = y_0 \in Y_0. \end{aligned} \tag{4}$$

The variational interpretation of (4) means that

$$(\dot{y}(t) - Ay(t) - B\xi(t), \eta - y(t))_{-1,1} = 0,$$
$$\forall \eta \in Y_1,\ w(t) = Cy(t),\ \xi(t) = \phi(w(t)),\ y(0) = y_0.$$

A function $y \in W(T_1, T_2) \cap C(T_1, T_2; Y_0)$ is said to be a solution of (4) on $(T_1, T_2)$ if $y(0) = y_0$ and the equation (4) is satisfied for a. a. $t \in [T_1, T_2]$.

In order to have an existence and uniqueness property for solutions of (4) we state the following assumption.

**(A1)** The nonlinearity $\phi : W \to \Xi$ has the property that the operator $\mathbb{A} := -A - B\phi(C\cdot) : Y_1 \to Y_{-1}$ is monotone hemicontinuous such that the inequality

$$\|\mathbb{A}y\|_{-1} \le c_1\|y\|_1 + c_2,\ \ \forall y \in Y_1 \tag{5}$$

is satisfied, where $c_1 > 0$ and $c_2 \in \mathbb{R}$ are constants.

Suppose also that

$$(\mathbb{A}y, y)_{-1,1} \ge c_3\|y\|_1^2 + c_4,\ \ \forall y \in Y_1 \tag{6}$$

where $c_3 > 0$ and $c_4 \in \mathbb{R}$ are again constants. Then it follows from ([3, 19]) that for arbitrary $y_0 \in Y_0$ there exists a unique weak solution $y \in L^2_{\mathrm{loc}}(\mathbb{R}_+; Y_1) \cap C(\mathbb{R}_+; Y_0)$ with $y(0) = y_0$.

Furthermore, the solution satisfies the inequalities

$$\|y\|_{L^2(0,T;Y_1)} \le g_1(\|y_0\|_0) \ \text{ and}$$
$$\|y\|_{C(0,T;Y_0)} \le g_2(\|y_0\|_0),$$

where $g_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, 2$, are continuous and monotonically increasing functions.

In the following $L \ge 0$ for a linear operator $L \in \mathcal{L}(Z)$, $Z$ a Hilbert space, means that $L$ is positive, i. e. $(z, Lz)_Z > 0, \forall z \in Z\backslash\{0\}$. $L \le 0$ means that $-L$ is positive.

## 3 Frequency-domain methods

A basic tool for the construction of Lyapunov functionals is the following version of Kalman-Yakubovich-Popov lemma ([4, 14, 16]). In the infinite dimensional setting certain regularity conditions are necessary which we formulate at the beginning of this section.

Suppose that in the following $\lambda > 0$ is a fixed number.

**(F1)** For any $T > 0$ and any $f \in L^2(0, T; Y_1)$ the problem

$$\dot{y} = (A + \lambda I)y + f(t), \quad y(0) = y_0 \tag{7}$$

is *well-posed*, i. e., for arbitrary $y_0 \in Y_0$, $f \in L^2(0, T; Y_{-1})$ there exists a unique solution $y \in W(0, T)$ satisfying (7) in the sense that

$$(\dot{y}, \eta)_{-1,1} = ((A + \lambda I)y, \eta)_{-1,1} + (f(t), \eta)_{-1,1}, \quad \forall \eta \in Y_1, \ a.\ a.\ t \in [0, T]$$

and depending continuously on the initial data, i. e.

$$\|y(\cdot)\|^2_{W(0,T)} \leq c_1\|y_0\|^2_0 + c_2\|f\|^2_{2,-1}, \tag{8}$$

where $c_1 > 0$ and $c_2 > 0$ are some constants.

**(F2)** The operator $A + \lambda I \in \mathcal{L}(Y_1, Y_{-1})$ is *regular* ([16]), i. e. for any $T > 0$, $y_0 \in Y_1, \psi_T \in Y_1$ and $f \in L^2(0, T; Y_0)$ the solutions of the direct problem

$$\dot{y} = (A + \lambda I)y + f(t), \quad y(0) = y_0, \quad \text{a. a.} \quad t \in [0, T]$$

and of the dual problem

$$\dot{\psi} = -(A + \lambda I)^*\psi + f(t), \quad \psi(T) = \psi_T, \quad \text{a. a. } t \in [0, T]$$

are strongly continuous in $t$ in the norm of $Y_1$. Here $(A + \lambda I)^* \in \mathcal{L}(Y_{-1}, Y_0)$ denotes the adjoint to $A + \lambda I$, i. e.

$$((A + \lambda I)y, \eta)_{-1,1} = (y, (A + \lambda I)^*\eta)_{-1,1}, \quad \forall y, \eta \in Y_1.$$

Note that assumption **(F2)** is satisfied if the embedding $Y_1 \subset Y_0$ is completely continuous ([16]).

**(F3)** The pair $(A+\lambda I, B)$ is $L^2$-*controllable* ([16]), i. e. for arbitrary $y_0 \in Y_0$ there exist a control $\xi(\cdot) \in L^2(0, \infty; \Xi)$ such that the problem

$$\dot{y} = (A + \lambda I)y + B\xi, \quad y(0) = y_0 \tag{9}$$

is well-posed on the semiaxis $[0, +\infty)$, i. e. there exists a solution $y(\cdot) \in L_\infty$ with $y(0) = y_0$.

It is easy to see that a pair $(A + \lambda I, B)$ is $L^2$-controllable if this pair is *exponentially stabilizable*, i. e. if an operator $K \in \mathcal{L}(Y_0, \Xi)$ exists such that the solution $y(\cdot)$ of the Cauchy problem $\dot{y} = (A + \lambda I + BK)y,\ y(0) = y_0$, decreases exponentially as $t \to \infty$, i. e.

$$\exists c > 0,\ \ \exists \varepsilon > 0 : \|y(t)\|_0 \le ce^{-\varepsilon t \|y_0\|_0},\ \ \forall t \ge 0. \tag{10}$$

Let us denote by $H^c$ and $L^c$ the complexification of a real linear space $H$ and a real linear operator $L$, respectively, and introduce by

$$\chi(p) = C^c(pI^c - A^c)^{-1}B^c,\ \ p \in \rho(A^c) \tag{11}$$

the transfer operator function of the triple $(A^c, B^c, C^c)$.

**(F4)** Assume $\Xi = W$ and there exists an operator $M = M^* \in \mathcal{L}(\Xi, \Xi)$ such that

$$\begin{aligned}(\phi(Cy_1) - \phi(Cy_2), &M(\phi(Cy_1) - \phi(Cy_2)))_\Xi \\ &\le (\phi(Cy_1) - \phi(Cy_2), C(y_1 - y_2))_\Xi,\ \ \forall y_1, y_2 \in Y_1.\end{aligned} \tag{12}$$

**Theorem 1.** *Suppose that for the nonlinearity $\varphi$ from (4) the condition (F4) is satisfied and there exists a number $\lambda > 0$ such that the following holds:*

1. *The pair $(A + \lambda I, B)$ is exponentially stabilizable;*

2. *Consider the equation*
$$\dot{y} = (A + \lambda I)y \tag{13}$$
*in $Y_0$. The space $Y_0$ can be decomposed into $Y_0 = Y_0^- \oplus Y_0^+$ where $\dim Y_0^- =: k < \infty$. Denote by $y(\cdot, y_0)$ the (global) solution of (13) satisfying $y(0, y_0) = y_0$. For any $y_0 \in Y_0^-$ we assume that $\lim\limits_{t \to -\infty} y(t, y_0) = 0$ and for any $y_0 \in Y_0^+$ we assume that $\lim\limits_{t \to +\infty} y(t, y_0) = 0$;*

3. $$\mathrm{Re}(\chi(i\omega - \lambda)\xi, \xi)_{\Xi^c} - (\xi, M^c\xi)_{\Xi^c} < 0, \tag{14}$$
*for all $\omega \in \mathbb{R}$ with $i\omega \notin \sigma(A^c)$ and all $\xi \in \Xi^c$, $\xi \ne 0$.*

*Then there exists a real operator $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ which is negative on $Y_0^-$ and positive on $Y_0^+$ and there exists a number $\varepsilon > 0$ such that with*
*$V(y) := (y, Py)_0,\ \forall y \in Y_0$, the inequality*

$$\begin{aligned}\frac{d}{dt}V(y_1(t) - y_2(t)) &+ 2\lambda V(y_1(t) - y_2(t)) \\ &\le -2\varepsilon \|y_1(t) - y_2(t)\|_1^2 \ \ a.\ a.\ \ t \ge 0\end{aligned} \tag{15}$$

*is satisfied for any two solutions $y_1$ and $y_2$ of (4).*

*Proof.* Suppose that $y_1$ and $y_2$ are two arbitrary solutions of (4). Then $y := y_1 - y_2$ is a solution of

$$\dot{y}(t) = Ay(t) + B\psi(t) \tag{16}$$

with $\psi(t) := \xi_1(t) - \xi_2(t)$, where $\xi_i(t) = \phi(Cy_i(t)),\ t \in \mathbb{R}_+,\ i = 1, 2$.

By assumption **(F4)** we have with $\sigma_i(t) = Cy_i(t),\ i = 1, 2$,

$$(\xi_1(t)-\xi_2(t), M(\xi_1(t)-\xi_2(t)))_\Xi \leq (\xi_1(t)-\xi_2(t), \sigma_1(t)-\sigma_2(t))_\Xi \ a.\ a.\ t \geq 0. \tag{17}$$

Because of conditions 1) and 3) the Likhtarnikov-Yakubovich theorem ([16]) with the Hermitian form

$$F^c(y, \xi) = \mathrm{Re}(\xi, C^c y)_{\Xi^c} - (\xi, M^c\xi)_{\Xi^c} \tag{18}$$

on $Y_1^c \times \Xi^c$ is applicable. It follows from this theorem that there exists a real operator $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ and a number $\varepsilon > 0$ such that the inequality

$$\begin{aligned} ((A + \lambda I)y + B\xi, Py)_{-1,1} + (\xi, Cy)_\Xi - (\xi, M\xi)_\Xi \\ \leq -2\varepsilon[\|y\|_1^2 + \|\xi\|_\Xi^2] \quad \forall y \in Y_1, \xi \in \Xi \end{aligned} \tag{19}$$

holds.

For $\xi = 0$ we get from (19) the inequality

$$((A + \lambda I)y, Py)_{-1,1} \leq -2\varepsilon\|y\|_1^2, \ \ \forall y \in Y_1. \tag{20}$$

From (20) it follows by Lyapunov's theorem ([1]) that the operator $P$ is negative on $Y_0^-$ and positive on $Y_0^+$, respectively.

Putting in (19) $y = y_1 - y_2,\ \xi = \xi_1 - \xi_2$ and using the fact that the inequality

$$(\xi, Cy)_\Xi - (\xi, M\xi)_\Xi \geq 0$$

along the solutions $y_1(\cdot),\ y_2(\cdot)$ and the associated functions $\xi_i = \phi(Cy_i)$ is satisfied, we derive from (19) the inequality

$$\begin{aligned} \frac{d}{dt}V(y_1(t) - y_2(t)) + 2\lambda V(y_1(t) - y_2(t)) \\ \leq -2\varepsilon\|y_1(t) - y_2(t)\|_1^2, \ \ a.\ a.\ t \geq 0. \end{aligned}$$

□

# 4 Construction of homeomorphic maps from the attractor onto a subset of a finite-dimensional space

Under the assumptions of Sect. 2 the evolutionary equation (4) generates a semi-dynamical system $\{\varphi^t\}_{t\geq 0}$ on the phase space $Y_0$. Suppose that this system has a global $\mathcal{B}$-attractor $\mathcal{A}$ ([2, 22]). One way to show the existence of such an attractor is to use a frequency-domain theorem for the compact dissipativity of the semi-dynamical system. From the property of compact dissipativity it is easy to derive the existence of a global $\mathcal{B}$-attractor $\mathcal{A}$ ([22]). Our aim is to use the properties of Theorem 1 for the construction of a homeomorphic map from this attractor onto a subset of the finite-dimensional space. Note that we do not use any information about the fractal dimension of $\mathcal{A}$.

**Theorem 2.** *Suppose that the assumptions of Theorem 1 are satisfied with a parameter* $\lambda > 0$ *and with respect to the decomposition* $Y_0 = Y_0^- \oplus Y_0^+$, *where* $\dim Y_0^- = k$. *Then there exists a linear operator* $\Pi : Y_0 \to Y_0^-$ *such that* $\Pi : \mathcal{A} \to \Pi\mathcal{A}$ *is a homeomorphism.*

*Proof.* Suppose that $y_1$ and $y_2$ are two solutions on $\mathcal{A}$, i. e. we can assume that $y_1(t), y_2(t) \in \mathcal{A},\ t \in \mathbb{R}$.

It follows from (15) that

$$\frac{d}{dt}[e^{2\lambda t}V(y_1(t) - y_2(t))] \leq -2\varepsilon e^{2\lambda t}\|y_1(t) - y_2(t)\|_1^2,\ \forall t \leq \tau. \tag{21}$$

Integration of (21) on $[s, \tau]$ gives

$$\begin{aligned} e^{2\lambda\tau}V(y_1(\tau) - y_2(\tau)) \leq e^{2\lambda s}&V(y_1(s) - y_2(s)) \\ &- 2\varepsilon \int_s^\tau e^{2\lambda t}\|y_1(t) - y_2(t)\|_1^2 dt, \end{aligned} \tag{22}$$

since $e^{\lambda t}\|y_1(t)\|_1$ and $e^{\lambda t}\|y_2(t)\|_1$ are in $L^2(-\infty, \tau; Y_1)$ the function $e^{\lambda t}\|y_1(t) - y_2(t)\|_1$ is also in $L^2(-\infty, \tau; Y_1)$. It follows that there exists a sequence of times $s_i \to -\infty$ as $j \to \infty$ with

$$\|y_1(s_j) - y_2(s_j)\|_1 e^{\lambda s_j} \to 0.$$

Putting in (22) $s = s_j$ and assuming $j \to \infty$ we get

$$e^{2\lambda\tau}V(y_1(\tau) - y_2(\tau)) \leq -2\varepsilon \int_{-\infty}^{\tau} e^{2\lambda t}\|y_1(t) - y_2(t)\|_1^2 dt \leq 0. \tag{23}$$

Choose now an invertible linear map $Q : Y_1 \to Y_1$ such that the function $V$ takes the form

$$V(y) = -\|u\|_1^2 + \|v\|_1^2$$

where $\binom{u}{v} = Q^{-1}y,\ y \in Y_1$ and $(u, v) \in Y_0^- \oplus Y_0^+$.

We define the map $\Pi$ by $\Pi y := u \in Y_0^-$.

$\square$

*Remark* 1. The construction of a homeomorphism $\Pi : \mathcal{A} \to \Pi\mathcal{A}$ is closely related to the existence of a finite number of determining modes. Let us recall the following definition which goes back to O. Ladyzhenskaya ([7, 13, 20]). Suppose that the variational system (4) generates a semigroup $\{\varphi^t\}_{t\geq 0}$ in the phase space $Y_0$. Suppose also that this semigroup has an attractor $\mathcal{A}$ and a finite dimensional projector $\Pi$ with the following property: For any two orbits $\gamma_1, \gamma_2$ on the attractor $\mathcal{A}$ the condition $\Pi\gamma_1 = \Pi\gamma_2$ implies that $\gamma_1 = \gamma_2$. In this case we say that the number of determining modes of $\{\varphi^t\}_{t\geq 0}$ is finite. It follows that under the conditions of Theorem 2 the number of determining modes is finite. The properties of the quadratic form $V(y)$ from the proof of Theorem 2 are related to the cone condition ([8, 21]).

In the next section we consider some questions connected with determining functionals.

## 5 Constructing a reduced system from measurements

In this section we describe an algorithm for the construction of homeomorphic maps $\Pi$ in the sense of Sect. 4. For simplicity we consider the finite-dimensional case.

Suppose that

$$\dot{y} = f(y) \tag{24}$$

is a given (unknown) differential equation in $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field which generates the semi-group $\{\varphi^t\}_{t\geq 0}$. Suppose that $\mathcal{A}$ is the global $\mathcal{B}$-attractor of the semi-group. Our aim is to construct a homeomorphic map
$\Pi : \mathcal{A} \to \Pi\mathcal{A} \subset \mathbb{R}^2$. (The more general case $\Pi : \mathcal{A} \to \Pi\mathcal{A} \subset \mathbb{R}^k, k = 2, 3, \ldots, n-1$, can be described similarly.)

*Step 1*: **Choice of the linear part**

Choose a number $\lambda > 0$ and matrices $A, B$ and $C$ of order $n \times n, n \times 1$ and $1 \times n$, respectively, such that $(A + \lambda I, B)$ is stabilizable, and $A + \lambda I$ has 2 eigenvalues with positive real part and $n - 2$ eigenvalues with negative real part.

*Step 2*: **Reconstruction of the class of nonlinearities**

Calculate on $[0, T]$ the linear semigroup $\varphi^t = e^{At}$ with $A$ from Step 1. Take an $\varepsilon > 0$ (tolerance), a natural number $N$ and observe near the attractor the solutions $y_i(\cdot), i = 1, 2, \ldots, N$, of (24) on $[0, T]$. Find for any $i = 1, 2, \ldots, N$ a solution $\xi_i \in L^\infty(0, T; \mathbb{R}^n)$ of the linear inequality

$$\sup_{t \in [0,T]} |Cy_i(t) - CS(t)y_i(0) - \int_0^t CS(t-s)B\phi_i(s)ds| < \varepsilon. \tag{25}$$

It follows that $\xi_i(t) \approx \xi(Cy_i(t))$ in the sense of $L^2(0, T)$, where
$\dot{y}_i(t) = Ay_i + B\xi(Cy_i(t))$ on $[0, T]$.

Determine two constants $-\infty \le \mu_1 < \mu_2 \le +\infty$ ($\mu_2 < +\infty$ if $\mu_1 = -\infty$ and $\mu_1 > -\infty$ if $\mu_2 = +\infty$) such that

$$\begin{aligned} \mu_1[C(y_i(t) - y_j(t))]^2 &\le [\xi_i(t) - \xi_j(t)]C[y_i(t) - y_j(t)] \\ &\le \mu_2[C(y_i(t) - y_j(t))]^2, \; i, j = 1, \ldots, N, \; t \in [0, T]. \end{aligned} \tag{26}$$

*Step 3*: **Graphic test of the frequency-domain condition**

Compute the frequency-domain characteristic $\chi(i\omega - \lambda) = C((i\omega - \lambda)I - A)^{-1}B$ and compare with the circle $C[\mu_1, \mu_2]$ with $\mu_1 < \mu_2$ from Step 2 (1).

If there is no intersection between $\chi(i\omega - \lambda)$ and $C[\mu_1, \mu_2]$ go to Step 4.

In other case change $A, B, C$ and begin again with Step 1.

*Step 4*: **Calculation of a homeomorphism $\Pi : \mathcal{A} \to \Pi\,\mathcal{A}$**

Find with $A, B, C$ from Step 1 and $\mu_1 < \mu_2$ from Step 3 an $n \times n$ matrix $P = P^*$ of the matrix inequality

$$\begin{aligned} 2\,y^*P\,[(A + AI)y + B\psi] + (\mu_2 Cy - \psi)\;(\psi - \mu_1 Cy) < 0\,, \\ \forall\, y \in \mathbb{R}^n, \; \forall\, \psi \in \mathbb{R}, |y| + |\psi| \ne 0. \end{aligned} \tag{27}$$

Such a solution exists by the frequency theorem and is computable in a finite number of steps. Any solution $P = P^*$ of (27) has 2 negative and $n - 2$ positive eigenvalues. Define a regular matrix $Q = Q^*$ through

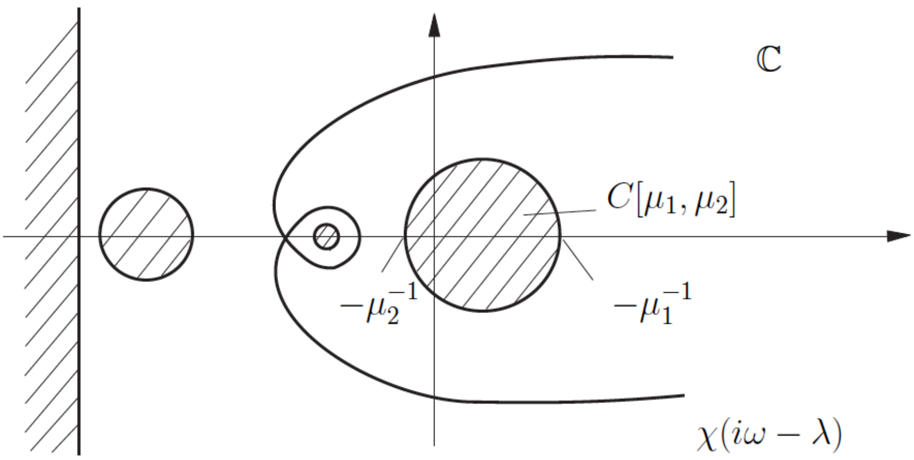


Figure 1: The frequency-domain characteristic

$$Q^*PQ = \begin{pmatrix} -1 & & & & \\ & -1 & & & 0 \\ & & +1 & & \\ 0 & & & \ddots & \\ & & & & +1 \end{pmatrix}.$$

Then the projection $\Pi : \mathbb{R}^n \to \mathbb{R}^2$ is defined by $\Pi\, y = u, y \in \mathbb{R}^n, u \in \mathbb{R}^2$, $v \in \mathbb{R}^{n-2}$, s. th. $\binom{u}{v} = Q^{-1}y$.

It follows from Theorem 2 that of $\mathcal{A}$ is the global $\mathcal{B}$-attractor of (24) then $\Pi : \mathcal{A} \to \Pi\mathcal{A}$ is a homeomorphism.

*Step 5*: **Determination of a reduced ODE for the full equation**

Let $\Pi : \mathcal{A} \to \Pi\mathcal{A}$ be the homeomorphism from Step 4. Determine a reduced 2-dimensional ODE $\dot{u} = \underbrace{\Pi\, f(\tilde{h}(u))}_{\tilde{g}(u)}$ from the observations $\Pi\, y_i(t)$, where $y_i(t)$ are arbitrary solutions of (24) near the attractor and use constructively the extension theorem of Stein to extend this vector field from the closed set $\Pi\mathcal{A} \subset E \cong \mathbb{R}^2$ to a Lipschitz vector field on the whole $E$.

## 6 Determining observations for variational equations

Suppose that $F$ and $G$ are quadratic forms on $Y_1 \times \Xi$. The class $\mathcal{N}(F, G)$ of nonlinearities for (4) consists of all maps $\varphi$ such that the following conditions are satisfied:

For any $T > 0$ and any two functions $y(\cdot) \in L^2(0,T;Y_1)$ and $\xi(\cdot) \in L^2(0,T;\Xi)$ with

$$\xi(t) = \varphi(Cy(t)),\ a.\ a.\ t \in [0,T] \tag{28}$$

it follows that

$$F(y(t),\xi(t)) \geq 0,\ a.\ a.\ t \in [0,T] \tag{29}$$

and there exists a continuous function $\Phi : Y_1 \to \mathbb{R}$ (generalized potential) and numbers $\lambda > 0$ and $\gamma > 0$ such that

$$\begin{aligned} \int_s^t G(y(\tau),\xi(\tau))d\tau \geq \frac{1}{2}[\Phi(y(t)) - \Phi(y(s))] \\ + \lambda \int_s^t \Phi(y(\tau))d\tau \ \text{ for all } \ 0 \leq s < t \leq T \end{aligned} \tag{30}$$

and

$$\Phi(y) \geq \gamma \|y\|_0^2,\ \forall y \in Y_0. \tag{31}$$

Suppose that $S$ is an other Hilbert space which we call *observation space.* Any bounded linear operator $M : Y_1 \to S$ is called *observation operator.*

Assume that $P \in \mathcal{L}(Y_{-1},Y_0) \cap \mathcal{L}(Y_0,Y_1),\ P = P^*$ in $Y_0$ and introduce the function

$$V(y) := \frac{1}{2}(y,Py)_0 + \frac{1}{2}\Phi(y),\ \forall y \in Y_0.$$

Assume that there exists number $\lambda > 0$ and $\mu > 0$ such that for an arbitrary solution $y(\cdot)$ of (4) we have

$$\frac{d}{dt}V(y(t)) + 2\lambda V(y(t)) \leq \mu \|My(t)\|_S^2,\ \text{a. a. } t \geq 0.$$

Then the observation

$$\sigma(t) := \mu \|My(t)\|_S^2 \tag{32}$$

is called by us *determining for the dissipativity with domain $\mathcal{D}$ of (4)*, i. e. the property

$$\int_t^\tau \|My(\tau)\|_S^2 d\tau \to 0 \text{ for } t \to +\infty$$

implies that $\limsup_{t\to+\infty} V(y(t)) \leq C$ and, consequently, (4) is dissipative with domain of dissipativity $\mathcal{D} := \{y \in Y_0 | \|y_0\|_0^2 \leq \frac{2C}{\gamma}\}$.

For an equivalent definition of an observation determining for dissipativity of a variational inequality, see ([9]).

In order to state a frequency-domain theorem for the existence of an observation determining for dissipativity we need the following frequency-domain condition.

**(F5)** There exists numbers $\lambda > 0$ and $\mu > 0$ such that the following three properties hold:

a) Any solution of $\dot{y} = (A + \lambda I)y,\ y(0) = y_0$ is exponentially decreasing for $t \to +\infty$, i. e., there exist constants $c_3 > 0$ and $\varepsilon > 0$ such that

$$\|y(t)\|_0 \le c_3 e^{-\varepsilon t}\|y_0\|_0,\ t > 0; \tag{33}$$

b) $F^c(y, \xi) + G^c(y, \xi) - \mu\|M^c\|^2_{S^c} \le 0,\ \forall (y, \xi) \in Y^c_1 \times \Xi^c : \exists \omega \in \mathbb{R}$ with $i\omega y = (A^c + \lambda I^c)y + B^c\xi$;

c) The functional

$$\mathcal{J}(y(\cdot), \xi(\cdot)) := \int\limits_0^\infty [F^c(y(\tau), \xi(\tau)) + G^c(y(\tau), \xi(\tau)) - \mu\|M^c y(\tau)\|^2_{S^c}]d\tau$$

is bounded from above on the set

$$\begin{aligned}\mathcal{M}_{y_0} := \{y(\cdot), \xi(\cdot) | \dot{y} =& (A^c + \lambda I^c)y + B^c\xi,\\ & y(0) = y_0, y(\cdot) \in W^c(0, \infty), \xi(\cdot) \in L^2(0, \infty; \Xi^c)\}\end{aligned}$$

for any $y_0 \in Y^c_0$.

**Theorem 3.** *Suppose that there are numbers $\lambda > 0$ and $\mu > 0$ such that the assumptions (F1)-(F3), (F5) are satisfied for the variational equation (4) with nonlinearity $\phi \in \mathcal{N}(F, G)$. Then the observation given by (32) is determining for the dissipativity of the variational equation (4).*

*Proof.* From the assumptions **(F1)-(F3)**, **(F5)** it follows ([16]) that there exists an operator $P = P^* \in \mathcal{L}(Y_{-1}, Y_0) \cap \mathcal{L}(Y_0, Y_1)$ such that

$$\begin{aligned}((A + \lambda I)y + B\xi, Py)_{-1,1} + F(y, \xi) + G(y, \xi)&\\ \le \mu\|My\|^2_S,\ \forall y \in Y_1, \forall \xi \in \Xi.&\end{aligned} \tag{34}$$

From (34) and assumption **(F5)** it follows ([5]) that $(y, Py)_0 \ge 0,\ \forall y \in Y_0$. For an arbitrary solution $y(\cdot)$ of (4) and $\xi(t) = \phi(Cy(t))$ we get from (34) the inequality

$$\begin{aligned}(\dot{y}(t), Py(t))_{-1,1} + \lambda(y(t), Py(t))_0 + F(y(t), \xi(t))&\\ + G(y(t), \xi(t)) - \mu\|My(t)\|^2_S \le 0,\ \text{a. a. } t > 0.&\end{aligned} \tag{35}$$

Integration of (35) on the time interval $0 < s < t$ gives

$$\begin{aligned}\frac{1}{2}(y(t), Py(t))_0 - \frac{1}{2}(y(s), Py(s))_0 \\ + \lambda \int_s^t (y(\tau), Py(\tau))_0 d\tau + \int_s^t F(y(\tau), \xi(\tau)) d\tau \\ + \int_s^t G(y(\tau), \xi(\tau)) d\tau \le \mu \int_s^t \|My(\tau)\|^2 d\tau.\end{aligned} \tag{36}$$

From the inequalities (29) and (30) it follows that

$$\int_s^t F(y(\tau), \xi(\tau)) d\tau \ge 0 \tag{37}$$

and

$$\begin{aligned}\int_s^t G(y(\tau), \xi(\tau)) d\tau \ge \frac{1}{2}[\Phi(y(t)) - \Phi(y(s))] \\ + \lambda \int_s^t \Phi(y(\tau)) d\tau, \ 0 < s < t.\end{aligned} \tag{38}$$

Taking into account (36) – (38) we obtain that

$$\begin{aligned}\frac{1}{2}(y(t), Py(t))_0 + \frac{1}{2}\Phi(y(t)) - \frac{1}{2}(y(s), Py(s))_0 - \frac{1}{2}\Phi(y(s)) \\ + 2\lambda \int_s^t [\frac{1}{2}(y(\tau), Py(\tau))_0 - \frac{1}{2}\Phi(y(\tau))] d\tau \\ \le \mu \int_s^t \|My(\tau)\|_S^2 d\tau.\end{aligned} \tag{39}$$

Introduce the functions

$$m(t) := \frac{1}{2}(y(t), Py(t))_0 + \frac{1}{2}\Phi(y(t))$$

and

$$g(t) := -\mu \|My(t)\|_S^2.$$

Then we get from (39) the inequality

$$m(t) - m(s) + 2\lambda \int_s^t m(\tau)d\tau \leq \int_s^t g(\tau)d\tau.$$

The last inequality implies the assertion.

$\square$

## 7 Maxwell's equation in one-space dimension with thermal effect

Consider the coupled system of Maxwell's equation and heat transfer equation in one space dimension ([18])

$$\begin{aligned}
&w_{tt} = w_{xx} - \sigma(\theta)w_t, && (x,t) \in Q_T, && (40)\\
&\theta_t = \theta_{xx} + \sigma(\theta)\omega_t^2, && (x,t) \in Q_T, && (41)\\
&w(0,t) = 0, w(1,t) = 0, && t \in [0,T], && (42)\\
&\theta(0,t) = \theta(1,t) = 0, && t \in [0,T], && (43)\\
&w(x,0) = w_0(x), w_t(x,0) = w_1(x), && x \in \Omega, && (44)\\
&\theta(x,0) = \theta_0(x), && x \in \Omega, && (45)
\end{aligned}$$

where $T > 0, \Omega = (0,1)$ and $Q_T = \Omega \times (0,T]$.

In order to write this system in the form (4) we introduce the following notations:

$$y(x,t) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} w_t(x,t) \\ w(x,t) \\ \theta(x,t) \end{pmatrix}, \; y_0(x) = \begin{pmatrix} w_1(x,t) \\ w_0(x,t) \\ \theta_0(x,t) \end{pmatrix}$$

$$\xi(x,t) = \begin{pmatrix} \xi_1(x,t) \\ \xi_2(x,t) \end{pmatrix} = \begin{pmatrix} \bar\sigma(\theta)w_t(x,t) \\ \sigma(\theta)w_t^2(x,t) \end{pmatrix}.$$

In the last expression we have used the new function $\bar\sigma$ from the decomposition

$$\sigma(\theta) = \sigma_0 + \bar\sigma(\theta),$$

where $\sigma_0 > 0$ is a constant and $\bar\sigma(\theta) > 0, \theta > 0$.

Let $\Lambda$ be the selfadjoint positive operator generated on $L^2(0,1)$ by the differential expression $\Lambda v = -v_{xx}$ for homogeneous Dirichlet boundary conditions.

Consider the spaces $Y_0 = L^2(0,1) \times H_0^1(0,1) \times L^2(0,1)$, $Y_1 = H_0^1(0,1)$ and $\Xi = L^2(0,1) \times L^2(0,1)$. Suppose that the norm in $Y_0$ is given by $\|(v_1, v_2, v_3)\|_0 = \max_i \|v_i\|_{L^2(\Omega)}$ and $(\cdot,\cdot)_0$ is the associated scalar product. Similar norm and scalar product are considered in $\Xi$. Using the operator $\Lambda$ we can define the Hilbert space rigging structure $Y_1 \subset Y_0 \subset Y_{-1}$ by

$$Y_1 = \mathcal{D}(\Lambda) = H_0^1(0,1) \times H_0^2(0,1) \times H_0^1(0,1),$$

using the norm $\|\cdot\|_1$ generated by the scalar product $(\eta_1, \eta_2)_1 = (\Lambda^{-1}\eta_1, \Lambda^{-1}\eta_2)_0$ for arbitrary $\eta_1, \eta_2 \in \mathcal{D}(\Lambda)$.

The pairing $(\cdot,\cdot)_{-1,1}$ on $Y_{-1} \times Y_1$ is introduced as continuous extension of the functional $(\cdot, \eta)_0$ onto $Y_{-1}$. This procedure was described in Sect. 2.

Let us now define linear operators $A : Y_1 \to Y_{-1}$ and $B : \Xi \to Y_{-1}$ by

$$A = \begin{pmatrix} -\sigma_0 I & -\Lambda & 0 \\ I & 0 & 0 \\ 0 & 0 & -\Lambda \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -I & 0 \\ 0 & 0 \\ 0 & I \end{pmatrix}.$$

The original initial-boundary problem (40) – (45) can be written as

$$\dot{y} = Ay + B\xi, \quad y(0) = y_0. \tag{46}$$

Let us show that the pair $(A, B)$ is $L^2$-controllable.

For this we show that the spectrum of $A$ lies in the left-hand side of the complex plane.

Consider for this the eigenvalue problem

$$Av = \lambda v, \tag{47}$$

where $v = (v_1, v_2, v_3)^T$ is an eigenvector and $\lambda$ the associated eigenvalue.

Equation (47) can be written in the form

$$\begin{cases} -\sigma_0 v_1 - \Lambda v_2 = \lambda v_1, \\ \quad v_1 = \lambda v_2, \\ -\Lambda v_3 = \lambda v_3. \end{cases} \tag{48}$$

Suppose that $\lambda_k$ are the eigenvalues of $\Lambda$ and $e_k$ the associated eigenfunctions. It is well-known that the system $\{e_k\}_k$ forms a basis of $L^2(\Omega)$. Thus every element $v_i, i = 1, 2, 3$, can be written as

$$v_i = \sum_k c_i^k e_k, \quad i = 1, 2, 3,$$

where $c_i^k$ are certain coefficients. Now the equation (48) is equivalent to the new system

$$-\sigma_0 \sum_k c_1^k e_k - \sum_k \lambda_k c_2^k e_k = \lambda \sum_k c_1^k e_k, \tag{49}$$

$$\sum_k c_1^k e_k = \lambda \sum_k c_2^k e_k, \tag{50}$$

$$-\sum_k \lambda_k c_3^k e_k = \lambda \sum_k c_3^k e_k. \tag{51}$$

Let us show that any $\lambda$ satisfying (49) – (51) has a negative real part. From (49), (50) it follows that for any $k$ either $c_1^k = c_2^k = 0$ (in this case is $c_3^k \neq 0$ for certain $k$) or

$$\lambda^2 + \sigma_0 \lambda + \lambda_k = 0. \tag{52}$$

It is clear that any $\lambda$ satisfying (52) has a negative real part.

From (51) it follows that for any $k$ either $c_3^k = 0$ (in this case we have $c_1^k \neq 0$ or $c_2^k \neq 0$) or $\lambda = -\lambda_k$.

Thus we have shown that the spectrum of $A$ lies on the left-hand side of the imaginary axis. From this it follows that the pair $(A, B)$ is $L^2$-controllable.

Let us consider the quadratic form

$$F(y, \xi) = \left( \begin{pmatrix} y_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right)_\Xi = \int_\Omega y_1 \xi_1 dx = \int_0^1 \bar{\sigma}(\theta) w_t^2 dx$$

and verify the conditions from Sect. 6.

Suppose that $F^c$ is the Hermitian extension of $F$ on $Y_1^c \times \Xi^c$. In order to show the boundedness of the functional $\mathcal{J}(y(\cdot), \xi(\cdot))$ on the set $\mathcal{M}_{y_0}$ it is sufficient to use the equation

$$\frac{d}{dt} \int_0^1 (w_t^2 + w_x^2) dx + \int_0^1 \sigma(\theta) w_t^2 dx = 0, \tag{53}$$

which follows from (40) – (45). Integration of (53) gives

$$\int_0^1 (w_t^2 + w_x^2) dx + \int_0^\infty \int_0^1 \sigma(\theta) w_t^2 dx dt = C,$$

where $C > 0$ is some constant. The last equation implies that

$$\int_0^\infty \int_0^1 \sigma(\theta) w_t^2 dx dt = \int_0^\infty F^c(y(\tau), \xi(\tau)) d\tau \leq C.$$

Let us check now the frequency-domain condition. Suppose that $\{\lambda_k\}$ are the eigenvalues of $\Lambda$ and $\{e_k\}$ the associated eigenfunctions which form a basis of $L^2(0,1)$. Using this property we can write

$$w(x,t) = \sum_k w^k(t) e_k, \theta(x,t) = \sum_k \theta^k(t) e_k, \xi(x,t) = \sum_k \xi^k(t) e_k,$$

where $w^k(t), \theta^k(t)$ and $\xi^k(t)$ are the associated Fourier coefficients.

Consider $F^c(y,\xi)$ for $i\omega y = A^c y + B^c \xi, \omega \in \mathbb{R}, \xi \in \Xi^c$, i. e. the form

$$F^c(y,\xi) = (\Pi_0(i\omega)\xi, \xi). \tag{54}$$

Suppose that $\tilde{w}^k, \tilde{\theta}^k$ and $\tilde{\xi}^k$ are the Fourier transforms of $w^k, \theta^k$ and $\xi^k$, respectively. It follows from (54) that

$$(\Pi_0(i\omega)\tilde{\xi}, \tilde{\xi}) = \sum_k (\Pi_0^k(i\omega)\tilde{\xi}^k, \tilde{\xi}^k). \tag{55}$$

In order to calculate $\Pi_0(i\omega)$, $\omega \in \mathbb{R}$ we take the formal Fourier transformation in (40), (41). As result we get the equations

$$-\omega^2 \tilde{w}^k(i\omega) + i\omega\sigma_0 \tilde{w}^k(i\omega) - \lambda_k \tilde{w}^k(i\omega) + \tilde{\xi}_1^k(i\omega) = 0, \tag{56}$$

$$i\omega\tilde{\theta}^k(i\omega) + \lambda_k \tilde{\theta}^k(i\omega) - \tilde{\xi}_2^k(i\omega) = 0, \quad k = 1, 2, \ldots. \tag{57}$$

From (56), (57) we get

$$\tilde{w}^k(i\omega) = \chi_0(i\omega, \lambda_k)\xi_1^k(i\omega) \text{ and}$$
$$\tilde{\theta}^k(i\omega) = \chi_1(i\omega, \lambda_k)\xi_2^k(i\omega),$$

where

$$\chi_0(i\omega, \lambda_k) = (-\omega^2 - i\omega\sigma_0 + \lambda_k)^{-1} \text{ and}$$
$$\chi_1(i\omega, \lambda_k) = (i\omega - \lambda_k)^{-1}, \quad k = 1, 2, \ldots.$$

From this formula and (55) it follows that

$$(\Pi_0^k(i\omega)\tilde{\xi}^k, \tilde{\xi}^k) = \mathrm{Re}(\tilde{w}_t^k \tilde{\xi}_1^k) = \mathrm{Re}(i\omega\chi_0)|\tilde{\xi}_1^k(i\omega)|^2.$$

Thus we have the representation

$$\Pi_0^k(i\omega) = \begin{pmatrix} \mathrm{Re}(i\omega\chi_0) & 0 \\ 0 & 0 \end{pmatrix}.$$

In order to get the inequality $\Pi_0^k(i\omega) \le 0$, we have to show that

$$\mathrm{Re}(i\omega\chi_0) \le 0, \quad \omega \in \mathbb{R}. \tag{58}$$

Inequality (58) means that

$$\mathrm{Re}\left(\frac{i\omega}{\omega^2 - i\omega\sigma_0 + \lambda_k}\right) = \mathrm{Re}\left(\frac{(\lambda_k\omega + \omega^3)i - \omega^2\sigma_0}{(\lambda_k + \omega^2)^2 + \omega^2\sigma_0^2}\right) \le 0,$$

i. e. $-\omega^2\sigma_0 \le 0, \quad \forall\omega \in \mathbb{R}$. The last inequality is satisfied since $\sigma_0 > 0$.

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