DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 1, 2010
Electronic Journal,
reg. N P2375 at 07.03.97
ISSN 1817-2172

http://www.neva.ru/journal
http://www.math.spbu.ru/diffjournal/
e-mail: jodiff@mail.ru

New Results on The Stability of Solution of Some Non-autonomous Delay Differential Equations of the Third Order

M.O. Omeike

Department of Mathematics,
University of Agriculture,
Abeokuta, Nigeria.
E-mail: moomeike@yahoo.com

Abstract: Sufficient conditions are established for the asymptotic stability of the zero solution of some non-autonomous delay differential equations of the third order. Our result improves on Sadek's [A.I. Sadek, On the stability of solutions of some non-autonomous delay differential equations of the third order, Asymptotic Analysis 43(2005) 1-7].

Keywords: Stability, Lyapunov functional, non-autonomous differential equations of third order with delay

1. Introduction Consider the third-order nonautonomous delay differential equations

$$\dddot{x} + a(t)\ddot{x} + b(t)\dot{x} + c(t)f(x(t-r)) = 0 \tag{1.1}$$

or its equivalent system form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -a(t)z - b(t)y - c(t)f(x) + c(t)\int_{t-r}^{t} f'(x(s))y(s)ds, \end{aligned} \tag{1.2}$$

where $a(t), b(t)$ and $c(t)$ are positive and continuously differentiable functions on $[0,\infty); r$ is positive constant; $f(x)$ is continuous function and $f(0) = 0$.

In recent years many books and papers dealt with the delay differential equations and obtained many good results, for example, [1,2,3,6,4,9,5,8,7,10-20] , etc. In many references, the authors dealt with the problems by considering Lyapunov functions or functionals and obtained the criteria for the stability.

Recently, Sadek [18] discussed the asymptotic stability of the zero solution of (1.1) and the following result was proved.

Theorem A (Sadek [18]).*Suppose that $a(t), b(t)$ and $c(t)$ are continuously differentiable on $[0,\infty)$ and the following conditions are satisfied:*

(i) $A \geq a(t) \geq a_0 > 0,\ B \geq b(t) \geq b_0 > 0, C \geq c(t) \geq c_0 > 0$ *for* $t \in [0,\infty)$;

(ii) $f(0) = 0, \dfrac{f(x)}{x} \geq f_0 > 0 (x \neq 0)$, *and* $f'(x) \leq f_1 \leq 1$ *for all* x;

(iii) $a_0b_0 - C > 0$;

(iv) $\mu a'(t) + b'(t) - \frac{1}{\mu}c'(t) < (a_0b_0 - C)/2, \mu = (a_0b_0 + C)/2b_0$;

(v) $\displaystyle\int_0^\infty |c'(t)|dt < \infty, c'(t) \to 0$ *as* $t \to \infty$.

Then the zero solution of (1.1) is uniformly asymptotically stable, provided that
$$r < \min\left\{\frac{2c_0f_0}{f_1C}, \frac{a_0b_0 - C}{(1+a_0)b_0f_1C}, \frac{a_0b_0 - C + 4_{a_0}C(1-f_1)}{2f_1C\{1 + 2\mu + 2a_0^2 + a_0 + (a_0b_0 - C)C\}}\right\}.$$

Obviously, this is a very interesting result but Theorem A has some hypotheses which are not necessary for the stability of solutions of (1.1).

Our aim in this paper is to further study the stability of the zero solution of (1.1). In the next section, we establish a criterion for the asymptotic stability of the zero solution of (1.1), which extends and improves Theorem A.

Our main result is the following theorem.

Theorem 1.1. *Suppose that $a(t), b(t)$ and $c(t)$ are continuously differentiable on $[0,\infty)$ and the following conditions are satisfied;*

(1) $1 \le c(t) \le b(t), -L \le b'(t) \le c'(t) \le 0, 0 < a \le a(t) \le L, t \in [0,\infty)$;

(2) $f(0) = 0, \dfrac{f(x)}{x} \ge \delta_0 > 0 \ \ (x \ne 0)$, *and* $f'(x) \le c$ *for all* x;

(3) $\dfrac{1}{2}a'(t) \le \delta_1 < 1 - \alpha c, \ \ t \in [0,\infty)$;

(4) $\displaystyle\int_0^\infty |c'(t)|dt < \infty, \quad c'(t) \to 0$ *as* $t \to \infty$.

Then the zero solution of (1.1) is uniformly asymptotically stable provided that

$$r < \min\left\{\frac{2\left((1-\alpha c) - \delta_1\right)}{(2+\alpha)Lc}, \frac{2(\alpha a - 1)}{\alpha L c}\right\}.$$

From (first term of) (1), it follows that $b(t)$ and $c(t)$ are non-decreasing functions on $[0,\infty)$ and the limit of each exists as $t \to \infty$. Since L in (1) is an arbitrary selected bound, we can also assume that

$$1 \le c(t) \le b(t) \le L,$$

$$\lim_{t\to\infty} c(t) = c_0, \lim_{t\to\infty} b(t) = b_0 \tag{1.3}$$

$$1 \le c_0 \le b_0 \le L.$$

Remark 1.1. If (1.1) is the constant coefficient delay differential equation $\dddot{x} + a\ddot{x} + \dot{x} + cx(t-r) = 0$, then conditions (1)-(2) reduce to the Routh-Hurwitz conditions $a > 0, c > 0$ and $a > c$. To show this we let $a(t) = a, b(t) = 1$ and

$c(t) = 1$ and $f(x(t-r)) = cx(t-r)$.

2. Preliminaries and Stability Results

We shall in this section give the stability results for (1.1), (hence for system (1.2)). First, we will give the stability criteria for the general non-autonomous delay differential system. We consider

$$\dot{x} = f(t, x_t), \quad x_t = x(t+\theta), \quad -r \le \theta \le 0, \tag{2.1}$$

where $f : [0, \infty) \times \mathcal{C}_H \to I\!R^n$ is continuous and takes bounded sets into bounded sets and $f(t, 0) = 0$. Here $(\mathcal{C}, \|\cdot\|)$ is the Banach space of continuous functions $\phi : [-r, 0] \to I\!R^n$ with the supremum norm, $\mathcal{C}_H$ is the open H-ball in $\mathcal{C}$. Standard existence theory [4]shows that if $\phi \in \mathcal{C}_H$ and $t \ge 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ on $[t_0, t_0+\alpha)$ satisfying (2.1) for $t > t_0, x_t(t_0, \phi) = \phi$ and α some positive constant; if there is a closed subset $\mathcal{B} \subset \mathcal{C}_H$ such that the solution remains in B, then $\alpha = \infty$. Also, (1.1) will denote the norm in $I\!R^n$ with $|x| = \max\limits_{1 \le i \le n} |x_i|$.

We are concerned here with stability in the context of Lyapunov's direct method. Thus, we are concerned with continuous, strictly increasing functions $W_i : [0, \infty) \to [0, \infty)$ with $W_i(0) = 0$, called wedges, and with Lyapunov functionals, V.

Definition 2.1 [8]. A continuous functional $V : [0, \infty) \times \mathcal{C}_H \to [0, \infty)$ which is locally Lipschitz in ϕ is called a Lyapunov functional for (2.1) if there is a wedge W with

(i) $W(|\phi(0)|) \le V(t, \phi), \quad V(t, 0) = 0$, and

(ii) $\dot{V}_{(1.2)}(t, x_t) = \limsup_{h \to 0} \frac{1}{h}\{V(t+h, x_{t+h}(t_0, \phi)) - V(t, x_t(t_0, \phi))\} \le 0$.

We have the following fundamental definitions:

Definition 2.2 [8]. (Stability definitions). Since $f(t, 0) = 0, \quad x(t) \equiv 0$ is a solution of (2.1) and its is said to be

(a) stable if for each $\epsilon > 0$ there is a $\delta > 0$ such that $[t \geq 0, \|\phi\| < \delta, t \geq t_0]$ implies that $|x(t, t_0, \phi)| < \epsilon$;

(b) uniformly stable if for each $\epsilon > 0, t \geq 0$ there is a $\delta > 0$ such that $[\|\phi\| < \delta, t \geq t_0]$ implies that $|x(t, t_0, \phi)| < \epsilon$;

(c) asymptotically stable if it is stable and if for each $t \geq 0$ thee is a $\gamma > 0$ such that $\|\phi\| < \gamma$ implies that $|x(t, t_0, \phi)| \to 0$ as $t\infty$;

(d) uniformly asymptotically stable if it is uniformly stable and if there is a $\gamma > 0$ and for each $\mu > 0$ there is a $T > 0$ such that $[t \geq 0, \|\phi\| < \gamma, t \geq t_0 + T]$ implies that $|x(t, t_0, \phi)| < \mu$.

The following is the classical theorem on uniform stability for the solution of (2.1). It goes back to Krasovskii [14].

Theorem 2.1 [7]. *If there is a Lyapunov functional for (2.1) and wedges satisfying:*

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$ *and*

(ii) $\dot{V}_{(2.1)}(t, x_t) \leq 0$.

Then $x = 0$ *is uniformly stable.*

The basic conjecture for (2.1) on uniform asymptotic stability also goes back to Krasovskii [14] and may be stated as follows.

Theorem 2.2 [7]. *If there is a Lyapunov functional for (2.1) and wedges such that:*

(i) $W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$ *and*

(ii) $\dot{V}_{(2.1)}(t, x_t) \leq -W_3(|x(t)|)$.

Then the zero solution of (2.1) is uniformly asymptotically stable.

3. Proof of Theorem 1.1.

We consider, in place of (1.1) the equivalent system form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -a(t)z - b(t)y - c(t)f(x) + c(t)\int_{t-r}^{t} f'(x(s))y(s)ds, \end{aligned} \tag{3.1}$$

and denote $\gamma(t) = \int_0^t |c'(s)|ds$. It may be assumed that $\int_0^\infty |c'(t)|dt \le N < \infty$. We define the Lyapunov functional $V(t, x_t, y_t, z_t)$ as:

$$V(t, x_t, y_t, z_t) = e^{-\gamma(t)} U(t, x_t, y_t, z_t), \tag{3.2}$$

where

$$\begin{aligned} U(t, x_t, y_t, z_t) &= c(t)\int_0^x f(\xi)d\xi + \frac{1}{2}\alpha\{b(t)y^2 + z^2\} + \alpha c(t)f(x)y \\ &+\frac{1}{2}a(t)y^2 + yz + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds, \end{aligned} \tag{3.3}$$

$\alpha > 0$ is any number chosen such that

$$\frac{1}{c} > \alpha > \frac{1}{a} \tag{3.4}$$

where λ is a positive constant which will be determined later. So that, from (3.3) and (3.1),

$$\begin{aligned} \frac{d}{dt}U(t, x_t, y_y, z_t) &= c'(t)\int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y + \frac{1}{2}a'(t)y^2 \\ &- \left\{b(t)y^2 - \alpha c(t)f'(x)y^2 - \lambda r y^2\right\} - \{\alpha a(t) - 1\}\, z^2 \\ &+c(t)y\int_{t-r}^{t} f'(x(s))y(s)ds + \alpha c(t)z\int_{t-r}^{t} f'(x(s))y(s)ds \\ &+\lambda\int_{t-r}^{t} y^2(\theta)d\theta. \end{aligned}$$

By (2) and using $2uv \le u^2 + v^2$, we obtain

$$c(t)y\int_{t-r}^{t} f'(x(s))y(s)ds \le \frac{1}{2}Lcry^2 + \frac{1}{2}Lc\int_{t-r}^{t} y^2(s)ds$$

and

$$\alpha c(t) z \int_{t-r}^{t} f'(x(s)) y(s) ds \leq \frac{1}{2}\alpha Lcrz^2 + \frac{1}{2}\alpha Lc \int_{t-r}^{t} y^2(s) ds.$$

Therefore

$$\begin{aligned}\frac{d}{dt}U(t, x_t, y_t, z_t) \;\; &\leq \;\; c'(t)\int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y + \frac{1}{2}a'(t)y^2 \\ &- \left\{ b(t) - \alpha c(t)f'(x) - \frac{1}{2}(Lc + 2\lambda)r \right\} y^2 \\ &- \left\{ \alpha a(t) - 1 - \tfrac{1}{2}\alpha Lcr \right\} z^2 \\ &+ \left\{ \frac{1}{2}Lc(1+\alpha) - \lambda \right\} \int_{t-r}^{t} y^2(\theta)d\theta.\end{aligned}$$

If we take $\lambda = \dfrac{Lc(1+\alpha)}{2} > 0$, we obtain

$$\begin{aligned}\frac{d}{dt}U(t, x_t, y_t, z_t) \;\; &\leq \;\; c'(t)\int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y + \frac{1}{2}a'(t)y^2 \\ &- \left\{ b(t) - \alpha c(t)f'(x) - \frac{1}{2}Lc(2+\alpha)r \right\} y^2 \\ &- \left\{ \alpha a(t) - 1 - \frac{1}{2}\alpha Lcr \right\} z^2.\end{aligned}$$

By (1)-(3), we obtain

$$\begin{aligned}\frac{d}{dt}U(t,x_t,y_t,z_t) &\le c'(t)\int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y \\ &\quad -\left\{c(t)\left[\frac{b(t)}{c(t)} - \alpha f'(x)\right] - \frac{1}{2}a'(t) - \frac{1}{2}(2+\alpha)Lcr\right\}y^2 \\ &\quad -\left\{\alpha a(t) - 1 - \frac{1}{2}\alpha Lcr\right\}z^2 \\ &\le c'(t)\int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y \\ &\quad -\left\{(1-\alpha c) - \delta_1 - \frac{1}{2}(2+\alpha)Lcr\right\}y^2 \\ &\quad -\left\{\alpha a - 1 - \frac{1}{2}\alpha Lcr\right\}z^2.\end{aligned}$$

If we choose

$$r < \min\left\{\frac{2[(1-\alpha c)-\delta_1]}{(2+\alpha)Lc}, \frac{2(\alpha a - 1)}{\alpha Lc}\right\},$$

we have that there exists $\delta_2 > 0$ such that

$$\frac{d}{dt}U(t,x_t,y_t,z_t) \le c'(t)\int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y - \delta_2(y^2+z^2).$$

Next, we show that

$$c'(t)\int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y \le 0$$

for all x, y and $t \ge 0$. From (1), $-L \le b'(t) \le c'(t) \le 0$ for $t \ge 0$, if $c'(t) = 0$, then

$$c'(t)\int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y = \frac{1}{2}\alpha b'(t)y^2 \le 0$$

since $b'(t) \le 0$. For those t's such that $c'(t) < 0$, we have

$$\begin{aligned}&c'(t)\int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y \\ &= \frac{1}{2}\alpha c'(t)\left\{2\alpha^{-1}\int_0^x f(\xi)d\xi + \frac{b'(t)}{c'(t)}y^2 + 2f(x)y\right\}\end{aligned}$$

$$\leq \frac{1}{2}\alpha c'(t)\left\{2\alpha^{-1}\int_0^x f(\xi)d\xi + y^2 + 2f(x)y\right\}$$

$$= \frac{1}{2}\alpha c'(t)\left\{(y+f(x))^2 + 2\alpha^{-1}\int_0^x \{1-\alpha f'(\xi)\}f(\xi)d\xi\right\}.$$

By hypotheses (2) and (3), $1-\alpha f'(x) \geq 1-\alpha c > 0$, in view of (3.4); and so using hypothesis (2), we find that

$$2\alpha^{-1}\int_0^x \{1-\alpha f'(\xi)\}f(\xi)d\xi \geq \alpha^{-1}(1-\alpha c)\delta_0 x^2.$$

Hence

$$c'(t)\int_0^x f(\xi)d\xi + \frac{1}{2}\alpha b'(t)y^2 + \alpha c'(t)f(x)y \leq 0.$$

Therefore

$$\frac{d}{dt}U(t,x_t,y_t,z_t) \leq -\delta_2(y^2+z^2). \tag{3.5}$$

Since

$$c(t)\int_0^x f(\xi)d\xi + \frac{1}{2}\alpha\{b(t)y^2+z^2\} + \alpha c(t)f(x)y + \frac{1}{2}a(t)y^2 + yz$$

$$= \frac{1}{2}c(t)\left\{2\int_0^x f(\xi)d\xi + \alpha\frac{b(t)}{c(t)}y^2 + 2\alpha f(x)y\right\} + \frac{1}{2}\left\{a(t)y^2 + 2yz + \alpha z^2\right\}.$$

$$\geq \frac{1}{2}c(t)\left\{2\int_0^x f(\xi)d\xi + \alpha y^2 + 2\alpha f(x)y\right\} + \frac{1}{2}\left\{a(t)y^2 + 2yz + \alpha z^2\right\}$$

$$= \frac{1}{2}c(t)\left\{\alpha(y+f(x))^2 + \int_0^x \{1-\alpha f'(\xi)\}f(\xi)d\xi\right\}$$

$$+\frac{1}{2}\left\{a(t)\left(y+\frac{1}{a(t)}z\right)^2 + \frac{1}{a(t)}(\alpha a(t)-1)z^2\right\}$$

By (1)-(3), hence there exists some $\delta_3 > 0$ (small enough) such that

$$c(t)\int_0^x f(\xi)d\xi + \frac{1}{2}\alpha\{b(t)y^2+z^2\} + \alpha c(t)f(x)y + \frac{1}{2}a(t)y^2 + yz \geq \delta_3(x^2+y^2+z^2).$$

Thus,

$$U(t,x_t,y_t,z_t) \geq \delta_3(x^2+y^2+z^2) \tag{3.6}$$

Therefore we can find a continuous function $W_1(|\phi(0)|)$ with

$$W_1(|\phi(0)|) \geq 0 \text{ and } W_1(|\phi(0)|) \leq V(t,\phi).$$

The existence of a continuous function $W_2(\|\phi\|)$ which satisfies the inequality $V(t,\phi) \leq W_2(\|\phi\|)$, is easily verified.

From (3.2), we find

$$\frac{d}{dt}V(t,x_t,y_t,z_t) = e^{-\gamma(t)}\left(\frac{d}{dt}U(t,x_t,y_t,z_t) - |c'(t)|U(t,x_t,y_t,z_t)\right).$$

Using the inequalities (3.5) and (3.6), and the fact that $|c'(t)| \geq 0$, we have

$$\frac{d}{dt}U(t,x_t,y_t,z_t) - |c'(t)|U(t,x_t,y_t,z_t) \leq -\delta_2(y^2+z^2) - \delta_4(x^2+y^2+z^2),$$

therefore, if

$$r < \min\left\{\frac{2((1-\alpha c)-\delta_1)}{(2+\alpha)Lc}\ ,\ \frac{2(\alpha a-1)}{\alpha Lc}\right\},$$

we have

$$\frac{d}{dt}V(t,x_t,y_t,z_t) \leq -\delta e^{\gamma(t)}(x^2,x_y^2+z^2) \leq -W_3(|x(t)|) \text{ for some } \delta > 0 \qquad (3.7)$$

As a result of (3.7) and the existence of the functions $W_1(|\phi(0)|)$ and $W_2(\|\phi\|)$ we note that the zero of (3.1) is uniformly asymptotically stable.

Remark 3.1. Clearly, our theorem is an improvement and extension of Theorem A. In particular, from our theorem we see that (iv) assumed in Theorem A is not necessary, and (i) (ii) and (iii) can be replaced by (1),(2) and (3) of Theorem 1.1 respectively, for the uniform asymptotic stability of the zero solution of (1.1).

References

[**1**] A. U. Afuwape and M. O. Omeike, On the stability and boundedness of solutions of a kind of third order delay differential equations, Appl. Math. Comput. (2007), doi:10.1016/j.amc.2007.11.037.

[**2**] T.A. Burton, Uniform asymptotic stability in functional differential equations, Proc. Amer. Math Soc. 68 (1978), 195-199.

[**3**] T.A. Burton, Volterra Integral and Differential Equations, Academic Press, New York,1983.

[**4**] T. A. Burton, Stability and Periodic Solutions of Ordinary and Functional Differential Equations, Academic Press, New York, 1985.

[**5**] T. A. Burton, An extension of Krasovskii's stability theory, Engrg. Cybernetics 1 (1993), 44-51.

[**6**] T. A. Burton and L. Hatvani, Stability theorems for nonautonomous functional differential equations by Liapunov functional, Tohoku Math. J. 41 (1984), 295-306.

[**7**] T. A. Burton and R. Hering, Liapunov theory for functional differential equations, Rocky Mountain J. Math. 24 (1994), 3-17.

[**8**] T. A. Burton and G. Makey, Asymptotic stability for functional differential equations, Acta Math. Hungar, 65 (1994), 243-251.

[**9**] T.A. Burton and S. Zhang, Unified boundedness, periodicity, and stability in ordinary and functional differential equations, Ann. Mat. Pura Appl. 145 (1986), 129-158.

[**10**] T. Furumochi, Stability and boundedness in functional differential equations, J. Math. Anal. Appl. 113 (1986), 473-489.

[**11**] K. Gopalsamy, Stability and Oscillations in Delay Differentials Equations of Population Dynamics, Kluwer-Academic, Dordrecht 1992.

[**12**] J. K. Hale, Theory of Functional Differential Equations, Springer, New York, 1977.

[**13**] V. Kolmanovskii and A. Myshkis, Introduction to the Theory and Applications of Functional Differential Equations, Kluwer-Academic, Dordrecht, 1999.

[**14**] N. N. Krasovskii, Stability of Motion, Stanford Univ. Press, Stanford, CA, 1963.

[**15**] S. Li and L. Wen, Functional Differential Equations, Human Science and Technology Press, 1987.

[**16**] Y. Qin, Y. Liu and L. Wang, Stability of Motion for Dynamic Systems with Delay,Academic Press, Beijing, 1966.

[**17**] A. I. Sadek, Stability and boundedness of a kind of third-order delay differential system, Appl. Math. Lett. 16 (2003), 657-662.

[18] A. I. Sadek, On the stability of solutions of some non-autonomous delay differential equations of the third order, Asymptotic Analysis 43 (2005) 1-7.

[19] T. Yoshizawa, Stability Theorem by Liapunov's Second Method, Mathematical Society of Japan, 1966.

[20] Y. Zhu, On stability, boundedness and existence of periodic solution of a kind of third-order nonlinear delay differential system, Ann. Differential Equations 8(2), 249-259.