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Existence of solutions of fractional abstract integro-differential equation with impulsive nonlocal conditions

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Abstract

In this paper we prove the existence of solutions to fractional abstract integro-differential equation with impulsive nonlocal conditions in Banach spaces. The results are obtained using fractional calculus and fixed point theorem. An example is provided to illustrate the theory.

1 Introduction

Fractional differential equations have recently been addressed by several researchers for a variety of problems. The interest in the study of differential equations of fractional order lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. With this advantage, the fractional-order models become more realistic and practical than the classical integer-order models, in which such effects are not taken into account. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology,

economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. [17, 24, 25, 26] . For some recent development on the topic, see [5, 20, 19] and the references therein.
In [14, 16] the authors have proved the existence of solutions of abstract differential equations by using semigroup theory and fixed point theorem. Many partial fractional differential equations can be expressed as fractional differential equations in some Banach spaces [15].
The study of nonlocal Cauchy problem for abstract evolution differential equations has been initiated by Byszewski [11]. Subsequently several authors [12, 21] have discussed the problem for different types of nonlinear differential and integrodifferential equations including functional differential equations in Banach spaces. Balachandran et al. [6, 3, 4] established the existence of solutions of quasilinear integrodifferential equations with nonlocal conditions where the quasilinear operator is unbounded. The existence of solutions of fractional abstract differential equations with nonlocal initial condition was recently investigated by *ŃGuérékata* [22] and Balachandran and Park [3]. Numerical experiments for fractional models on population dynamics are discussed in [22]. On the other hand, the theory of impulsive differential equations has undergone rapid development over the years and played a very important role in modern applied mathematical models of real processes arising in phenomena studied in physics, population dynamics, chemical technology, biotechnology and economics: see for instance the monograph by Lakshmikantham et al. [18]. Benchohra and Seba [8] studied the existence of fractional impulsive differential equations in Banach spaces while Balachandran and Kiruthika [2] discussed the existence of nonlocal Cauchy problem for semilinear fractional evolution equations where as Chang and Nieto [13] studied the same type of problem for neutral integrodifferential inclusions via fractional operators. Belmekki et al. [7] proved the existence of periodic solutions of nonlinear fractional differential equations where as Ahmad and Nieto [1] discussed the existence results for nonlinear boundary value problem of fractional integrodifferential equations with integral boundary conditions. Recently, the study of impulsive differential equations has attracted a great deal of attention in fractional dynamics and its theory has been treated in several works [10, 18]. Balachandran and Trujillo [10] investigated the nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces which motivates to study the existence of solutions of fractional quasilinear impulsive integrodifferential equations in Banach spaces by using the fractional calculus and fixed point theorems. This work initiates new avenues for obtaining numerical solutions of impulsive fractional integrodifferential equations.

2 Preliminaries

In this section, we shall introduce some basic definitions, notations and lemmas which are used throughout this paper.

Definition 2.1 *The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}^+)$ of order $\alpha \in \mathbb{R}^+$ is defined by*

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.2 *For a function h given on the interval $[a,b]$, the Caputo fractional order derivative of h, is defined by,*

$$({}^c D_{a^+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)\, ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Let $\mathbf{B}(X)$ be the Banach space of all linear and bounded operators on X. Now let us consider the set of functions

$$\begin{aligned} PC(I,X) = \{u : I = [0,T] \to X : u \in \mathcal{C}((t_{k-1}, t_k], X), k = 1,2,\cdots,m \quad \text{and} \\ \text{there exist} \quad u(t_k^-) \quad \text{and} \quad u(t_k^+), k = 1,2,\cdots,m \quad \text{with} \quad u(t_k^-) = u(t_k)\}. \end{aligned}$$

endowed with the norm

$$\|u\|_{PC} = \sup_{t \in I} \|u(t)\|$$

$(PC(I,X), \|.\|_{PC})$ is a Banach space.

3 Semilinear Evolution problem

Consider the linear fractional impulsive evolution equation

$$\begin{aligned} u^{(\alpha)} &= A(t)u(t) + f(t,u(t)) \quad t \in [0,T], \quad t \ne t_k \quad 0 < \alpha \le 1 \\ \Delta u|_{t=t_k} &= I_k(u(t_k^-)) \quad k = 1,\ldots,m. \qquad\qquad (3.1) \\ u(0) &= u_0 \in X \end{aligned}$$

where $A(t)$ is a bounded linear operator on a Banach space X, $I_k : X \to X$, $k = 1,\ldots,m$. and $u_0 \in X$, $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = T$. $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-), u(t_k^+) = \lim_{h \to 0^+} u(t_k + h)$ and $u(t_k^-) = \lim_{h \to 0^-} u(t_k + h)$ represent the right and left limits of $u(t)$ at $t = t_k$.

Definition 3.1 *A function $u(t)$ is said to be a solution of the abstract Cauchy problem (3.1) if the following conditions are satisfied*

(i) $u \in PC(I,X)$ and $u \in D(A(t))$ for all $t \in I$;

(ii) $u^{(\alpha)}$ *exists on* I *where* $0 < \alpha < 1$;

(iii) u *satisfies Eq (3.1) on* I *and satisfy the conditions*

$$\begin{aligned} \Delta u|_{t=t_k} &= I_k(u(t_k^-)) \quad k = 1, ..., m. \\ u(0) &= u_0 \in X \end{aligned}$$

Now, we assume the following conditions to prove the existence of a solution of the Eq. (3.1).

(H1) $A(t)$ is a bounded linear operator on X for each $t \in I$ and the function $t \to A(t)$ is continuous in the uniform operator topology and there exists a constant M such that

$$M = \max_{t \in I} \|A(t)\|.$$

(H2) The function $I_k : X \to X$ are continuous and there exists a constant $L_1 > 0$ and $\rho_1 > 0$ such that

$$\|I_k(u) - I_k(v)\| \le L_1 \|u - v\| \text{ and } \|I_k(u)\| < \rho_1 \|u\| \text{ for each } u, v \in X \text{ and } k = 1, 2, ..., m,$$

and $f : I \times X \to X$ is continuous and there exists a constant L_2 such that

$$\|f(t, u) - f(t, v)\| \le L_2 \|u - v\| \qquad \text{for all} \qquad u, v \in X.$$

For brevity let us take $\dfrac{T^\alpha}{\Gamma(\alpha + 1)} = \gamma$.

Definition 3.2 *A function* $u \in PC(I, X)$ *solution of the fractional integral equation*

$$u(t) = \begin{cases} u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(A(s)u(s) + f(s, u(s))\right) ds, \quad \text{if } t \in [0, t_1]; \\[2ex] u_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} (A(s)u(s) + f(s, u(s)))ds \\[2ex] + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} (A(s)u(s) + f(s, u(s)))\, ds \\[2ex] + \sum_{0<t_k<t} I_k(u(t_k^-)) \quad \text{if } t \in (t_k, t_{k+1}]. \end{cases} \tag{3.2}$$

will be called a mild solution of Problem (3.1).

Theorem 3.3 *If the hypotheses (H1) and (H2) are satisfied, then Equation (3.1) has a unique solution on* I*. Moreover, the mapping* $u_0 \to u$ *is a lipschitz continuous from* X *into* $\mathcal{C}([0, T], X)$.

Proof

The proof is based on the application of Picards iteration method. Let $M = max_{0\le t\le T}\|A(t)\|$ and define a mapping $F : PC([0,T] : X) \to PC([0,T] : X)$ by

$$\begin{aligned} Fu(t) = & u_0 + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}(A(s)u(s)+f(s,u(s)))ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}(A(s)u(s)+f(s,u(s)))\,ds \\ & + \sum_{0<t_k<t} I_k(u(t_k^-)) \qquad \text{if} \qquad t\in(t_k,t_{k+1}] \end{aligned} \tag{3.3}$$

Let $u, v \in PC(I, X)$. Then from Eq. (3.3), we have for each $t \in I$

$$\|Fu(t)-Fv(t)\| \le \frac{T^{\alpha}(m+1)}{\Gamma(\alpha+1)}(M+L_2)\|u-v\| + mL_1\|u-v\|.$$

Then by induction we have

$$\|F^n u(t)-F^n v(t)\| \le \frac{(\gamma(m+1)(M+L_2)+mL_1)^n}{n!}\|u-v\|.$$

Since $\frac{(\gamma(m+1)(M+L_2)+mL_1)^n}{n!} < 1$ for large n, then by the well-known generalization of the Banach contraction principle, F has a unique fixed point $u \in PC([0,T] : X)$. This fixed point is the solution of Eq. (3.1).

The uniqueness of u and Lipschitz continuity of the map $u_0 \to u$ are consequence of the following argument. Let v be a mild solution of Eq.(3.1) on [0,T] with the initial value u_0. Then

$$\begin{aligned} \|u(t)-v(t)\| \le & \|T(t)u_0 - T(t)v_0\| \\ & + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\left(\|A(s)\|\,\|u(s)-v(s)\| + \|f(s,u(s))-f(s,v(s))\|\right)ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}(\|A(s)\|\,\|u(s)-v(s)\| + \|f(s,u(s))-f(s,v(s)\|)\,ds \\ & + \sum_{0<t_k<t}\left\|I_k(u(t_k^-)-I_k(v(t_k^-))\right\| \end{aligned}$$

The last equation can be written in the form

$$\|u(t)-v(t)\| \le M\|u_0-v_0\| + M^*\int_0^t \|u(s)-v(s)\|\,ds + mL_1\|u(t)-v(t)\|,$$

where M^* is a suitable constant. Hence

$$\|u(t)-v(t)\| \le M_1\|u_0-v_0\| + M_1^*\int_0^t\|u(s)-v(s)\|\,ds$$

which implies by Grownwall's inequality, that

$$\|u(t) - v(t)\| \le M_1 e^{M_1^* T} \|u_0 - v_0\|,$$

and therefore

$$\|u - v\|_\infty \le M_1 e^{M_1^* T} \|u_0 - v_0\|$$

which yields both the uniqueness of u and the lipsctiz continuity of the map $u_0 \to u$.

4 Impulsive Semilinear Cauchy Problem

Let $A(t)$ is a bounded linear operator and the nonlinear functions $f, g : I \times X \times X \to X$ and $h : D \times X \to X$ are continuous. Here $D = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t \le T\}$. For brevity let

$$Hu(t) = \int_0^t h(t, s, u(s))\, ds.$$

(H3) h is continuous function and there exist positive constants α_1, α_2 and α such that

$$\|h(t,s,u) - h(t,s,v)\| \le \alpha_1 \|u - v\|$$

for all $u, v \in \mathcal{C}(I, X)$, $\alpha_2 = \max_{(t,s)\in D} \|h(t,s,0)\|$, and $\alpha = \max\{\alpha_1, \alpha_2\}$.

Lemma 4.1 *If $(H3)$ is satisfied, then the estimate*

$$\|Hu(t)\| \le t(\alpha_1 \|u\| + \alpha_2)$$
$$\|Hv(t) - Hw(t)\| \le \alpha_1 t \|v - w\|.$$

is satisfied for any $t \in I$, and $u, v, w \in \mathcal{C}(I, X)$.

Proof

By $(H3)$, we have

$$\begin{aligned}\|Hu(t)\| &\le \int_0^t \|h(t,s,u(s))\|\, ds \\ &\le \int_0^t \|h(t,s,u(s)) - h(t,s,0)\|\, ds + \int_0^t \|h(t,s,0\|\, ds \\ &\le \alpha_1 t \|u\| + \alpha_2 t = t(\alpha_1 \|u\| + \alpha_2).\end{aligned}$$

On the other hand,

$$\begin{aligned}\|Hv(t) - Hw(t)\| &\le \int_0^t \|h(t,s,v(s)) - h(t,s,w(s)\|\, ds \\ &\le \alpha_1 \int_0^t \|v - w\|\, ds \\ &\le \alpha_1 t \|v - w\|.\end{aligned}$$

We need the following additional assumptions to prove the existence of solution of the Eq. (3.1).
(H4) f, g are continuous and there exist a positive constants L, L_2 ,L_3, K, K_2 K_3 and N such that

$$\|f(t,u(t),Hu(t)) - f(t,v(t),Hv(t))\| < L_2\|u-v\| + L_3\|Hu-Hv\| \\ < (L_2 + L_3\alpha_1 t)\|u-v\| < L\|u-v\| \quad \text{for all} \quad u,v \in X$$

$$\|g(t,u(t),Hu(t)) - g(t,v(t),Hv(t))\| < K_2\|u-v\| + K_3\|Hu-Hv\| \\ < (K_2 + K_3\alpha_1 t)\|u-v\| < K\|u-v\| \quad \text{for all} \quad u,v \in X$$

and $N = max_{t\in I}\|f(t,0,0)\|$, $\lambda = max_{t\in I}\|g(t,0,0)\|$
(H4*) The function $q : I \to X$ is continuous and there exist constant q_t such that $\int_0^t |q(t-s)|\, ds = q_t$.

Now consider the semilinear fractional impulsive evolution equation

$$\begin{aligned} u^{(\alpha)} &= A(t)u(t) + f(t,u(t),Hu(t)) + \int_0^t q(t-s)g(s,u(s),Hu(s))\, ds, \\ & t \in [0,T], \quad 0 < \alpha \le 1 \\ \Delta u|_{t=t_k} &= I_k(u(t_k^-)) \quad k = 1,...,m. \\ u(0) &= u_0 \in X \end{aligned} \tag{4.1}$$

The Eq. (4.1) is equivalent to the integral equation

$$\begin{aligned} u(t) = u_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}[A(s)u(s) + f(s,u(s),Hu(s)) \\ &+ \int_0^s q(s-\tau)g(\tau,u(\tau),Hu(\tau))\, d\tau]\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1}[A(s)u(s) + f(s,u(s),Hu(s))) \\ &+ \int_0^s q(s-\tau)g(\tau,u(\tau),Hu(\tau))\, d\tau]\, ds + \sum_{0<t_k<t} I_k(u(t_k^-)) \end{aligned}$$

Theorem 4.2 *If the hypotheses H1, H2 and H4 are satisfied and if*

$$[\gamma(m+1)(M + L + q_t K) + m\rho_1] < \frac{1}{2}$$

then the fractional impulsive equation (4.1) has a unique solution on I.

Proof Let $Z = PC(I, X)$. Define the mapping $\Phi : Z \to Z$ by

$$\begin{aligned}\Phi u(t) = u_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}[A(s)u(s) + f(s, u(s), Hu(s))\\
&+ \int_0^s q(s-\tau)g(\tau, u(\tau))\, d\tau]\, ds\\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}[A(s)u(s) + f(s, u(s), Hu(s))\\
&+ \int_0^s q(s-\tau)g(\tau, u(\tau), Hu(\tau))\, d\tau]\, ds + \sum_{0<t_k<t} I_k(u(t_k^-))\end{aligned}$$

and we have to show that Φ has a fixed point. This fixed point is the solution of Eq. (4.1). Choose $r \geq 2\,[\|u_0\| + \gamma(m+1)(N + q_t\lambda)]$.
Then we can show that $\Phi B_r \subseteq B_r$, where $B_r := \{u \in Z : \|u\| \leq r\}$. From the assumption we have

$$\begin{aligned}\|\Phi u(t)\| \leq \|u_0\| &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}[\|A(s)\|\, \|u(s)\| + \|f(s, u(s), Hu(s))\|\\
&+ \left\| \int_0^s q(s-\tau)g(\tau, u(\tau), Hu(\tau))\, d\tau \right\|]\, ds\\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}[\|A(s)\|\, \|u(s)\| + \|f(s, u(s), Hu(s))\|\\
&+ \left\| \int_0^s q(s-\tau)g(\tau, u(\tau), Hu(\tau))\, d\tau \right\|]\, ds + \sum_{0<t_k<t} \left\| I_k(u(t_k^-)) \right\|\\
\leq \|u_0\| &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}[\|A(s)\|\, \|u(s)\|\\
&+ \|f(s, u(s), Hu(s)) - f(s, 0, 0)\| + \|f(s, 0, 0)\|\\
&+ \left\| \int_0^s q(s-\tau)[g(\tau, u(\tau), Hu(\tau)) - g(\tau, 0, 0)]\, d\tau \right\|\\
&+ \left\| \int_0^s q(s-\tau)g(\tau, 0, 0)\, d\tau \right\|]\, ds\\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}[\|A(s)\|\, \|u(s)\|\\
&+ \|f(s, u(s), Hu(s)) - f(s, 0, 0)\| + \|f(s, 0, 0)\|\\
&+ \left\| \int_0^s q(s-\tau)[g(\tau, u(\tau), Hu(\tau)) - g(\tau, 0, 0)]\, d\tau \right\|\\
&+ \left\| \int_0^s q(s-\tau)g(\tau, 0, 0)\, d\tau \right\|]\, ds + \sum_{0<t_k<t} \left\| I_k(u(t_k^-)) \right\|\\
\leq \|u_0\| &+ \gamma\, \|u\|\, [(m+1)[M + L + q_t K] + \rho_1 m] + \gamma(m+1)(N + q_t\lambda)\\
\leq \|u_0\| &+ \gamma(m+1)(N + q_t\lambda) + r\,[\gamma(m+1)(M + L + q_t K) + m\rho_1]\\
&\leq r\end{aligned}$$

Thus Φ maps B_r into itself. Now, for u_1, $u_2 \in Z$, we have

$$\begin{aligned}\|\Phi u_1(t) - \Phi u_2(t)\| \le \|u_0\| + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} &(t_k - s)^{\alpha-1}[\|A(s)(u_1(s) - u_2(s))\| \\ &+ \|f(s, u_1(s), Hu_1(s)) - f(s, u_2(s), Hu_2(s))\| \\ &+ \left\|\int_0^s q(s-\tau)[g(\tau, u_1(\tau), Hu_1(\tau)) - g(\tau, u_2(\tau), Hu_2(s))]\, d\tau\right\|]\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1}[\|A(s)(u_1(s) - u_2(s))\| \\ &+ \|f(s, u_1(s), Hu_1(s)) - f(s, u_2(s), Hu_2(s))\|\end{aligned}$$

$$\begin{aligned}&+ \left\|\int_0^s q(s-\tau)\left[g(\tau, u_1(\tau), Hu_1(\tau)) - g(\tau, u_2(\tau), Hu_2(\tau))\right]\, d\tau\right\|]\, ds \\ &+ \sum_{0<t_k<t} \left\|I_k(u_1(t_k^-)) - I_k(u_2(t_k^-))\right\| \\ &\le \left[\frac{T^\alpha}{\Gamma(\alpha+1)}(m+1)\left[M + L + q_t K\right] + mL_1\right] \|u_1 - u_2\| \\ &\le \left[\gamma(m+1)\left[M + L + q_t K\right] + mL_1\right] \|u_1 - u_2\| \\ &\le \frac{1}{2} \|u_1 - u_2\|.\end{aligned}$$

Hence Φ is a contraction mapping and therefore there exists a unique fixed point $u \in B_r$ such that $\Phi u(t) = u(t)$. Any fixed point of Φ is the solution of Equation (4.1).

5 Nonlocal Impulsive Semilinear Cauchy Problem

In this section we discuss the existence of solution of the impulsive evolution equation (4.1) with nonlocal condition of the form

$$u(0) + \chi(u) = u_0 \tag{5.1}$$

where $\chi : PC(I, X) \to X$ is a given function which satisfies the following condition

(H5) $\chi : PC(I, X) \to X$ is continuous and there exists a constant $G > 0$ such that

$$\|\chi(u) - \chi(v)\| \le G\, \|u - v\|_{PC} \quad \text{for } u, v \in PC(I, X).$$

Theorem 5.1 *If the hypotheses $(H1), (H2), (H3), (H4)$ and $(H5)$ are satisfied and if*

$$\left[G + \gamma\left[(m+1)\left(M + L + q_t K\right)\right] + mL_1\right] < \frac{1}{2}$$

then the fractional impulsive evolution Equation (4.1) with nonlocal condition (5.1) has a unique solution on I.

Proof

We want to prove that the operator $\Psi : Z \to Z$ defined by

$$\Psi u(t) = u_0 - \chi(u) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} A(s)u(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} A(s)u(s)\, ds$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \left[f\left(s, u(s), Hu(s)\right) + \int_0^s q(s-\tau) g(\tau, u(\tau), Hu(\tau))\, d\tau \right] ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \left[f\left(s, u(s), Hu(s)\right) + \int_0^s q(s-\tau) g(\tau, u(\tau), Hu(\tau))\, d\tau \right] ds$$
$$+ \sum_{0<t_k<t} I_k(u(t_k^-))$$

has a fixed point. This fixed point is then a solution of Equation (4.1) with nonlocal condition (5.1). Choose $r \geq 2\left(\|u_0\| + \|\chi(0)\| + (m+1)N\gamma\right)$. Then it is easy to see that $\Psi B_r \subset B_r$. Further, for $u, v \in Z$, we have

$$\|\Psi u - \Psi v\| \leq G\,\|u(t) - v(t)\| + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|A(s)\|\, \|u(s) - v(s)\|\, ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t_k - s)^{\alpha-1} \|A(s)\|\, \|u(s) - v(s)\|\, ds$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} [\|f\left(s, u(s), Hu(s)\right) - f\left(s, v(s), Hv(s)\right)\|$$
$$+ \int_0^s \|q(s-\tau)\|\, \|g(\tau, u(\tau), Hu(\tau)) - g(\tau, v(\tau), Hv(\tau))\|\, d\tau]\, ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} [\|f\left(s, u(s), Hu(s)\right) - f\left(s, v(s), Hv(s)\right)\|$$
$$+ \int_0^s \|q(s-\tau)\|\, \|g(\tau, u(\tau), Hu(\tau)) - g(\tau, v(\tau), Hv(\tau))\|\, d\tau]\, ds$$
$$+ \sum_{0<t_k<t} \left\| I_k(u(t_k^-) - I_k(v(t_k^-) \right\|$$
$$\leq \left[G + \frac{T^\alpha}{\Gamma(\alpha+1)} \left[(m+1)\left(M + L + q_t K\right) \right] + mL_1 \right] \|u - v\|$$
$$\leq \left[G + \gamma \left[(m+1)\left(M + L + q_t K\right) \right] + mL_1 \right] \|u - v\|$$
$$\leq \frac{1}{2} \|u - v\|.$$

The result follows by the application of contraction mapping principle. Our next result is based on the following well-known fixed point theorem.

Krasnoselskii Theorem. *Let S be closed convex nonempty subset of a Banach space X. Let P, Q be two operators such that*

(i) $Px + Qx \in S$ whenever $x, y \in S$;

(ii) P is a contraction mapping;

(iii) Q is compact and continuous.

Then there exists $z \in S$ such that $z = Pz + Qz$.

Now, we assume the following condition with $(H4)$ and apply the above fixed point theorem.

(H6) $f : I \times X \times X \to X$ is continuous and there exists a continuous function $\mu \in L^1(I)$ such that

$$\|f(t, u, Hu)\| \leq \mu(t), \quad \text{for all} \quad (t, u, Hu) \in I \times X \times X.$$

(H7) $g : I \times X \times X \to X$ is continuous and there exists a continuous function $\mu \in L^1(I)$ such that

$$\|g(t, u, Hu)\| \leq \rho(t), \quad \text{for all} \quad (t, u, Hu) \in I \times X \times X.$$

Theorem 5.2 *Assume that $(H1), (H2), (H4), (H5), (H6)$, and $(H7)$ hold. If $G + \gamma(m+1)M + mL_1 < 1$, then the fractional evolution Equation (4.1) with nonlocal condition (5.1) has a solution on I.*

Proof

Choose

$$r \geq \frac{\|u_0\| + \|\chi(0)\| + \gamma(m+1)(\mu_0 + q_t\rho_0)}{1 - (G + [\gamma(m+1)M] + m\rho_1)}$$

where $\mu_0 = \sup\limits_{t \in I} \mu(t), \rho_0 = \sup\limits_{t \in I} \rho(t)$ and define the operators P and Q on B_r as

$$\begin{aligned} Pu(t) = u_0 - \chi(u) &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} A(s)u(s)\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} A(s)u(s)\, ds + \sum_{0<t_k<t} I_k(u(t_k^-)) \end{aligned}$$

and

$$\begin{aligned} Qu(t) = &\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \left[f\left(s, u(s), Hu(s)\right) + \int_0^s q(s-\tau) g(\tau, u(\tau), Hu(\tau))\, d\tau \right] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} \left[f\left(s, u(s), Hu(s)\right) + \int_0^s q(s-\tau) g(\tau, u(\tau), Hu(\tau))\, d\tau \right] ds. \end{aligned}$$

For any $u, v \in B_r$, we have

$$\|Pu(t) + Qv(t)\| \leq \|u_0\| + \|\chi(u) - \chi(0)\| + \|\chi(0)\| + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|A(s)\| \|u(s)\| \, ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \|A(s)\| \|u(s)\| \, ds + \sum_{0<t_k<t} \left\| I_k(u(t_k^-)) \right\|$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \left[\|f(s, u(s), Hu(s))\| + \int_0^s \|q(s-\tau)\| \|g(\tau, u(\tau), Hu(\tau))\| \, d\tau \right] ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \left[\|f(s, u(s), Hu(s))\| + \int_0^s \|q(s-\tau)\| \|g(\tau, u(\tau), Hu(\tau))\| \, d\tau \right] ds$$
$$\leq \|u_0\| + \|\chi(0)\| + r[G + [\gamma(m+1)M] + m\rho_1] + \gamma(m+1)(\mu_0 + q_t \rho_0)$$
$$\leq r.$$

Hence, we deduce that $\|Pu + Qv\| \leq r$.
Next, for any $t \in I$, $u, v \in X$ we have

$$\|Pu - Pv\| \leq \|\chi(u) - \chi(v)\| + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|A(s)\| \|u(s) - v(s)\| \, ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \|A(s)\| \|u(s) - v(s)\| \, ds + \sum_{0<t_k<t} \left\| I_k(u(t_k^-)) - I_k(v(t_k^-)) \right\|$$
$$\leq G \|u - v\| + \gamma(m+1)M \|u - v\| + mL_1 \|u - v\|$$
$$\leq [G + \gamma(m+1)M + mL_1] \|u - v\| .$$

And since $G + \gamma(m+1)M + mL_1 < 1$, then P is a contraction mapping.
Now, let us prove that Q is continuous and compact.

Let $\{u_n\}$ be a sequence in B_r, such that $u_n \to u$ in B_r. Then

$$f(s, u_n(s), Hu_n(s)) \to f(s, u(s), Hu(s)), \quad \text{and} \quad g(s, u_n(s), Hu_n(s)) \to g(s, u(s), Hu(s)) \quad n \to \infty$$

because the functions f and g are continuous on $I \times X \times X$. Now, for each $t \in I$, we have

$$\|Qu_n - Qu\| \leq \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} [\|f(s, u_n(s), Hu_n(s)) - f(s, u(s), Hu(s))\|$$
$$+ \int_0^s \|q(s-\tau)\| \|g(\tau, u_n(\tau), Hu_n(\tau)) - g(\tau, u(\tau), Hu(\tau))\| \, d\tau] \, ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} [\|f(s, u_n(s), Hu_n(s)) - f(s, u(s), Hu(s))\|$$
$$+ \int_0^s \|q(s-\tau)\| \|g(\tau, u_n(\tau), Hu_n(\tau)) - g(\tau, u(\tau), Hu(\tau))\| \, d\tau] \, ds$$

Consequently, $\lim_{n\to\infty} \|Qu_n(t) - Qu(t)\| = 0$. In other word, Q is continuous.
Let's now note that Q is uniformly bounded on B_r. This follows from the inequality

$$\begin{aligned}
\|Qu(t)\| &\le \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \left[\|f(s, u(s), Hu(s))\| + \int_0^s \|q(s-\tau)\| \, \|g(\tau, u(\tau), Hu(\tau))\| \, d\tau \right] ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \left[\|f(s, u(s), Hu(s))\| + \int_0^s \|q(s-\tau)\| \, \|g(\tau, u(\tau), Hu(\tau))\| \, d\tau \right] ds \\
&\le \gamma[(m+1)(\mu_0 + q_t\rho_0)].
\end{aligned}$$

Now, lets prove that Qu, $u \in B_r$ is equicontinuous. Let $t_1, t_2 \in I$, $t_1 < t_2$, and let B_r be a bounded set in X. Let $u \in B_r$, we have

$$\begin{aligned}
\|Qu(t_2) - Qu(t_1)\| \le &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} [\|f(s, u(s), Hu(s))\| \\
&+ \int_0^s \|q(s-\tau)\| \, \|g(\tau, u(\tau), Hu(\tau))\| \, d\tau] \, ds \\
&= \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t_2} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} [\|f(s, u(s), Hu(s))\| \\
&+ \int_0^s \|q(s-\tau)\| \, \|g(\tau, u(\tau), Hu(\tau))\| \, d\tau] \, ds \\
&- \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - s)^{\alpha-1} [\|f(s, u(s), Hu(s))\| \\
&+ \int_0^s \|q(s-\tau)\| \, \|g(\tau, u(\tau), Hu(\tau))\| \, d\tau] \, ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] [\|f(s, u(s), Hu(s))\| \\
&+ \int_0^s \|q(s-\tau)\| \, \|g(\tau, u(\tau), Hu(\tau))\| \, d\tau] \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} [\|f(s, u(s), Hu(s))\| \\
&+ \int_0^s \|q(s-\tau)\| \, \|g(\tau, u(\tau), Hu(\tau))\| \, d\tau] \, ds \\
&\le \frac{\mu_0 + q_t\rho_0}{\Gamma(\alpha+1)} [2(t_2 - t_1)^{\alpha} - t_1^{\alpha} + t_2^{\alpha}]
\end{aligned}$$

As $t_2 \to t_1$, the right hand side of the above inequality tends to zero. In short we have proven that $Q(B_r)$ is relatively compact for $t \in I$. By Arzela Ascoli's theorem, Q is compact. Hence by the Krasnoselskii theorem there exists a solution of the problem (4.1) with nonlocal condition (5.1).

6 Example

In this section we provide with an example to illustrate our main result. Consider the following fractional integrodifferential equation with impulsive of the form.

$$\begin{aligned}
{}^cD^\alpha u(t) &= \frac{1}{100}e^{-t}|u(t)| + f(t,u(t),Hu(t)) + \int_0^t q(t-s)g(s,u(s)Hu(s))ds\\
t \in I &= [0,1], \qquad x \in (0,\pi) \qquad t \neq \frac{1}{2}, \qquad 0<\alpha\leq 1.\\
\Delta u|_{t=\frac{1}{2}} &= \frac{|u(\frac{1}{2}^-)|}{9+|u(\frac{1}{2}^-)|}\\
u(0) &= u_0,
\end{aligned}$$

where

$$A(t) = \frac{1}{100}e^{-t}I, \qquad Hu(t) = \int_0^t e^{-\frac{1}{5}u(s)}\,ds$$

$$f(t,u,Hu) = \frac{e^{-t}|u|}{(49+e^t)(1+|u|)} + \frac{1}{10}Hu(t), \quad g(t,u,Hu) = \frac{|u|}{(t+5)^2(1+|u|)} + \frac{1}{20}Hu(t)$$

$$q(t-s) = e^{t-s} \quad \text{and} \quad I_k(u) = \frac{|u|}{9+|u|}, \quad u \in X$$

Let $u, v \in \mathcal{C}(I,X)$ and $t \in I$. Then we have

$$\|Hu - Hv\| = \left|\int_0^t e^{-\frac{1}{5}u(s)}\,ds - \int_0^t e^{-\frac{1}{5}v(s)}\,ds\right| \leq \frac{1}{5}\|u-v\|$$

$$\begin{aligned}
\|f(t,u,Hu) - f(t,v,Hv)\| &= \frac{e^{-t}}{49+e^t}\left|\frac{u}{1+u} - \frac{v}{1+v}\right| + \frac{1}{10}\|Hu-Hv\|\\
&= \frac{e^{-t}|u-v|}{(49+e^t)(1+u)(1+v)} + \frac{1}{10}\|Hu-Hv\|\\
&\leq \frac{e^{-t}}{49+e^t}|u-v| + \frac{1}{10}\|Hu-Hv\|\\
&\leq \frac{1}{50}\left[|u-v| + \|Hu-Hv\|\right] \leq \frac{2}{50}\|u-v\|.
\end{aligned}$$

$$\begin{aligned}
\|g(t,u,Hu) - g(t,v,Hv)\| &= \frac{1}{(t+5)^2}\left|\frac{u}{1+u} - \frac{v}{1+v}\right| + \frac{1}{20}\|Hu-Hv\|\\
&= \frac{|u-v|}{(t+5)^2(1+u)(1+v)} + \frac{1}{20}\|Hu-Hv\|\\
&\leq \frac{1}{(t+5)^2}|u-v| + \frac{1}{20}\|Hu-Hv\|\\
&\leq \frac{1}{25}|u-v| + \frac{1}{20}\|Hu-Hv\| \leq \frac{5}{100}\|u-v\|.
\end{aligned}$$

Hence the condition $(H1), (H4)$, and $(H4^*)$ hold with $M = \frac{1}{100}, L = \frac{2}{50}, K = \frac{5}{100}, q_t = 2$. Let $u, v \in X$. Then we have by $(H2)$

$$\|I_k(u) - I_k(v)\| = \left|\frac{u}{9+u} - \frac{v}{9+v}\right| = \frac{9|u-v|}{(9+u)(9+v)} \leq \frac{1}{9}|u-v|$$

Note that $L_1 = \frac{1}{9}$. Choose $r = 1, m = 1$ we shall check the condition

$$[\gamma(m+1)(M + L + q_t K) + m\rho_1] < \frac{1}{2}$$

is satisfied. Indeed

$$[\gamma(m+1)(M + L + q_t K) + m\rho_1] < \frac{1}{2} \Leftrightarrow \Gamma(\alpha+1) > \frac{75}{100}, \tag{6.1}$$

which is satisfied for some $0 < \alpha \leq 1$. Then by Theorem 4.2 the problem has a unique solution on $[0, 1]$ for some values of α satisfied condition (6,1).

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