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New results on the stability and boundedness of solutions of certain third order nonlinear vector differential equations

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Abstract

We investigate in this paper, the asymptotic stability of the zero solution and boundedness of all solutions of a certain third order nonlinear ordinary vector differential equation. Our results revise and improve those results obtained by Tunc and Ates [Tunc C., Ates, M., Stability and boundedness results for solutions of certain third order nonlinear vector differential equations,Nonlinear Dynamics 45 (2006); 273-281].

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1. Introduction

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Recently, Tunc and Ates [11] considered the differential equation

$$\dddot{X} + F(X, \dot{X}, \ddot{X})\ddot{X} + B(t)\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}), \tag{1.1}$$

or the equivalent system form

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= Z \\ \dot{Z} &= -F(X, Y, Z)Z - B(t)Y - H(X) + P(t, X, Y, Z) \end{aligned} \tag{1.2}$$

where F and B are $n \times n$-symmetric continuous matrix functions, H and P are continuous vector functions, $t \in [0, \infty)$ and $X \in I\!R^n, I\!R^n$ denotes the real n-dimensional Euclidean space $I\!R \times I\!R \times \cdots \times I\!R$ (n factors). It is also assumed that the Jacobian matrix $J_h(X)$ and the matrix $\dot{B}(t)$ exist, and are symmetric and continuous. Hence the following theorems were proved.

In the case $P \equiv 0$, the following result was established.

Theorem A (Tunc and Ates[11]). *In addition to the fundamental assumptions on F, B and H suppose that:*

(i) there exists an $n \times n$-real continuous operator $A(X, Y)$ for any vectors X, Y in $I\!R^n$ such that

$$H(X) = H(Y) + A(X, Y)(X - Y), \quad (H(0) = 0),$$

whose eigenvalues $\lambda_i(A(X, Y))$, $(i = 1, 2, \cdots, n)$, satisfy

$$0 < \delta_h \le \lambda_i(A(X, Y)) \le \Delta_h$$

for fixed constants δ_h and Δ_h;

(ii) there exists a real $n \times n$-constant symmetric matrix A such that the matrices $A, B(t), \dot{B}(t), (F(X, Y, Z) - A)$ have positive eigenvalues and pairwise commute with themselves as well as with operator $A(X, Y)$ for any X, Y in $I\!R^n$, and that

$$\delta_a = \min_{1\le i\le n}\{\lambda_i(A), \lambda_i(F(X,Y,Z))\}, \Delta_a = \max_{1\le i\le n}\{\lambda_i(A), \lambda_i(F(X,Y,Z))\},$$

$$\delta_b = \min_{1\le i\le n, t\in[0,\omega]}(\lambda_i(B(t))), \quad \Delta_a = \max_{1\le i\le n, t\in[0,\omega]}(\lambda_i(B(t)))$$

$$\Delta_h \le k\delta_a\delta_b \text{ (where } k \text{ is a positive constant)},$$

$$0 \le \lambda_i(F(X,Y,Z)-A) \le \tfrac{\sqrt{\epsilon}}{2} \text{ and } \epsilon = \max|\lambda_i(\dot{B}(t))|, \ (i = 1,2,\cdots,n),$$

$$\text{where } \epsilon \le \frac{1}{2}\min\left\{\left(\frac{\delta_b\delta_h}{4\Delta_b+4}\right)^2, \ \left(\frac{\delta_a\delta_b}{6\Delta_a+7}\right)^2, \ \frac{\delta_a^2}{4}, \ 1\right\}.$$

Then, the zero solution of system (1.2) is asymptotically stable.

In the case $P \neq 0$, the following result was established.

Theorem B (Tunc and Ates[11]). *Let all the conditions of Theorem A be satisfied, and in addition we assume that there exist a finite constant $K > 0$ and a non-negative and continuous function $\theta = \theta(t)$ such that the vector P satisfies*

$$\|P(t,X,Y,Z)\| \le \theta(t) + \theta(t)(\|X\| + \|Y\| + \|Z\|),$$

where $\int_0^t \theta(s)ds \le K < \infty$ for all $t \ge 0$. Then the exists a constant $D > 0$ such that any solution $(X(t), Y(t), Z(t))$ of (1.2) determined by

$$X(0) = X_0, \quad Y(0) = Y_0, \quad Z(0) = Z_0$$

satisfies

$$\|X\| \le D, \quad \|Y\| \le D, \quad \|Z\| \le D$$

for all $t \ge 0$.

These are very interesting results obtained by the authors [11]. However, these results contain certain conditions which are not necessary for the stability and boundedness of (1.2). Our aim in this paper is to further study the stability (when $P \equiv 0$) and boundedness (when $P \neq 0$) of solutions of Eq. (1.1). In the next section, we establish criteria for the stability of the zero solution of Eq. (1.1) when $P \equiv 0$, and the boundedness of solutions of Eq. (1.1) when $P \neq 0$, which extend and improve Theorems A and B,respectively. An effective method for studying the stability and boundedness of nonlinear differential equations is the second method of Lyapunov (See [1-11]).

2. Statement of the results

Let $H(0) = 0$ and $J_h = J_h(X)$ denote the Jacobian matrix $(\partial h_i/\partial x_j)$ derived from the vector $H(X)$ in (1.1). Our first theorem is given for the case in which $P \equiv 0$.

Theorem 1. *Assume that $F(X,Y,Z), B(t), \dot{B}(t)$ and $J_h(X)$ are symmetric for all X, Y, Z in $I\!R^n$ and $t \in [0,\infty)$, and let $\delta_a, \delta_b, \delta_h, \Delta_a, \Delta_b, \Delta_h$ and ϵ be positive constants.*

(i) The matrices $F(X,Y,Z), B(t), \dot{B}(t)$ and $J_h(X)$ are associative and commute pairwise. The eigenvalues $\lambda_i(F(X,Y,Z)), \lambda_i(B(t)), \lambda_i(\dot{B}(t))$, and $\lambda_i(J_h(X))$ $(i = 1,2,\cdots,n)$ of $F(X,Y,Z), B(t), \dot{B}(t)$ and $J_h(X)$ satisfy

$$0 < \delta_a < \lambda_i(F(X,Y,Z)) < \Delta_a \tag{2.1}$$

$$0 < \delta_b \leq \lambda_i(B(t)) \leq \Delta_b \tag{2.2}$$

$$0 < \delta_h \leq \lambda_i(J_h(X)) \leq \Delta_h \tag{2.3}$$

$$\epsilon = \max|\lambda_i(\dot{B}(t))| \tag{2.4}$$

with $\delta_a\delta_b - \Delta_h > \epsilon$.

Then, the zero solution of system (1.2) is asymptotically stable.

In the case $P \neq 0$ we have the following result.

Theorem 2. *Let all the conditions of Theorem 1 be satisfied, and in addition we assume that there exists a finite constant $K > 0$ and a non-negative and continuous function $\theta = \theta(t)$ such that the vector P satisfies*

$$\|P(t,X,Y,Z)\| \leq \theta(t) + \theta(t)(\|X\| + \|Y\| + \|Z\|), \tag{2.5}$$

where $\int_0^t \theta(s)ds \leq K < \infty$ *for all* $t \geq 0$*. Then there exists a constant* $D > 0$ *such that any solution* $(X(t), Y(t), Z(t))$ *of (1.2) determined by*

$$X(0) = X_0, \quad Y(0) = Y_0, \quad Z(0) = Z_0$$

satisfies

$$\|X(t)\| \leq D, \quad \|Y(t)\| \leq D, \quad \|Z(t)\| \leq D$$

for all $t \geq 0$*.*

3. Some Preliminaries

The following results will be basic to the proofs of Theorems 1 and 2. We do not give the proofs since they are found in [1-7,9,10,11].

Lemma 1. *Let* D *be a real symmetric* $n \times n$ *matrix, then for any* X *in* $\mathbb{R}^n$*, we have*

$$\Delta_d \|X\|^2 \geq \langle DX, X \rangle \geq \delta_d \|X\|^2,$$

where δ_d, Δ_d *are the least and greatest eigenvalues of* D*, respectively.*

Lemma 2. *Let* Q, D *be any two real* $n \times n$ *commuting symmetric matrices. Then*

(i) The eigenvalues $\lambda_i(QD)(i = 1, 2, \cdots, n)$ *of the product matrix* QD *are real and satisfy*

$$\max_{1 \leq j,k \leq n} \lambda_j(Q)\lambda_k(D) \geq \lambda_i(QD) \geq \min_{1 \leq j,k \leq n} \lambda_j(Q)\lambda_k(D).$$

(ii) The eigenvalues $\lambda_i(Q + D)$ $(i = 1, 2, \cdots, n)$ *of the sum of matrices* Q *and* D *are real and satisfy*

$$\left\{ \max_{1 \leq j \leq n} \lambda_j(Q) + \max_{1 \leq k \leq n} \lambda_k(D) \right\} \geq \lambda_i(Q+D) \geq \left\{ \min_{1 \leq j \leq n} \lambda_j(Q) + \min_{1 \leq k \leq n} \lambda_k(D) \right\},$$

where $\lambda_j(Q)$ *and* $\lambda_k(D)$ *are, respectively, the eigenvalues of* Q *and* D*.*

4. The Function V

Our main tool in the proof of our result is the Lyapunov function $V = V(t, X, Y, Z)$ defined by

$$\begin{aligned} 2V &= 2\delta_a \int_0^1 \langle H(\sigma X), X\rangle d\sigma + \delta_a \int_0^1 \langle \sigma F(X, \sigma Y, Z)Y, Y\rangle d\sigma \\ &+\alpha\beta\delta_b\langle X, X\rangle + \langle Z, Z\rangle + \langle B(t)Y, Y\rangle + 2\alpha\beta\delta_a\langle X, Y\rangle \\ &+2\alpha\beta\langle X, Z\rangle + 2\delta_a\langle Y, Z\rangle + 2\langle Y, H(X)\rangle - \alpha\beta\langle Y, Y\rangle \end{aligned} \tag{4.1}$$

where $\beta = \delta_a\delta_b$ and α satisfies

$$\alpha < \min\left\{\frac{1}{\delta_a},\ \frac{\delta_h}{\beta(\Delta_a - \delta_a)},\ \frac{\beta - \delta_h - \epsilon}{\beta[\delta_a + \delta_h^{-1}(\Delta_b - \delta_b)^2]}\right\} \tag{4.2}$$

The function V above can be written thus,

$$\begin{aligned} 2V &= \|Z + \delta_a Y + \alpha\beta X\|^2 + \delta_a \int_0^1 \langle \sigma F(X, \sigma Y, Z)Y, Y\rangle d\sigma - \delta_a^2\langle Y, Y\rangle \\ &+\langle B(t)Y, Y\rangle - \beta\delta_a^{-1}\langle Y, Y\rangle + \alpha\beta(\delta_b - \alpha\beta)\langle X, X\rangle \\ &+2\delta_a \int_0^1 \langle H(\sigma X), X\rangle d\sigma - \beta^{-1}\delta_a\|H(X)\|^2 \\ &+\beta\|\delta_a^{-\frac{1}{2}}Y + \beta^{-1}\delta_a^{\frac{1}{2}}H(X)\|^2 \end{aligned} \tag{4.3}$$

The following result is immediate from (4.3).

Lemma 3. *Assume that all the hypotheses on matrices $F(X, Y, Z)$, $B(t)$ and vector $H(X)$ in Theorems 1 and 2 are satisfied. Then there exists a positive constant δ_1 such that*

$$V(t, X, Y, Z) \geq \delta_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2), \tag{4.4}$$

for arbitrary X, Y, Z in $I\!R^n$.

Proof of Lemma 3. We shall make use of the result:

$$H(X) = \int_0^1 J_h(\sigma_1 X)X d\sigma_1 \tag{4.5}$$

for arbitrary X in $\mathbb{R}^n$, which follows from integrating the equality

$$\frac{d}{d\sigma_1} H(\sigma_1 X) = J_h(\sigma_1 X) X$$

with respect to σ_1 and then using the fact that $H(0) = 0$.

By (4.5), we can rewrite (4.3) thus,

$$\begin{aligned} 2V &= \|Z + \delta_a Y + \alpha\beta X\|^2 + \delta_a \int_0^1 \sigma \langle \{F(X, \sigma Y, Z) - \delta_a I\} Y, Y \rangle d\sigma \\ &+ \langle \{B(t) - \beta \delta_a^{-1} I\} Y, Y \rangle + \alpha\beta(\delta_b - \alpha\beta) \langle X, X \rangle \\ &+ 2\delta_a \int_0^1 \int_0^1 \sigma_1 \langle \{I - J_h(\sigma_1 X)\beta^{-1}\} J_h(\sigma_1 \sigma_2 X) X, X \rangle d\sigma_1 d\sigma_2 \\ &+ \beta \|\delta_a^{-\frac{1}{2}} Y + \beta^{-1} \delta_a^{\frac{1}{2}} H(X)\|^2. \end{aligned}$$

By (2.1), (2.2) and (2.3) of Theorem 1, and Lemma 1, we have that

$$2V \geq \|Z + \delta_a Y + \alpha\beta X\|^2 + \alpha\beta(\delta_b - \alpha\beta)\|X\|^2 + 2\delta_a(1 - \Delta_h \beta^{-1})\delta_h \|X\|^2.$$

By (2.5) and (4.2), we have that there is a constant $\delta_2 > 0$ such that

$$2V \geq \|Z + \delta_a Y + \alpha\beta X\|^2 + \delta_2 \|X\|^2.$$

Hence we can find a positive number δ_1 small enough such that (4.4) holds.

This completes the proof of Lemma 3.

The following lemma is instrumental in the proof of the next result.

Lemma 4. *Subject to earlier conditions on F and H the following are true.*

(i) $\displaystyle \frac{d}{dt} \int_0^1 \langle \sigma F(X, \sigma Y, Z) Y, Y \rangle d\sigma = \langle F(X, Y, Z) Y, Z \rangle,$

(ii) $\displaystyle \frac{d}{dt} \int_0^1 \langle H(\sigma X), X \rangle d\sigma = \langle H(X), Y \rangle.$

Proof. See [4,5,9].

Lemma 5. *Assume that all the conditions of Theorem 1 are satisfied. Then*

$$\dot{v}(t) \le 0 \quad \textit{for all} \;\; t \ge 0 \tag{4.6}$$

and especially

$$\dot{v}(t) = \frac{d}{dt}V(t, X, Y, Z) \le 0 \;\; \textit{provided} \;\; \|X\|^2 + \|Y\|^2 + \|Z\|^2 > 0 \tag{4.7}$$

Proof of Lemma 5. A straightforward calculation from (4.1), (1.2) and Lemma 4 give

$$\dot{v} = \frac{d}{dt}V(t, X(t), Y(t), Z(t)) = -V_1 - V_2 - V_3$$

where

$$\begin{aligned}
V_1 &= -\frac{1}{2}\alpha\beta\int_0^1 \langle X, J_h(\sigma X)X\rangle d\sigma \\
& -\langle Y, \{\delta_a B(t) - \dot{B}(t) - \{[J_h(X) + \alpha\beta\delta_a]\}Y\} \\
& -\langle Z, \{F(X, Y, Z) - \delta_a I\}Z\rangle \\
V_2 &= -\frac{1}{4}\alpha\beta\int_0^1 \{\langle J_h(\sigma X)X, X\rangle + 4\langle X, [B(t) - \delta_b I]Y\rangle\} d\sigma \\
V_3 &= -\frac{1}{4}\alpha\beta\int_0^1 \{\langle J_h(\sigma X)X, X\rangle + 4\langle X, \{F(X, Y, Z) - \delta_a I]\}Z\rangle\} d\sigma.
\end{aligned}$$

Since $J_h(X)$ is symmetric and positive definite, we have that

$\langle J_h(\sigma X)X, X\rangle + 4\langle X, [B(t) - \delta_b I]Y\rangle$

$= \|J_h^{\frac{1}{2}}X + 2J^{-\frac{1}{2}}[B(t) - \delta_b I]Y\|^2 - \|2[B(t) - \delta_b I]J_h^{-\frac{1}{2}}Y\|^2$

and

$\langle J_h(\sigma X)X, X\rangle + 4\langle X, \{F(X, Y, Z) - \delta_a I\}Z\rangle$

$= \|J_h^{\frac{1}{2}}X + 2J_h^{-\frac{1}{2}}[F(X, Y, Z) - \delta_a I]Z\|^2 - \|2[F(X, Y, Z) - \delta_a I]J_h^{-\frac{1}{2}}Z\|^2.$

Using the fact that

$$\int_0^1 \|2[B(t) - \delta_b I]J_h^{-\frac{1}{2}}Y\|^2 d\sigma = 4\int_0^1 \langle J_h^{-1}[B(t) - \delta_b I]Y, [B(t) - \delta_b I]Y\rangle d\sigma$$

and

$$\int_0^1 \|2[F(X,Y,Z)-\delta_a I]J_h^{-\frac{1}{2}}Z\|^2 d\sigma$$

$$= 4\int_0^1 \langle J_h^{-1}[F(X,Y,Z)-\delta_a I]Z, [F(X,Y,Z)-\delta_a I]Z\rangle d\sigma,$$

we have,

$$\begin{aligned}
\dot{v}(t) \;\leq\; & -\frac{1}{2}\alpha\beta\int_0^1 \langle X, J_h(\sigma X)X\rangle d\sigma \\
& -\int_0^1 \langle Y, \{\delta_a B(t) - J_h(X) - [\dot{B}(t)+\alpha\beta\delta_a I] - \alpha\beta J_h^{-1}[B(t)-\delta_b I]^2\}Y\rangle d\sigma \\
& -\int_0^1 \langle Z, [F(X,Y,Z)-\delta_a I]\{I-\alpha\beta J_h^{-1}[F(X,Y,Z)-\delta_a I]\}Z\rangle d\sigma \\
\leq\; & -\frac{1}{2}\alpha\beta\delta_h\|X\|^2 \\
& -\{\delta_a\delta_b - \delta_h - \epsilon - \alpha\beta\delta_a - \alpha\beta\delta_h^{-1}(\Delta_b-\delta_b)^2\}\|Y\|^2 \\
& -\gamma\{1-\alpha\beta\delta_h^{-1}(\Delta_a-\delta_a)\}\|Z\|^2 \\
\leq\; & -\delta_3\|X\|^2 - \delta_4\|Y\|^2 - \delta_5\|Z\|^2
\end{aligned}$$

where $\delta_3 = \frac{1}{2}\alpha\beta\delta_h, \delta_4 = \delta_a\delta_b - \delta_h - \epsilon - \alpha\beta[\delta_a + \delta_h^{-1}(\Delta_b-\delta_b)^2]$ and $\delta_5 = 1-\alpha\beta\delta_h^{-1}(\Delta_a-\delta_a)$.

By (4.2),δ_3, δ_4 and δ_5 are positive. This completes the proof.

Proof of Theorem 2. Consider the function V defined by (4.1). Then, under the assumptions of Theorem 2 the conclusion of Lemma 3 can be obtained, that is,

$$V \geq \delta_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \tag{4.8}$$

and since $P(t,X,Y,Z) \neq 0$, then the conclusion of Lemma 5 can be revised as follows

$$\dot{v} = \frac{d}{dt}V \leq \langle \alpha\beta X + \delta_a Y + Z, P(t,X,Y,Z)\rangle.$$

Next, by noting the assumption of Theorem 2 on $P(t, X, Y, Z)$ and using Schwarz's inequality, we obtain

$$\begin{aligned}\dot{v} &\le (\alpha\beta\|X\| + \delta_a\|Y\| + \|Z\|) \times \|P(t, X, Y, Z)\| \\ &\le (\alpha\beta\|X\| + \delta_a\|Y\| + \|Z\|) \times (\theta(t) + \theta(t)(\|X\| + ||Y\| + \|Z\|)) \\ &\le \delta_6(\|X\| + \|Y\| + \|Z\|) \times (\theta(t) + \theta(t)(\|X\| + ||Y\| + \|Z\|))\end{aligned}$$

where $\delta_6 = \max\{\alpha\beta, \delta_a, 1\}$.

Hence, by using the inequalities

$$\|X\| \le 1 + \|X\|^2, \quad \|Y\| \le 1 + \|Y\|^2, \quad \|Z\| \le 1 + \|Z\|^2$$

and (4.8), we obtain

$$\dot{v} \le \delta_7\theta(t) + \delta_8\theta(t)v, \tag{4.9}$$

where $\delta_7 = 3\delta_6$ and $\delta_8 = 4\delta_6\delta_1^{-1}$.

Integrating both sides of (4.9) from 0 to $t(t \ge 0)$, leads to the inequality

$$v(t) - v(0) \le \delta_7 \int_0^t \theta(s)ds + \delta_8 \int_0^t v(s)\theta(s)ds.$$

On putting $\delta_9 = v(0) + \delta_7 K$, it follows that

$$v(t) \le \delta_9 + \delta_8 \int_0^t v(s)\theta(s)ds.$$

Gronwall-Bellman inequality yields

$$v(t) \le \delta_9 \exp\left(\delta_8 \int_0^t \theta(s)ds\right).$$

The proof of the theorem is now complete.

5. Remarks

(i). Clearly, Theorems 1 and 2 are improvement and extension of Theorems A and B respectively. Particularly, from Theorems 1 and 2, we see that hypothesis (i) of Theorems A and B is not necessary since $H(X)$ is assumed differentiable.

(ii). Also, from Theorems 1 and 2, it is clear that we do not need any symmetric matrix A (as assumed in Theorems A and B), thus the condition $0 \leq \lambda_i(F(X,Y,Z) - A) \leq \frac{\sqrt{\epsilon}}{2}$, $(i = 1, 2, \ldots, n)$, is not necessary.

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