This paper presents a simple and efficient method for determining the solution of Riccati differential equation with coefficients rational. In case the differential Galois group of the differential equation \((E_l) : y'' = ry, \ r \in \mathbb{C}(x)\) is reducible, we look for the rational solutions of Riccati differential equation \(\theta' + \theta^2 = r\), by reducing the number of check to be made and by accelerating the search for the partial fraction decomposition of the solution reserved for the poles of \(\theta\) which are false poles of \(r\). This partial fraction decomposition of solution can be used to code \(r\). The examples demonstrate the effectiveness of the method.

1 Introduction

The quadratic Riccati differential equation:

\[
(E_R) : \sigma' = p_2\sigma^2 + p_1\sigma + p_0
\]

where \(p_0, p_1\) and \(p_2\) are in a differential field \(\mathbb{K}, p_2 \neq 0\). The quadratic Riccati differential equation is first converted to a reduced Riccati differential equation:

\[
(E_r) : \theta' + \theta^2 = r
\]

where \(\theta = -p_2\sigma - \frac{1}{2}a\), with \(a = \frac{p_1}{p_2} + p_1\) and \(r = \frac{1}{4}a^2 - \frac{1}{2}a' + p_2p_0\).

Furthermore, we put: \(\frac{\gamma'}{\gamma} = \theta\), reduced Riccati differential equation (2) is converted to a second-order linear ordinary differential equation:

\[
(E_l) : y'' = ry
\]
If we have a particular solution non-zero of \((E_l)\) then general solution is :  \(y = cu\) where \(c' = \frac{\lambda}{u}\), \(\lambda\) constant (see\([6,9,14]\)).

In paper, we base ourselves mainly on the work of J.J. Kovačić \([9]\) where differential Galois group of the differential equation \((E_l)\) is reducible and we take : \(\mathbb{K} = \mathbb{C}(X)\).

In case where every solution of \((E_l)\) is Liouvillian corresponds to the case where reduced Riccati differential equation \((E_r)\) have algebraic solution over \(\mathbb{K}\). The case where differential Galois group is reducible corresponds to the case where the Riccati differential equation \((E_r)\) have the rational solution \(\frac{u'}{u}\) solution of \((E_l)\). The solution \(u\) of \((E_l)\) is rational fraction if only if \(\frac{u'}{u}\) the fraction of simples poles with the integers residues and negative degree.

The field \(\mathbb{C}(X)[u]\) is differential extension of \(\mathbb{C}(X)\) by exponential of an integral and if \(u' = \frac{1}{u^2}\) then \((u, v)\) two solutions of \((E_l)\) linearly independent over field of constants \(\mathbb{C}\). The ordinary extension \(\mathbb{C}(X)[u, v]\) is differential extension of \(\mathbb{C}(X)[u]\), by a integral. \(\mathbb{C}(X)[u, v]\) is Picard-Vessiot extension of \(\mathbb{C}(X)[u]\) for the differential equation \((E_l)\)(see\([8-9-10]\)). The existence of rational solution \(\frac{u'}{u}\) of Riccati differential equation \((E_r)\) given all solutions of \((E_r)\) of course research primitive of \(\frac{1}{\nu^2}\).

This paper presents a simple and efficient method for determining the solution of Riccati differential equation with coefficients rational. In case the differential Galois group of the differential equation \((E_l) : y'' = ry, r \in \mathbb{C}(x)\) is reducible, we look for the rational solutions of Riccati differential equation \(\theta' + \theta^2 = r\), by reducing the number of check to be made and by accelerating the search for the partial fraction decomposition of the solution reserved for the poles of \(\theta\) which are false poles of \(r\). This partial fraction decomposition of solution can be used to code \(r\). The examples demonstrate the effectiveness of the method.

2 Form of rational solution of equation : \((E_r)\)

Let \(r \in \mathbb{C}(x) \neq 0\) rational fraction and \(\theta \in \mathbb{C}(x)\) the rational solution of Riccati differential equation \((E_r) : \theta' + \theta^2 = r\).

2.1 Study in the pole \(c\) of multiplicity \(\nu\) of \(\theta\)

We put:

\[\theta = \frac{\tau}{(x-c)^\nu} \quad \text{where} \quad \tau(c) \neq 0\]

We have:

\[r = \theta' + \theta^2 = (x-c)^{-2\nu}[(\tau - \frac{\nu}{2}(x-c)^{\nu-1})^2 - \frac{\nu^2}{4}(x-c)^{2\nu-2} + \tau'(x-c)^{\nu}]\]

Thus:

\[(\tau - \frac{\nu}{2}(x-c)^{\nu-1})^2 = (x-c)^{2\nu}r + (x-c)^{\nu}[\frac{\nu^2}{4}(x-c)^{2\nu-2} - \tau']\]

1. Case 1 : \(\nu \geq 2\)

The function \((x-c)^{2\nu}r\) define and equal \(\tau(c)^2\) at \(c\).

Thus \(c\) is pole of multiplicity \(2\nu\) of \(r\) where:

\[
\lim_{x \to c}(x-c)^{2\nu}r = (\lim_{x \to c}(x-c)^{\nu} \theta)^2
\]
2. Case 2: \( \nu = 1 \)
We have:
\[
(\tau - \frac{1}{2})^2 = (x - c)^2 r + \frac{1}{4} - (x - c)\tau'
\]
The function \((x - c)^2 r\) define equal \(\tau(c)(\tau(c) - 1)\) at \(c\).

**Situation 1:** \(\lim_{x \to c} (x - c)^2 r \neq 0, -\frac{1}{4}\)
\(c\) is double pole of \(r\) and the residue \(\tau(c)\) of \(\theta\) at \(c\) have tow possibility values following:
\[
(\tau(c) - \frac{1}{2})^2 = \lim_{x \to c} (x - c)^2 r + \frac{1}{4}
\]
Thus, \(c\) is double pole of \(r\) and the residue of \(\theta\) at simple pole \(c\) equal:
\[
\tau(c) = \alpha_c + \frac{1}{2}
\]
where
\[
\alpha_c^2 = \lim_{x \to c} (x - c)^2 r + \frac{1}{4}
\]

**Situation 2:** \(\lim_{x \to c} (x - c)^2 r = -\frac{1}{4}\)
\(c\) is double pole of \(r\) and the residue of \(\theta\) at \(c\) is \(\frac{1}{2}\).

**Situation 3:** \(\lim_{x \to c} (x - c)^2 r = 0\)
\(c\) is simple pole of \(r\) or not pole of \(r\) and the residue of \(\theta\) at simple pole \(c\) equal 1.

**Proposition 1** Let \(\theta \in \mathbb{C}(x)\) such as: \(\theta' + \theta^2 = r\).

1. The fraction : \(r = \frac{N}{D}\) with \(N\) and \(D\) polynomials relatively prime.
\[
D = D_1 D_2^2 D_3^2 D_4^2
\] (4)
where \(D_1, D_2, D_3\) and \(D_4\) polynomials relatively prime pair-wise. \(D_1, D_2\) and \(D_3\) which simples roots, \(D_4\) without simple root.

\(\forall c \in \text{Root}(D_2)\) \(\lim_{x \to c} (x - c)^2 r = -\frac{1}{4}\), \(\forall c \in \text{Root}(D_3)\) \(\lim_{x \to c} (x - c)^2 r \neq -\frac{1}{4}\)

2. (a) Let \(\nu \geq 2\). \(c\) pole of multiplicity \(\nu\) of \(\theta\) \(\iff\) \(c \in \text{Root}(D_4)\)

(b) \(c\) simple pole of \(\theta\) with residue \(\neq 1, \frac{1}{2}\) \(\iff\) \(c \in \text{Root}(D_3)\)
The residue of \(\theta\) there \(c\) equal \(\alpha_c + \frac{1}{2}\) where
\[
\alpha_c^2 = \lim_{x \to c} (x - c)^2 r + \frac{1}{4}
\] (5)

(c) \(c\) simple pole of \(\theta\) with residue \(\frac{1}{2}\) \(\iff\) \(c \in \text{Roots}(D_2)\)

(d) \(c\) simple pole of \(\theta\) with residue \(1\) \(\iff\) \(c \in \text{Roots}(D_1)\) or \(c\) pole of \(\theta\) and false pole of \(r\)

**Corollary 2** We assume that \(r = \frac{N}{D}\) with \(N\) and \(D\) polynomials relatively prime, \(D = D_1 D_2^2 D_3^2 D_4^2\) where \(D_1, D_2, D_3\) and \(D_4\) polynomials relatively prime pair-wise, \(D_1, D_2\) and \(D_3\) which simples roots, \(D_4\) without simple root.
\(\forall c \in \text{Root}(D_2)\) \(\lim_{x \to c}(x - c)^2 r = -\frac{1}{4}\), \(\forall c \in \text{Root}(D_3)\) \(\lim_{x \to c}(x - c)^2 r \neq -\frac{1}{4}\)

A rational fraction \(\theta\) Verify \(\theta' + \theta^2 = r\) is the shape:

\[
\theta = E(\theta) + \sum_{c \in \text{Roots}(D_4)} \alpha_c + \sum_{c \in \text{Roots}(D_3)} \frac{\alpha_c + \frac{1}{2}}{x - c} + \frac{1}{2} \frac{D'_2}{D_2} + \frac{D'_1}{D_1} + \frac{D'_0}{D_0}
\]

with \(D_0\) monic polynomial which simples roots and the roots are false poles of \(r\).

\[4\]

2.2 Study in the infinity

Let: \(t \in \mathbb{C}(x); t \neq 0\) rational fraction such as: \(\frac{d}{dx} + t^2 = 0\).

**case 1:** We assume that: \(d^o(\theta) < 0\)

We have: \(d^o(\theta') < 0\) and \(d^o(\theta^2) < 0\) thus: \(d^o(r) < 0\).

We put:

\[
\theta = t \sigma(t) \quad \sigma \text{ rational fraction defined at 0.}
\]

We have:

\[
\sigma(0) = \lim_{x \to \infty} x \theta = \text{sum the residues of } \theta,
\]

\[
r = \theta' + \theta^2 = (\sigma^2 - \sigma)t^2 - \sigma't^3 \quad \text{and} \quad \lim_{x \to \infty} x^2r = (\sigma(0)^2 - \sigma(0))
\]

Thus: \(d^o r \leq -2\) and \((\sigma(0) - \frac{1}{2})^2 = \lim_{x \to \infty} x^2r + \frac{1}{4}\)

If: \(\lim_{x \to \infty} x^2r = -\frac{1}{4}\) then the sum of residues of \(\theta\) : \(\sigma(0) = \frac{1}{2}\)

If: \(\lim_{x \to \infty} x^2r \neq -\frac{1}{4}\) then the sum of residues of \(\theta\) : \(\sigma(0) = \alpha_\infty + \frac{1}{2}\)

where

\[
\alpha_\infty^2 = \lim_{x \to \infty} x^2r + \frac{1}{4}
\]

**case 2:** We assume that: \(d^o(\theta) = 0\).

\(E(\theta)\) constant \(\neq 0\). We put:

\[
\theta = E(\theta) + t \sigma(t); \quad \sigma \text{ rational fraction defined at 0.}
\]

We have:

\[
\sigma(0) = \lim_{x \to \infty} x(\theta - E(\theta)) = \text{the sum of residues of } \theta
\]

and

\[
r = \theta' + \theta^2
\]

\[
= (E(\theta))^2 + 2E(\theta)t\sigma + (\sigma^2 - \sigma)t^2 - \sigma't^3
\]

So \(E(r)\) constant equal \(E(\theta)^2\):

\[
2E(\theta)[\text{sum the residues of } \theta] = \text{sum the residues of } r
\]

**case 3:** We assume that: \(d^o(\theta) > 0\)

We put:

\[
\nu = d^o(\theta) \geq 1 \quad \text{and} \quad \theta = t^{-\nu}\sigma(t)
\]

\(\sigma\) rational fraction defined at 0. The scalar \(\sigma(0)\) is the dominant coefficient of \(E(\theta)\). We have:

\[
r = \theta' + \theta^2 = t^{-2\nu}[\sigma^2 + \nu t^{\nu+1}\sigma - t^{\nu+2}\sigma']
\]
Thus:

\[ t^{2\nu}r = \sigma(0)^2 + o(t) \]

So: \( d^\nu r = 2\nu = 2d^\nu \theta \) and \( \sigma^2(0) \) the dominant coefficient of \( E(r) \).

**Proposition 3** Let: \( \theta \in \mathbb{C}(x) \) such as: \( \theta' + \theta^2 = r \).

1. \( d^\nu(\theta) < 0 \iff d^\nu(r) < 0 \). Thus: \( d^\nu(r) \leq -2 \) and:

   \[
   \text{the sum of residues of } \theta = \begin{cases} 
   \frac{1}{2} \text{ if } \lim_{x \to \infty} x^2r = -\frac{1}{4} \\
   \alpha_\infty + \frac{1}{2} \text{ if } \lim_{x \to \infty} x^2r \neq -\frac{1}{4}
   \end{cases}
   \]

2. \( d^\nu(\theta) = 0 \iff d^\nu(r) = 0 \). In the case: \( E(\theta) \) is square root of \( E(r) \):

   \[
   2E(\theta)(\text{sum of residues of } \theta) = \text{sum of residues of } r = \lim_{x \to \infty} x(r - E(r))
   \]

3. We have: \( d^\nu(\theta) > 0 \iff d^\nu(r) > 0 \). In the case:

   (a) \( d^\nu(r) = 2d^\nu(\theta) \)
   (b) The dominant coefficient of \( E(\theta) \) is square root of \( E(r) \)

**2.3 Determination of \( E(\theta) \): \( d^\nu(r) = 2\nu > 0 \)**

We assume that \( r \) is a rational fraction of degree \( 2\nu > 0 \) and \( \theta \in \mathbb{C}(x) \) such as: \( \theta' + \theta^2 = r \). Let \( a \) the dominant coefficient of \( E(r) \). Thus: \( r \sim ax^{2\nu} \) if \( x \) tend to \( \infty \):

\[
\frac{t^{2\nu} E(r)}{a} = 1 + a_1 t + \ldots + a_{2\nu} t^{2\nu}
\]

The Taylor’s expansion of order \( \nu + 1 \) at 0:

\[
(t^{2\nu} E(r))^{\frac{1}{2}} = 1 + s_1 t + \ldots + s_{\nu+1} t^{\nu+1} + o(t^{\nu+1})
\]

We have:

\[
\begin{align*}
    t^{2\nu}r &= t^{2\nu} E(r) + o(t^{2\nu}) \\
    &= t^{2\nu} E(r) + o(t^{\nu+1}) \\
    &= a[(t^{2\nu} E(r))^{\frac{1}{2}}]^2 + o(t^{\nu+1})
\end{align*}
\]

We have: \( \theta = t^{-\nu} \sigma(t) \) with \( \sigma \) rational fraction defined at 0, \( \sigma(0)^2 = a \) and

\[
\begin{align*}
    (\sigma + \frac{\nu}{2} t^{\nu+1})^2 &= t^{2\nu} r + \frac{\nu^2}{4} t^{2\nu+2} + t^{\nu+2} \sigma' \\
    &= a[1 + s_1 t + \ldots + s_{\nu+1} t^{\nu+1}]^2 + o(t^{\nu+1}) \\
    \sigma + \frac{\nu}{2} t^{\nu+1} &= \sigma(0)[1 + s_1 t + \ldots + s_{\nu+1} t^{\nu+1}] + o(t^{\nu+1})
\end{align*}
\]

Thus:

\[
\theta = t^{-\nu} \sigma = \sigma(0)[t^{-\nu} + s_1 t^{-(\nu-1)} + \ldots + s_{\nu+1} t] - \frac{\nu}{2} t + o(t)
\]
Imply:

\[
\begin{align*}
E(\theta) &= \sigma(0) [t^{-\nu} + s_1 t^{-(\nu-1)} + \ldots + s_\nu] \\
\sigma(0) s_{\nu+1} - \frac{\nu}{2} &= \text{sum of residues of } \theta
\end{align*}
\]

**Proposition 4** Let \( r \) is a rational fraction of degree \( 2\nu > 0 \), \( \theta \in \mathbb{C}(x) \) such as: \( \theta' + \theta^2 = r \) and \( a \) the dominant coefficient of \( E(r) \). If:

\[
(t^{2\nu} \frac{E(r)}{a})^{\frac{1}{2}} = 1 + s_1 t + \ldots + s_{\nu+1} t^{\nu+1} + o(t^{\nu+1})
\]

Then:

\[
E(\theta) = \alpha [t^{-\nu} + s_1 t^{-(\nu-1)} + \ldots + s_\nu]
\]

\[
\alpha s_{\nu+1} - \frac{\nu}{2} = \text{sum of residues of } \theta
\]

where

\[
\alpha^2 = a
\]

3 Determination of partial fraction decomposition

Let \( r = \frac{N}{D} \) rational fraction with \( N \) and \( D \) polynomials relatively prime, \( D = D_1 D_2^2 D_3^2 D_4^2 \) where \( D_1, D_2, D_3 \) and \( D_4 \) polynomials relatively prime pair-wise, \( D_1, D_2 \) and \( D_3 \) which simples roots, \( D_4 \) without simple root.

\( \forall c \in \text{Root}(D_2) \) \( \lim_{x \to c} (x-c)^2 r = -\frac{1}{4} \). \( \forall c \in \text{Root}(D_3) \) \( \lim_{x \to c} (x-c)^2 r \neq -\frac{1}{4} \).

Let \( \theta \in \mathbb{C}(x) \) rational fraction Verify: \( \theta' + \theta^2 = r \)

3.1 Case \( d^a D_3 = 0 \) and \( d^a D_4 = 0 \)

We have: \( r = \frac{N}{D_1 D_2^2} \)

This case corresponds to the fact that one pole \( c \) of \( r \) is or simple or double with:

\( \lim_{x \to c} (x-c)^2 r = -\frac{1}{4} \)

**Proposition 5** We assume \( d^a r < 0 \) and \( d^a D_3 = d^a D_4 = 0 \). We have:

\[
\lim_{x \to \infty} x^2 r + \frac{1}{4} = (\frac{q}{2})^2
\]

with \( q \) positive integer of parity against that of \( d^a D_2 \)

\[
\theta = \frac{1}{2} \frac{D'}{D_2} + \frac{D'_1}{D_1} + \frac{D'_0}{D_0}
\]

with \( D_0 \) polynomial of degree:

\[
d^a D_0 = \frac{1}{2} (q + 1 - d^a D_2) - d^a D_1
\]
Proof. We have:

\[ \theta = \frac{1}{2} D'_2 + \frac{D'_1}{D_1} + \frac{D'_0}{D_0} \]

Sum of residues of \( \theta \) equal \( \frac{1}{2} d^\nu D_2 + d^\nu D_1 + d^\nu D_0 = \alpha_\infty + \frac{1}{2} \)

with \( \alpha_\infty^2 = \lim_{x \to \infty} x^2 r + \frac{1}{4} \). In particular: \( \alpha_\infty = \frac{q}{2} \) with \( q \in \mathbb{N} \)

Remark: if \( \lim_{x \to \infty} x^2 r = -\frac{1}{4} \) then: \( d^\nu D_2 = 1, d^\nu D_1 = d^\nu D_0 = 0, \theta = \frac{1}{2} \frac{1}{(x-c)} \) and \( r = -\frac{1}{4(x-c)^2} \)

Proposition 6 : We assume \( d^\nu r = 0 \) and \( d^\nu D_3 = d^\nu D_4 = 0 \).

1. \( E(\theta) \) square root of \( E(r) \) such as \( p = \frac{1}{E(\theta)} \lim_{x \to \infty} x[r - E(r)] \) positive integer of same parity as \( d^\nu D_2 \).

2. We have:

\[ \theta = E(\theta) + \frac{1}{2} D'_2 + \frac{D'_1}{D_1} + \frac{D'_0}{D_0} \]  \hspace{1cm} (14)

with \( D_0 \) polynomial of degree:

\[ d^\nu D_0 = \frac{1}{2} p - d^\nu D_1 - \frac{1}{2} d^\nu D_2 \]  \hspace{1cm} (15)

Proof. We have:

\[ \theta = E(\theta) + \frac{1}{2} D'_2 + \frac{D'_1}{D_1} + \frac{D'_0}{D_0} \]

We have:

\[ 2E(\theta)\text{sum of residues}(\theta) = \text{sum of residues}(r) = \lim_{x \to \infty} x[r - E(r)] \]

\[ 2E(\theta)[\frac{1}{2} d^\nu D_2 + d^\nu D_1 + d^\nu D_0] = \lim_{x \to \infty} x[r - E(r)] \]

Thus:

\[ d^\nu D_2 + 2d^\nu D_1 + 2d^\nu D_0 = \frac{1}{E(\theta)} \lim_{x \to \infty} x[r - E(r)] \]

If \( r \) constant then \( D_1 = D_2 = D_0 = 1 \) and \( \theta \) constant.

If \( r \) non-constant then \( r \) is not polynomial thus:

\[ p = \frac{1}{E(\theta)} \lim_{x \to \infty} x[r - E(r)] \]

positive integer of same parity as \( d^\nu D_2 \).

Proposition 7 :
We assume \( d^\nu r = 2\nu > 0 \) and \( d^\nu D_3 = d^\nu D_4 = 0 \)
Let a the dominant coefficient of \( E(r) \) and we consider the Taylor’s expansion at infinity :

\[ (\frac{r^{2\nu} E(r)}{a})^{\frac{1}{2}} = 1 + s_1 t + \ldots + s_{\nu+1} t^\nu + o(t^\nu) \]  \hspace{1cm} (16)
1. \[ 4a s_{\nu+1}^2 = p^2 \] 
where \( p \) positive integer, \( p \geq d^p D_2 + 2d^p D_1 + \nu \) and same parity of \( d^p D_2 + 2d^p D_1 + \nu \).

2. We have:
\[ \theta = E(\theta) + \frac{1}{2} \frac{D'_2}{D_2} + \frac{D'_1}{D_1} + \frac{D'_0}{D_0} \] 
with \( D_0 \) polynomial of degree:
\[ d^p D_0 = \frac{p - \nu - d^p D_2}{2} - d^p D_1 \] 
and \( \alpha \) the dominant coefficient of \( E(\theta) \) where:
\[ p = 2\alpha s_{\nu+1} \] 

**Proof.** We have:
\[ E(\theta) = \alpha [t^{-\nu} + s_1 t^{-(\nu-1)} + .. + s_\nu] \]
\[ \alpha s_{\nu+1} - \nu \frac{2}{2} = \text{sum of residues of } \theta \]
\[ \theta = E(\theta) + \frac{1}{2} \frac{D'_2}{D_2} + \frac{D'_1}{D_1} + \frac{D'_0}{D_0} \]

Thus:
\[ \frac{1}{2} d^p D_2 + d^p D_1 + d^p D_0 = \alpha s_{\nu+1} - \nu \frac{2}{2} \]
\[ \alpha s_{\nu+1} = \frac{p}{2} \] with \( p \) positive integer. Thus: \( p \geq d^p D_2 + 2d^p D_1 + \nu \) and same parity of \( d^p D_2 + 2d^p D_1 + \nu \).

\[ \square \]

3.2 Case: \( D_3 = X - c \) and \( d^p D_4 = 0 \)

This case corresponds to the fact that a pole of \( r \) is simple or double with a only double pole \( c \) such as:
\[ \lim_{x \rightarrow c} (x - c)^{\frac{3}{2}} r \neq -\frac{1}{4} \] 
\[ r = \frac{N}{D_1 D_2^2 (x - c)^2} \]

**Proposition 8**: Consider the Eq. (22) and let \( \theta \in \mathbb{C}(x) \) rational fraction Verify: \( \theta' + \theta^2 = r \)

We assume: \( d^p r < 0 \). Accordingly, in view of (5) and (7) we have:
\[ \theta = \alpha_c + \frac{1}{2} \frac{D'_2}{D_2} + \frac{D'_1}{D_1} + \frac{D'_0}{D_0} \] 
with \( D_0 \) polynomial of degree:
\[ d^p D_0 = \lambda - \frac{1}{2} d^p D_2 - d^p D_1 \] 
where
\[ \lambda = \alpha_{\infty} - \alpha_c \] 

one half positive integer of same parity as \( d^p D_2 \).
Proof. We have:

\[
\lim_{x \to \infty} x^\theta = \alpha_c + \frac{1}{2} + \frac{1}{2} d^u D_2 + d^u D_1 + d^u D_0
\]

\[
(\alpha_c + \frac{1}{2} d^u D_2 + d^u D_1 + d^u D_0)^2 = \lim_{x \to \infty} x^2 r + \frac{1}{4} = \alpha_1^2
\]

Thus:

\[
\frac{1}{2} d^u D_2 + d^u D_1 + d^u D_0 = \alpha_1 - \alpha_c
\]

\[\blacksquare\]

**Proposition 9**: Consider the Eq. (22) and let \( \theta \in \mathbb{C}(x) \) rational fraction. Verify: \( \theta' + \theta^2 = r \)

We assume: \( d^o r = 0 \). We have:

\[
\theta = E(\theta) + \frac{\alpha_c + \frac{1}{2}}{x - c} + \frac{1}{2} \frac{D_2'}{D_2} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}
\]

with \( E^2(\theta) = E(r) \) and \( D_0 \) polynomial of degree:

\[
d^o D_0 = \frac{1}{2} \lambda - \frac{1}{2} d^u D_2 - d^u D_1 - \frac{1}{2}
\]

where

\[
\lambda = \frac{1}{E(\theta)} \lim_{x \to \infty} x[r - E(r)] - 2 \alpha_c
\]

\( \lambda \) is positive integer of parity against that of \( d^u D_2 \).

Proof. We have:

\[
2E(\theta)[\alpha_c + \frac{1}{2} + \frac{1}{2} d^u D_2 + d^u D_1 + d^u D_0] = \lim_{x \to \infty} x[r - E(r)]
\]

Thus:

\[
1 + d^u D_2 + 2 d^u D_1 + 2 d^u D_0 = \frac{1}{E(\theta)} \lim_{x \to \infty} x[r - E(r)] - 2 \alpha_c
\]

\[\blacksquare\]

**Proposition 10**: Consider the Eq. (22) and let \( \theta \in \mathbb{C}(x) \) rational fraction. Verify: \( \theta' + \theta^2 = r \)

We assume: \( d^o r = 2 \nu > 0 \). Let \( a \) the dominant coefficient of \( E(r) \), \( \alpha \) the dominant coefficient of \( E(\theta) \), \( t = \frac{1}{x} \) and Taylor’s expansion:

\[
(t^{2\nu} \frac{E(r)}{a})^{\frac{1}{2}} = 1 + s_1 t + \ldots + s_{\nu+1} t^{\nu+1} + o(t^{\nu+1})
\]

We have:

\[
\theta = E(\theta) + \frac{\alpha_c + \frac{1}{2}}{x - c} + \frac{1}{2} \frac{D_2'}{D_2} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}
\]

\( D_0 \) polynomial of degree:

\[
d^o D_0 = \alpha s_{\nu+1} - \alpha_c - \frac{1 + \nu + d^o D_2}{2} - d^o D_1 \quad \text{and} \quad \alpha^2 = a
\]
Proof. We have:

\[ E(\theta) = \alpha [t^{-\nu} + s_1 t^{-(\nu-1)} + \ldots + s_\nu] \]

and

\[ \alpha s_{\nu+1} - \frac{\nu}{2} = \text{sum of residues of } \theta \]

Thus:

\[ \theta = E(\theta) + \frac{\alpha}{x-c} + \frac{1}{2} \left( \frac{D_0'}{D_0} + \frac{D_1'}{D_1} + \frac{D_2'}{D_2} \right) \]

\[ \alpha s_{\nu+1} - \frac{\nu}{2} = \alpha c + \frac{1}{2} [d^\nu D_2 + d^\nu D_1 + d^\nu D_0] \]

Thus:

\[ d^\nu D_0 = \alpha s_{\nu+1} - \alpha c - \frac{1 + \nu + d^\nu D_2}{2} - d^\nu D_1 \]

\[ \blacksquare \]

3.3 Case: \((d^\nu D_3 \neq 0 \text{ and } d^\nu D_4 \neq 0)\) or \((d^\nu D_3 \geq 2 \text{ and } d^\nu D_4 = 0)\)

\(D_1 D_2^2\) and \(D_3 D_4\) polynomials relatively prime. Thus there are two only polynomials \(N_1\) and \(N_2\) such as:

\[ \left\{ \begin{array}{l}
D_1 D_2^2 D_3^2 D_4^2 (r - E(r)) = N_1 (D_1 D_2^2) + N_2 (D_3 D_4) \\
d^\nu N_1 < d^\nu (D_3 D_4)
\end{array} \right. \]

Thus

\[ r = E(r) + \frac{N_1}{D_3^2 D_4^2} + \frac{N_2}{D_1 D_2^2 D_3 D_4} \] (31)

We have:

\[ d^\nu \left( \frac{N_1}{D_3^2 D_4^2} \right) = d^\nu \left( \frac{N_1}{D_3 D_4} \right) < d^\nu \left( \frac{1}{D_3 D_4} \right) \leq -2 \] (32)

Because \(D_4\) does not have simple roots verify: \(d^\nu D_4 = 0\) or \(d^\nu D_4 \geq 2\). Thus: \(d^\nu (D_3 D_4) \geq 2\). Thus:

\[ \lim_{x \to \infty} x^2 \frac{N_1}{D_3^2 D_4^2} = 0 \] (33)

\[ d^\nu (r - E(r)) < 0 \] (34)

\[ d^\nu \left( \frac{N_2}{D_1 D_2^2 D_3 D_4} \right) < 0 \] (35)

We consider the rational fraction:

\[ F = E(r) + \frac{N_1}{D_3^2 D_4^2} - \delta \] (36)

where

\[ \delta = \frac{1}{4} \left[ \frac{1}{k} \left( \frac{D_3'}{D_3} \right)^2 + \left( \frac{1}{k} + 1 \right) \left( \frac{D_3'}{D_3} \right)^2 \right] \] (37)

\[ k = d^\nu D_3 \neq 0 \] (38)

Proposition 11: We assume \(\left\{ \begin{array}{l}
d^\nu D_3 \neq 0 \\
d^\nu D_4 \neq 0
\end{array} \right. \) or \(\left\{ \begin{array}{l}
d^\nu D_3 \geq 2 \\
d^\nu D_4 = 0
\end{array} \right. \)
1. If \( d^0 r \geq 0 \) then: \( d^0 F = d^0 r \) and \( E(F) = E(r) \)

2. If \( d^0 r < 0 \) then: \( d^0 F = -2 \) where \( \lim_{x \to \infty} x^2 F = \frac{1}{4} \).

3. For all \( c \) root of \( D_3 \) we have:

\[
\lim_{x \to c} (x - c)^2 F = \lim_{x \to c} (x - c)^2 r + \frac{1}{4}
\]

4. For all \( c \) root of \( D_4 \) of multiplicity \( \nu \) we have:

\[
(x - c)^{2\nu} F = (x - c)^{2\nu} r + o((x - c)^{\nu - 1})
\]

**Proof.**

1. \( r - F = \frac{N_2}{D_1 D_2^2 D_3 D_4} + \delta \) is from negative degree.

2. \( \lim_{x \to \infty} x^2 F = \lim_{x \to \infty} x^2 \frac{N_1}{D_2^2 D_4} - \lim_{x \to \infty} x^2 \delta = -\lim_{x \to \infty} x^2 \delta = \frac{1}{4} \)

3. Let \( c \) root of \( D_3 \).

\[
\lim_{x \to \infty} (x - c)^2 (r - F) = \lim_{x \to c} \frac{N_2}{D_1 D_2^2 D_4} \frac{(x - c)^2}{D_3} + \lim_{x \to c} (x - c)^2 \delta
\]

\[
= \lim_{x \to c} (x - c)^2 \delta = -\frac{1}{4}
\]

4. Let \( c \) root of \( D_4 \).

\[
(x - c)^{2\nu} (r - F) = \frac{N_2}{D_1 D_2^2 D_4} \frac{(x - c)^{2\nu}}{D_4} + (x - c)^{2\nu} \delta
\]

\[
= o((x - c)^{\nu - 1})
\]

Lemma 12 Let \( Z \) be non-zero rational fraction . \( \Sigma \) is a finished set such as :

\[
\Sigma \cap [\text{Roots}(Z) \cup \text{poles}(Z)] = \emptyset
\]  \hspace{1cm} (39)

1. Exists \( O \) open connected on which we have a square root holomorphic of \( Z \), containing for every \( c \in \Sigma \) a half-right closed by origin \( c \).

2. If, besides, \( Z \) is from even degree then can choose \( O \) of complementary compact.

**Proof.** We fix \( c_0 \) not an element of \( \Sigma \). Let \( \Sigma' \) a finished set containing roots, poles of \( Z \) and \( c_0 \). They consider all rights linked to the different pairs of points of \( \Sigma \cup \Sigma' \). They choose a point \( w \) not being on these rights. For all \( c \in \Sigma \cup \Sigma' \), which joins right \( w \) in \( c \) does not contain any other point of \( \Sigma \cup \Sigma' \).

**case 1:** \( d^0 Z \) even: Replacing \( Z \) by rational fraction: \( \frac{Z}{(x - c_0)^{m_0}} \).
Assume $d^pZ = 0$. We put: $K = \bigcup_{c \in \Sigma'} [w, c]. K$ is compact connected.

We put: $O = \mathbb{C} \setminus K$. $O$ is open at infinity such as for all $c \in \Sigma$ reaching right $w$ in $c$ private of $w$ is contained in $O$.

If $\gamma$ a shoelace of $O$ then $K$ is in one connected component of $\mathbb{C} \setminus \gamma$. Thus:

$$\frac{1}{2i\pi} \int_{\gamma} \frac{Z'}{Z}(x)dx = \pm \sum_{c \in \Sigma'} \text{residue}(\frac{Z'}{Z}, c) = \pm d^pZ = 0$$

Thus exist the primitive of $\frac{Z'}{Z}$ and the determination of logarithm of $Z$ and the square root of $Z$ in $O$.

**case 2 : $d^pZ$ odd**: They use the previous case in replacing $Z$ by rational fraction $\frac{Z}{(x-c)^\nu}$. ■

**Notation**: We choose a square root of the polynomial of even degree : $N_1 + D_3^2 D_4^2 [E(r) - \delta]$ on a connected open at infinity, roots of $D_3$ and root of $D_4$. We put:

$$(F)\frac{1}{2} = \frac{1}{D_3 D_4} (N_1 + D_3^2 D_4^2 [E(r) - \delta])\frac{1}{2} \quad (40)$$

**Proposition 13**: We assume $\{d^p D_3 \neq 0 \det d^p D_4 \neq 0 \text{ or } d^p D_3 \geq 2 \det d^p D_4 = 0 \}$

We have:

$$\theta = E(\theta) + \sum_{c \in \text{Root}(D_3) \cup \text{Root}(D_4)} \varepsilon_c(\text{partial fraction of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} \frac{(D_2 D_3 D_4)^\nu}{D_2 D_3 D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \quad (41)$$

where $\varepsilon_c = \pm 1$

**Proof.** For $c$ root of multiplicity $\nu \geq 2$ of $D_4$

$$((x-c)^\nu \theta - \frac{\nu}{2}(x-c)^{\nu-1})^2 = ((x-c)^\nu (F)^\frac{1}{2})^2 + o((x-c)^{\nu-1})$$

Thus:

$$(x-c)^\nu \theta - \frac{\nu}{2}(x-c)^{\nu-1} = \varepsilon_c(x-c)^\nu (F)^\frac{1}{2} + o((x-c)^{\nu-1})$$

$$\theta - \frac{\nu}{2(x-c)} = \varepsilon_c (F)^\frac{1}{2} + o\left(\frac{1}{x-c}\right)$$

Polar part of $\theta$, associated in root $c$ of $(D_4)$, minus $\frac{\nu}{2(x-c)}$, is in sign meadows that of $(F)^\frac{1}{2}$.

For $c$ root of $D_3$ :

$$[(\text{residue of } \theta \text{ at } c) - \frac{1}{2}]^2 = \lim_{x->c}((x-c)(F)^{\frac{1}{2}})^2$$

Thus:

$$[(\text{residue of } \theta \text{ at } c) - \frac{1}{2}] = \varepsilon_c(\text{residue of } (F)^{\frac{1}{2}} \text{ at } c)$$

Polar part of $\theta$, associated in root $c$ of $(D_3)$, minus $\frac{1}{2(x-c)}$, is in sign meadows that of $(F)^\frac{1}{2}$.

Thus:

$$\theta = E(\theta) + \sum_{c \in \text{Root}(D_3) \cup \text{Root}(D_4)} \varepsilon_c(\text{partial fraction of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} \frac{(D_2 D_3 D_4)^\nu}{D_2 D_3 D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}$$

■
Proposition 14: We assume: $d^r r < 0$ and
\[
\begin{align*}
& d^o D_3 \neq 0 \quad \text{or} \quad d^o D_4 \neq 0 \\
& d^o D_3 \geq 2 \\
& d^o D_4 = 0
\end{align*}
\]
We have:
\[
\theta = \sum_{c \in \text{Root}(D_3) \cup \text{Root}(D_4)} \varepsilon_c (\text{partial fraction of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} \left(D_2 D_3 D_4\right)' + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}
\]
(42)
where $D_0$ polynomial of degree:
\[
\left[ \sum_{c \in \text{Roots}(D_3) \cup \text{Roots}(D_4)} \varepsilon_c \varepsilon_r \right]
\]
Proof. We have:
\[
\lim_{x \to \infty} x \theta = \sum_{c \in \text{Roots}(D_3) \cup \text{Roots}(D_4)} \varepsilon_c (\text{residue of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} d^o(D_2 D_3 D_4) + d^o D_1 + d^o D_0
\]
and \(\lim_{x \to \infty} x^2 r + \frac{1}{4} = (\lim_{x \to \infty} x \theta - \frac{1}{2})^2\). Thus:
\[
\left[ \sum_{c \in \text{Roots}(D_3) \cup \text{Zeros}(D_4)} \varepsilon_c \varepsilon_r \right]
\]
Proposition 15: We assume $d^r r = 0$. We have:
\[
\theta = E(\theta) + \sum_{c \in \text{Root}(D_3) \cup \text{Root}(D_4)} \varepsilon_c (\text{partial fraction of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} \left(D_2 D_3 D_4\right)' + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}
\]
(44)
where $E^2(\theta) = E(r)$ and $D_0$ polynomial of degree:
\[
d^o D_0 = \frac{1}{2E(\theta)} \lim_{x \to \infty} x[r-E(r)] - \left[ \sum_{c \in R(D_3) \cup R(D_4)} \varepsilon_c (\text{residue of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} d^o(D_2 D_3 D_4) + d^o D_1 \right]
\]
(45)
Proof.
\[
E^2(\theta) = E(r) \quad \text{and} \quad 2E(\theta) \sum_c \text{residue of } \theta \text{ at } c = \sum_c \text{residue of } r \text{ at } c
\]
Thus:
\[
2E(\theta) \left[ \sum_{c \in R(D_3) \cup R(D_4)} \varepsilon_c (\text{residue of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} d^o(D_2 D_3 D_4) + d^o D_1 + d^o D_0 \right] = \lim_{x \to \infty} x(r-E(r))
\]
\]

Proposition 16: We assume \( d^o r = 2\nu > 0 \). We have:

\[
\theta = E(\theta) + \sum_{c \in \text{Root}(D_3) \cup \text{Root}(D_4)} \varepsilon_c (\text{partial fraction of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} \left( \frac{D_2 D_3 D_4}{D_2 D_3 D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \right)
\]  

(46)

where \( D_0 \) polynomial of degree:

\[
d^o D_0 = \alpha s\nu + 1 - \nu^2 - \left[ \sum_{c \in \text{Roots}(D_3) \cup \text{Roots}(D_4)} \varepsilon_c \left( \text{residue of } (F)^{\frac{1}{2}} \text{ at } c \right) + \frac{1}{2} d^o(D_2 D_3 D_4) + d^o(D_1) + d^o(D_0) \right]
\]  

(47)

Proof. Let a dominant coefficient of \( E(r) \), and we have:

\[
\alpha s\nu - \frac{\nu}{2} = \text{sum of residues of } \theta.
\]

Thus:

\[
\alpha s\nu + 1 - \frac{\nu}{2} = \left[ \sum_{c \in R(D_3) \cup R(D_4)} \varepsilon_c \left( \text{residue of } (F)^{\frac{1}{2}} \text{ at } c \right) + \frac{1}{2} d^o(D_2 D_3 D_4) + d^o(D_1) + d^o(D_0) \right]
\]

Thus:

\[
d^o D_0 = \alpha s\nu + 1 - \frac{\nu}{2} - \left[ \sum_{c \in R(D_3) \cup R(D_4)} \varepsilon_c \left( \text{residue of } (F)^{\frac{1}{2}} \text{ at } c \right) + \frac{1}{2} d^o(D_2 D_3 D_4) + d^o(D_1) \right]
\]

\[
\boxed{\theta = E(\theta) + \sum_{c \in \text{Zero}(D_3) \cup \text{Zero}(D_4)} \varepsilon_c (\text{partial fraction of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} d^o(D_2 D_3 D_4) + d^o(D_1) + d^o(D_0) \]

(48)

3.4 Case \( d^o D_3 = 0 \) and \( d^o D_4 \neq 0 \)

\( D_1 D_2^2 \) and \( D_4 \) polynomials relatively prime. Thus there are two only polynomials \( N_1 \) and \( N_2 \) such as:

\[
\left\{ \begin{array}{l}
D_1 D_2^2 D_4^2 (r - E(r)) = N_1(D_1 D_2^2) + N_2 D_4 \\
\text{d}^o N_1 < \text{d}^o D_4
\end{array} \right.
\]

We have:

\[
r = E(r) + \frac{N_1}{D_1^2} + \frac{N_2}{D_1 D_2^2 D_4}
\]  

(49)

\[
\lim_{x \to \infty} x^2 \frac{N_1}{D_4^2} = 0
\]

(50)

We consider the rational fraction:

\[
F = E(r) + \frac{N_1}{D_4^2} - \frac{1}{4 d^o D_4} \left( \frac{D_4'}{D_4} \right)'
\]

Proposition 17: We assume \( \left\{ \begin{array}{l}
d^o D_3 = 0 \\
\text{d}^o D_4 \neq 0
\end{array} \right. \)

1. If \( d^o r \geq 0 \) then \( \text{d}^o F = \text{d}^o r \) and \( E(F) = E(r) \)
2. If : \(d^\nu r < 0\) then : \(d^\nu F = -2\) where \(\lim_{x \to \infty} x^2 F = \frac{1}{4}\).

3. For all \(c\) root of \(D_4\) of multiplicity \(\nu\) we have:

\[(x - c)^{2\nu} r = (x - c)^{2\nu} F + o((x - c)^{\nu-1})\]

**Proof.**

1. \(r - F = \frac{N_2}{D_1D_2D_4} + \frac{1}{4d^\nu D_4}(\frac{D_4'}{D_4})'\) is from negative degree.

2. 

\[
\lim_{x \to \infty} x^2 F = \lim_{x \to \infty} \frac{N_1}{D_2^2} - \lim_{x \to \infty} x^2 \frac{1}{4d^\nu D_4}(\frac{D_4'}{D_4})' \\
= - \lim_{x \to \infty} x^2 \frac{1}{4d^\nu D_4}(\frac{D_4'}{D_4})' = \frac{1}{4}
\]

3. Let \(c\) root of \(D_4\):

\[(x - c)^{2\nu}(r - F) = \frac{N_2}{D_1D_2^2} \frac{(x-c)^{2\nu}}{D_4} + (x - c)^{2\nu} \frac{1}{4d^\nu D_4}(\frac{D_4'}{D_4})' \]

\[= o((x - c)^{\nu-1})\]

**Notation:** We choose a square root of the polynomial of even degree : \(N_1 + D_4^2[E(r) - \frac{1}{4d^\nu D_4}(\frac{D_4'}{D_4})']\) on a connected open at infinity, root of \(D_4\). We put :

\[(F)^\frac{1}{2} = \frac{1}{D_4}(N_1 + D_4^2[E(r) - \frac{1}{4d^\nu D_4}(\frac{D_4'}{D_4})'])^\frac{1}{2} \quad (51)\]

**Proposition 18:** We assume : \(d^\nu r < 0\). We have :

\[
\theta = \sum_{c \in \text{Roots}(D_4)} \epsilon_c (\text{partial fraction of } (F)^\frac{1}{2} \text{ at } c) + \frac{1}{2} \frac{(D_2D_4)'}{D_2D_4} + \frac{D_4'}{D_1} + \frac{D_0'}{D_0} \quad (52)
\]

where \(\epsilon_c = \pm 1\) and \(D_0\) polynomial of degree :

\[
[\sum_{c \in \text{Root}(D_4)} \epsilon_c \text{ (residue of } ((F)^\frac{1}{2} \text{ at } c) + \frac{1}{2} d^\nu(D_2D_4) + d^\nu D_1 + d^\nu D_0 - \frac{1}{2})^2 = \lim_{x \to \infty} x^2 r + \frac{1}{4} \quad (53)
\]

**Proof.**

\[
\lim_{x \to \infty} x \theta = \sum_{c \in \text{Roots}(D_4)} \epsilon_c (\text{residue of } ((F)^\frac{1}{2} \text{ at } c) + \frac{1}{2} d^\nu(D_2D_4) + d^\nu D_1 + d^\nu D_0
\]

\[
\lim_{x \to \infty} x^2 r + \frac{1}{4} = (\lim_{x \to \infty} x \theta - \frac{1}{2})^2. \text{ Thus :}
\]

\[
[\sum_{c \in \text{Root}(D_4)} \epsilon_c \text{ residue of } (F)^\frac{1}{2} \text{ at } c) + \frac{1}{2} d^\nu(D_2D_4) + d^\nu D_1 + d^\nu D_0 - \frac{1}{2})^2 = \lim_{x \to \infty} x^2 r + \frac{1}{4}
\]
Proposition 19: We assume \( d^\nu r = 0 \). We have:

\[
\theta = E(\theta) + \sum_{c \in \text{Roots}(D_4)} \varepsilon_c \text{(partial fraction of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} \frac{(D_2 D_4)^{\prime}}{D_2 D_4} + \frac{D_1^\prime}{D_1} + \frac{D_0^\prime}{D_0} \tag{54}
\]

where \( \varepsilon_c = \pm 1 \) and \( D_0 \) polynomial of degree :

\[
d^\nu D_0 = \frac{1}{2E(\theta)} \lim_{x \to \infty} x[r - E(r)] - \left[ \sum_{c \in \text{Roots}(D_4)} \varepsilon_c \text{ (residue of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} d^\nu(D_2 D_4) + d^\nu D_1 \right] \tag{55}
\]

Proof.

\[
E^2(\theta) = E(r) \text{ and } 2E(\theta) \sum_c \text{ residue of } \theta \text{ at } c = \sum_c \text{ residue of } r \text{ at } c
\]

Thus:

\[
2E(\theta) \left[ \sum_{c \in \text{Roots}(D_4)} \varepsilon_c \text{ (residue of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} d^\nu(D_2 D_4) + d^\nu D_1 + d^\nu D_0 \right] = \lim_{x \to \infty} x(r - E(r))
\]

Proposition 20 We assume \( d^\nu r = 2\nu > 0 \). We have:

\[
\theta = E(\theta) + \sum_{c \in \text{Roots}(D_4)} \varepsilon_c \text{(partial fraction of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} \frac{(D_2 D_4)^{\prime}}{D_2 D_4} + \frac{D_1^\prime}{D_1} + \frac{D_0^\prime}{D_0} \tag{56}
\]

where \( \varepsilon_c = \pm 1 \) and \( D_0 \) polynomial of degree :

\[
d^\nu D_0 = \alpha s_{\nu+1} - \frac{\nu}{2} - \left[ \sum_{c \in \text{Roots}(D_4)} \varepsilon_c \text{ (residue of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} d^\nu(D_2 D_4) + d^\nu D_1 \right] \tag{57}
\]

Proof. Let a dominant coefficient of \( E(r) \). \( E(\theta) = \alpha[t^{-\nu} + s_1 t^{-(\nu-1)} + \ldots + s_{\nu}] \). \( \alpha s_{\nu+1} - \frac{\nu}{2} = \text{sum residue of } \theta \), where \( \alpha^2 = a \). Thus:

\[
\theta = E(\theta) + \sum_{c \in \text{Roots}(D_4)} \varepsilon_c \text{(partial fraction of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} \frac{(D_2 D_4)^{\prime}}{D_2 D_4} + \frac{D_1^\prime}{D_1} + \frac{D_0^\prime}{D_0}
\]

Thus:

\[
\alpha s_{\nu+1} - \frac{\nu}{2} = \left[ \sum_{c \in \text{Roots}(D_4)} \varepsilon_c \text{ (residue of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} d^\nu(D_2 D_4) + d^\nu D_1 + d^\nu D_0 \right]
\]

Thus:

\[
d^\nu D_0 = \alpha s_{\nu+1} - \frac{\nu}{2} - \left[ \sum_{c \in \text{Roots}(D_4)} \varepsilon_c \text{ (residue of } (F)^{\frac{1}{2}} \text{ at } c) + \frac{1}{2} d^\nu(D_2 D_4) + d^\nu D_1 \right]
\]
4 Recurrent method at infinity

4.1 Presentation of $D_0$ as determinant:

Proposition 21 : Let $c_1, \ldots, c_m$ complexes constants. We put:

$$P(x) = (x-c_1) \ldots (x-c_m) = x^m - p_1x^{m-1} + p_2x^{m-2} - \ldots + (-1)^mp_m$$

For $j \in \mathbb{N}$, we put: $\sigma_j = c_1^j + \ldots + c_m^j$ and $\Delta = \left| \begin{array}{cccc} 1 & c_1 & c_1^{m-1} \\ 1 & c_2 & c_2^{n-1} \\ \vdots & \vdots & \vdots \\ 1 & c_m & c_m^{n-1} \end{array} \right|$

1. 

$$\Delta^2 = \left| \begin{array}{cccc} \sigma_0 & \sigma_1 & \ldots & \sigma_{m-1} \\ \sigma_1 & \sigma_2 & \ldots & \sigma_m \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m-1} & \sigma_m & \ldots & \sigma_{2m-2} \end{array} \right|$$

2. 

$$\Delta^2 P(x) = \left| \begin{array}{cccc} \sigma_0 & \ldots & \sigma_{m-1} & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_m & \ldots & \sigma_{2m-1} & x^m \end{array} \right|$$

Proof. $\Delta$, polynomials at $c_1, \ldots, c_m$ with real coefficients. Thus, $\Delta^2 P(x)$ and polynomials at $c_1, \ldots, c_m, x$ with real coefficients. Thus to have both identities we can assume $c_1, \ldots, c_m, x$ reals. Furthermore, we can content themselves with the open of Zariski $c_1, \ldots, c_m, x$ distinct real non-zero.

We have: $\forall j = 1, \ldots, m, \quad c_j^m = p_1c_j^{m-1} - p_2c_j^{m-2} - \ldots - (-1)^mp_m$. 

We put:

\[
\Delta(x) = \begin{vmatrix}
1 & c_1 & \ldots & c_1^{m-1} & c_1^m \\
1 & c_2 & \ldots & c_2^{m-1} & c_2^m \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
1 & c_m & \ldots & c_m^{m-1} & c_m^m \\
1 & x & \ldots & x^{m-1} & x^m
\end{vmatrix} = \Delta P(x)
\]

We put:

\[
v_j = \begin{pmatrix}
c_0^{j-1} \\
\vdots \\
c_m^{j-1}
\end{pmatrix}, \quad \text{pour } j = 1, \ldots, m
\]

The scalar product:

\[
<v_j, v_k> = c_1^{j+k-2} + \ldots + c_m^{j+k-2} = \sigma_{j+k-2}
\]

Matrix of Gram of \(v_1, \ldots, v_m\) is:

\[
G = \begin{pmatrix}
\sigma_0 & \sigma_1 & \ldots & \sigma_{m-1} \\
\sigma_1 & \sigma_2 & \ldots & \sigma_m \\
\vdots & \ddots & \ddots & \ddots \\
\sigma_{m-1} & \sigma_m & \ldots & \sigma_{2m-2}
\end{pmatrix}
\]

Thus:

\[
\Delta^2 = \begin{vmatrix}
\sigma_0 & \sigma_1 & \ldots & \sigma_{m-1} \\
\sigma_1 & \sigma_2 & \ldots & \sigma_m \\
\vdots & \ddots & \ddots & \ddots \\
\sigma_{m-1} & \sigma_m & \ldots & \sigma_{2m-2}
\end{vmatrix}
\]

We put: 

\[
v_j(x) = \begin{pmatrix}
c_1^{j-1} \\
\vdots \\
c_m^{j-1} \\
x^{j-1}
\end{pmatrix}, \quad j = 1, \ldots, m + 1. \quad \text{We notice:}

\[
<v_j(x), v_k(x)> = <v_j, v_k> + x^{j+k-2} = \sigma_{j+k-2} + x^{j+k-2}
\]
The j-th column of matrix of Gram of matrix defines $\Delta(x)$ is:

$$
\begin{pmatrix}
\sigma_{j-1} \\
\sigma_j \\
. \\
. \\
\sigma_{j+m-1}
\end{pmatrix} + x^{j-1}
\begin{pmatrix}
1 \\
x \\
. \\
. \\
x^m
\end{pmatrix}
$$

Thus:

$$
\begin{vmatrix}
\sigma_0 & \sigma_1 & \ldots & \sigma_m \\
\sigma_1 & \sigma_2 & \ldots & \sigma_{m+1} \\
. & . & . & . \\
. & . & . & . \\
\sigma_m & \sigma_{m+1} & \ldots & \sigma_{2m}
\end{vmatrix} + x^m
\begin{vmatrix}
\sigma_0 \\
\sigma_1 \\
. \\
. \\
\sigma_{2m}
\end{vmatrix} + \ldots +
\begin{vmatrix}
\sigma_0 & \sigma_{m-1} \\
1 & \sigma_m \\
. & . \\
. & . \\
\sigma_m & \sigma_{2m-1}
\end{vmatrix} \cdot x^{2m}
$$

Thus:

$$
\Delta^2(x) = \sum_{i,j} x^{i+j-2} \text{cofactor}_{i,j}(M)
$$

where

$$M = 
\begin{pmatrix}
\sigma_0 & \sigma_1 & \ldots & \sigma_m \\
\sigma_1 & \sigma_2 & \ldots & \sigma_{m+1} \\
. & . & . & . \\
. & . & . & . \\
\sigma_m & \sigma_{m+1} & \ldots & \sigma_{2m}
\end{pmatrix}
$$

We obtain: $\Delta^2(x) = \Delta^2 P^2(x) = (1 \ldots x^m) \text{Com}(M) \cdot x^m$

The cofactor $(m+1, m+1)$ of $M$ is $\Delta^2$. The adjoint of $M$ is non-zero. We prove that adjoint of $M$ of rank 1.

$\text{det}(M) = 0$. Because: $\sigma_m = p_1 \sigma_{m-1} - p_2 \sigma_{m-2} - \ldots - (-1)^m p_m \sigma_0$

and $\forall k \geq m$: $\sigma_k = p_1 \sigma_{k-1} - p_2 \sigma_{k-2} - \ldots - (-1)^m p_m \sigma_{k-m}$

Thus, the $(m+1)$-th column of $M$ is a linear combination of other column.

We consider the matrix: $E_{1,1} = 
\begin{pmatrix}
1 & 0 & 0 \\
0 & . & . \\
0 & . & 0
\end{pmatrix}$

we obtain:

$$M + E_{1,1} = 
\begin{pmatrix}
\sigma_0 + 1 & \sigma_1 & \ldots & \sigma_m \\
\sigma_1 & \sigma_2 & \ldots & \sigma_{m+1} \\
. & . & . & . \\
. & . & . & . \\
\sigma_m & \sigma_{m+1} & \ldots & \sigma_{2m}
\end{pmatrix}$$

and
\[
\det(M + E_{1,1}) = \det(M) + \begin{vmatrix}
 1 & \sigma_1 & \ldots & \sigma_m \\
 0 & \sigma_2 & \ldots & \sigma_{m+1} \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & \sigma_{m+1} & \ldots & \sigma_{2m} \\
\end{vmatrix} = \begin{vmatrix}
 \sigma_2 & \sigma_3 & \ldots & \sigma_{m+1} \\
 \sigma_3 & \sigma_4 & \ldots & \sigma_{m+2} \\
 \vdots & \vdots & \ddots & \vdots \\
 \sigma_{m+1} & \sigma_{m+2} & \ldots & \sigma_{2m} \\
\end{vmatrix}
\]

It is the determinant of Gram of
\[
\begin{pmatrix}
  c_1 \\
  \vdots \\
  c_m \\
\end{pmatrix}, \ldots, \begin{pmatrix}
  c_1^m \\
  \vdots \\
  c_m^m \\
\end{pmatrix}
\]

Thus:
\[
\det(M + E_{1,1}) = \begin{vmatrix}
  c_1 & \ldots & c_1^m \\
  \vdots & \ddots & \vdots \\
  c_m & \ldots & c_m^m \\
\end{vmatrix}^2
\]

But:
\[
\begin{vmatrix}
  c_1 & \ldots & c_1^m \\
  \vdots & \ddots & \vdots \\
  c_m & \ldots & c_m^m \\
\end{vmatrix} = (-1)^{m-1}p_m \begin{vmatrix}
  c_1 & \ldots & 1 \\
  \vdots & \ddots & \vdots \\
  c_m & \ldots & 1 \\
\end{vmatrix} = p_m \Delta
\]

Thus : \( \det(M + E_{1,1}) = p_m^2 \Delta^2. \) Or \( p_m = c_1 \ldots c_m \) is scalar non-zero, \( M + E_{1,1} \) is invertible matrix. \( M \) symmetric matrix, \( \text{Com}(M) \) symmetric matrix, we have :
\[
M \text{Com}(M) = \text{Com}(M) M = \det(M) I = 0; \ I \text{ matrice identité}
\]

Thus:
\[
(M + E_{1,1}) \text{Com}(M) = E_{1,1} \text{Com}(M)
\]

Or \( M + E_{1,1} \) is invertible and \( \text{Com}(M) \neq 0, \text{Com}(M) \) of rank 1.
Thus:
\[
cofactor_{i,j}(M) = \frac{ \text{cofactor}_{i,m+1}(M) }{ \text{cofactor}_{m+1,m+1}(M) } \cdot \text{cofactor}_{m+1,j}(M)
\]

For \( i = 1, \ldots, m + 1, j = 1, \ldots, m + 1, \) we put : \( C_{i,j} = \text{cofactor}_{i,j}(M). \)
\[
\frac{C_{i,j}}{\Delta^2} = \frac{C_{i,m+1}}{\Delta^2} \frac{C_{m+1,j}}{\Delta^2}
\]

Thus:
\[
P(x)^2 = \sum_{1 \leq i,j \leq m+1} \frac{C_{i,j}}{\Delta^2} x^{i+j-2} = \sum_{1 \leq i,j \leq m+1} \frac{C_{i,m+1}}{\Delta^2} \frac{C_{j,m+1}}{\Delta^2} x^{i+j-2} = \left( \sum_{1 \leq i \leq m+1} \frac{C_{i,m+1}}{\Delta^2} x^{i-1} \right)^2
\]
Or $P$ is monic polynomial and: \[ \frac{C_{m+1,m+1}}{\Delta^2} = 1 \]

\[
P(x) = \sum_i^m \frac{C_{i,m+1}}{\Delta^2} x^{i-1} = \frac{1}{\Delta^2} \begin{vmatrix}
\sigma_0 & \ldots & \sigma_{m-1} & 1 \\
\sigma_1 & \ldots & \sigma_m & x \\
\vdots & \ddots & \vdots & \vdots \\
\sigma_m & \ldots & \sigma_{2m-1} & x^m
\end{vmatrix}
\]

We obtain:

\[
\Delta^2 P(x) = \begin{vmatrix}
\sigma_0 & \ldots & \sigma_{m-1} & 1 \\
\sigma_1 & \ldots & \sigma_m & x \\
\vdots & \ddots & \vdots & \vdots \\
\sigma_m & \ldots & \sigma_{2m-1} & x^m
\end{vmatrix}
\]

### 4.2 Taylor expansion of $\frac{P'}{P}$

**Proposition 22**: Let $c_1, \ldots, c_m$ are pairwise distinct constant. We put, for $j = 0, \ldots, m$: $\sigma_j = c_1^j + \ldots + c_m^j$ and $P(x) = (x - c_1) \ldots (x - c_m)$

1. \[
\left( \frac{P'}{P} \right)' + \left( \frac{P'}{P} \right)^2 = 2 \sum_{i \neq j} \frac{1}{c_i - c_j} \frac{1}{x - c_i}
\]

2. We put: $t = \frac{1}{x}$. Taylor expansion on the neighborhood of infinity:

   (a) \[
   \left( \frac{P'}{P} \right)' + \left( \frac{P'}{P} \right)^2 = \sum_{l=2}^{\infty} \left[ \sum_{\mu=0}^{l-2} \sigma_\mu \sigma_{l-2-\mu} - (l - 1) \sigma_{l-2} \right] t^l
   \]

   (b) \[
   \frac{x P'}{P} = \sum_{l=0}^{\infty} \sigma_l t^l
   \]

**Proof.** We have: \[
\frac{P'}{P} = \sum_{i=1}^m \frac{1}{x - c_i}
\]

1. \[
\left( \frac{P'}{P} \right)' + \left( \frac{P'}{P} \right)^2 = \sum_{i \neq j} \frac{1}{x - c_i} \frac{1}{x - c_j}
\]

\[
= \sum_{i \neq j} \left[ \frac{1}{x - c_i} + \frac{1}{x - c_j} \right]
\]

\[
= 2 \sum_{i \neq j} \frac{1}{c_i - c_j} \frac{1}{x - c_i}
\]
2. We put: \( t = \frac{1}{x} \). We have:

\[
(P'P)'' + (P'P)^2 = 2t \sum_{i \neq j} \frac{1}{c_i - c_j} \frac{1}{1 - c_it}
\]

\[
= 2t \sum_{i \neq j} \frac{1}{c_i - c_j} \sum_{k=0}^{\infty} (c_it)^k
\]

\[
= \sum_{k=0}^{\infty} (2 \sum_{i \neq j} \frac{c_i^k}{c_i - c_j}) t^{k+1}
\]

\[
= \sum_{k=1}^{\infty} \sum_{i \neq j} \frac{c_i^k - c_j^k}{c_i - c_j} t^{k+1}
\]

As:

\[
\sum_{i \neq j} \frac{c_i^k - c_j^k}{c_i - c_j} = \sum_{i \neq j} \sum_{\mu + \nu = k-1, 0 \leq \mu \leq k-1} c_i^\mu c_j^\nu
\]

\[
= \sum_{\mu + \nu = k-1, 0 \leq \mu \leq k-1} [\sigma_\mu \sigma_\nu - \sum_{i=1}^{m} \sigma_i^{\mu + \nu}]
\]

\[
= \sum_{\mu + \nu = k-1, 0 \leq \mu \leq k-1} \sigma_\mu \sigma_\nu - k\sigma_{k-1}
\]

Thus:

\[
(P'P)'' + (P'P)^2 = \sum_{k=1}^{\infty} \sum_{\mu + \nu = k-1, 0 \leq \mu \leq k-1} (\sigma_\mu \sigma_{k-1-\mu} - k\sigma_{k-1}) t^l
\]

\[
= \sum_{l=2}^{\infty} \sum_{\mu = 0}^{l-2} (\sigma_\mu \sigma_{l-2-\mu} - (l-1)\sigma_{l-2}) t^l
\]

4.3 Research of \( D_0 \)

Let \( r = \frac{N}{D_1 D_2 D_3 D_4} \) rational fraction and \( \theta \in \mathbb{C}(x) \) where: \( \theta' + \theta^2 = r \).

We put:

\[
\theta = S + \frac{D_0'}{D_0}
\]

where \( S = E(\theta) + \sum_{c \text{ poles of } r} \theta_c \) and \( D_0 \) monic polynomial of degree \( m \). We have:

\[
\theta' + \theta^2 = S' + S^2 + \left( \frac{D_0'}{D_0} \right)' + \left( \frac{D_0'}{D_0} \right)^2 + 2S \frac{D_0'}{D_0}
\]
\[ \theta' + \theta^2 = r \Leftrightarrow (\frac{D_0'}{D_0})' + (\frac{D_0'}{D_0})^2 + 2S \frac{D_0'}{D_0} = \frac{R}{B} \] \hspace{1cm} (66)

with \( B = D_1D_2D_3D_4 \) and \( R \) polynomials as: \( r - S' - S^2 = \frac{R}{B} \).

**Case 1:** \( d^\nu r = 2\nu \geq 0 \)

We have: \( d^\mu S = \nu \) and \( d^\nu \frac{R}{B} = \nu - 1 \). Taylors expansion of \( \frac{1}{x^\nu} S, \frac{1}{x^{\nu-1}} \frac{R}{B} \) and \( x \frac{D_0'}{D_0} \) at order \( \mu \).

We obtain the Taylors expansion of \( \frac{1}{x^\nu} [\frac{R}{B} - 2S \frac{D_0'}{D_0}] \) at order \( \mu \).

The Taylors expansion equal the Taylors expansion of: \( \frac{1}{x^\nu} [(\frac{D_0'}{D_0})' + (\frac{D_0'}{D_0})^2] \)

Let \( t = \frac{1}{x} \). In Taylors expansion of \( \frac{1}{x^\nu} [(\frac{D_0'}{D_0})' + (\frac{D_0'}{D_0})^2] \) [see proposition 23] the coefficient of \( t^{\nu+1} \) is: \( \sigma_0^2 - \sigma_0 = m^2 - m \). For \( \nu \geq 2 \), coefficient of \( t^{\nu+1} \) use: \( \sigma_0, \ldots, \sigma_{1-2} \).

Besides, we have:

\[
x \frac{D_0'}{D_0} = \sum_{i=1}^{m} \frac{1}{1 - c_i t} = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} c_i^k t^k = \sum_{k=0}^{\infty} \sigma_k t^k
\]

Thus, Taylors expansion of:

\[
\Phi = \frac{1}{x^{\nu-1}} [(\frac{D_0'}{D_0})' + (\frac{D_0'}{D_0})^2] - \frac{1}{x^{\nu-1}} [\frac{R}{B} - 2S \frac{D_0'}{D_0}] \]

and the coefficient of \( t^k \) is:

\[-2\alpha \sigma_k + \text{polynomial at } \sigma_0, \ldots, \sigma_{k-1} \]

\[ \Phi = 0 \] determine \( \sigma_k \) by recurrence at \( k = \nu \).

**Case 2:** \( d^\nu r < 0 \)

The coefficient of \( t^k \) in Taylors expansion of:

\[
\Psi = x^2 [(\frac{D_0'}{D_0})' + (\frac{D_0'}{D_0})^2] - x^2 \frac{R}{B} + 2xS(x \frac{D_0'}{D_0})
\]

is:

\[ 2m \sigma_k - (k + 1) \sigma_k + (2 \lim_{x \to \infty} xS) \sigma_k + \text{polynomial at } \sigma_0, \ldots, \sigma_{k-1} \]

By recurrence \( \sigma_k \) at condition:

\[ k \neq 2 \lim_{x \to \infty} xS + 2m - 1 \]

We have: \( \lim_{x \to \infty} xS + m = \alpha_1^2 + \frac{1}{2} \) with \( \alpha_1^2 = \lim_{x \to \infty} x^2 r + \frac{1}{4} \).

For \( k \neq 2\alpha_1 \), to be \( \sigma_k \) at function of \( \sigma_0, \ldots, \sigma_{k-1} \).

If \( 2\alpha_1 \in \mathbb{N} \) then coefficient of \( t^{2\alpha_1} \) non-zero thus is no solution or is zero thus \( \sigma_{2\alpha_1} \) is arbitrary and \( \sigma_k \); \( k > 2\alpha_1 \) depend in a unique way.

\( D_0 \) is polynomial determined by \( \sigma_0, \ldots, \sigma_{2m-1} \) by determinant formulæ, the problem of non-unique suite \( (\sigma_k)_{k \in \mathbb{N}} \) put if \( 2\alpha_1 \) is positive integer equal to or less than \( 2m - 1 \), then \( 4 \lim_{x \to \infty} x^2 r + 1 \) is square of integer and:

\[ \lim_{x \to \infty} x^2 r \leq m^2 - m \]

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Example 23 In this example we consider the Riccati differential equation (2) where:

\[ r = -1 + \frac{z^2 - \frac{1}{4}}{x^2}; \quad z \in \mathbb{R}^* \]  

(73)

We have \( D_1 = 1, \ D_2 = 1, \ D_3 = x \) and \( E^2(\theta) = -1 \).
We can assume \( E(\theta) = i \).

\[ \lim_{x \to 0} x^2 r + \frac{1}{4} = z^2 = \alpha_0^2 \]

\[ \theta = i + \frac{\alpha_0 + \frac{1}{2}}{x} + D_0' \]

\[ 2i(\alpha_0 + \frac{1}{2} + d''D_0) = 0. \] Thus : \( m = d''D_0 = -\alpha_0 - \frac{1}{2} \in \mathbb{N} \)

Thus : \( \alpha_0 = -m - \frac{1}{2}, \ r = -1 + \frac{m + \frac{1}{2} - \frac{1}{2}}{x^2} = -1 + \frac{m^2 + m}{x^2} \) and \( \theta = i - \frac{m}{x} + D_0' \).

\( \theta \) solution of Riccati equation if and only if \( D_0 \) verify :

\[ \left( \frac{D_0'}{D_0} \right)' + \left( \frac{D_0'}{D_0} \right)^2 + 2\left( i - \frac{m}{x} \right) \left( \frac{D_0'}{D_0} \right) = 2i \frac{m}{x} \]

Thus \( \Phi = 0 \) where

\[ \Phi = x \left[ \left( \frac{D_0'}{D_0} \right)' + \left( \frac{D_0'}{D_0} \right)^2 \right] + 2\left( i - \frac{m}{x} \right) \left( x \frac{D_0'}{D_0} \right) - 2im \]

Taylor's expansion of \( \Phi = 0 \) at \( t = \frac{1}{x} \):

\[ \Phi = \sum_{k=1}^{\infty} \left( \sum_{\mu=1}^{k-1} \sigma_\mu \sigma_{k-1} - k \sigma_{k-1} - 2m \sigma_{k-1} + 2i \sigma_k \right) t^k \]

As \( \sigma_0 = m \). Thus:

\[ \Phi = 0 \iff \forall k \geq 1, \ \left( \sum_{\mu=0}^{k-1} \sigma_\mu \sigma_{k-1} - k \sigma_{k-1} - 2m \sigma_{k-1} + 2i \sigma_k \right) = 0 \]

For \( k = 1, \ \sigma_0^2 - \sigma_0 - 2m \sigma_0 + 2i \sigma_1 = 0 \) equivalent to \( 2i \sigma_1 = m^2 + m \).

For all : \( k \geq 2, \) \( 2i \sigma_k = k \sigma_{k-1} - \sum_{\mu=1}^{k-2} \sigma_\mu \sigma_{k-1} - 2i \sigma_{k-1} \)

For example \( m = 3, \) :

\[ \begin{align*}
2i \sigma_3 &= 3 \sigma_2 - 2 \sigma_1 \\
2i \sigma_4 &= 4 \sigma_3 - 2 \sigma_1 \sigma_2 \\
2i \sigma_5 &= 5 \sigma_4 - 2 \sigma_1 \sigma_3 - \sigma_2^2 
\end{align*} \]

Thus, the polynomial \( D_0 \) partner to :

\[
\begin{pmatrix}
\sigma_0 & \sigma_1 & \sigma_2 & 1 \\
\sigma_1 & \sigma_2 & \sigma_3 & x \\
\sigma_2 & \sigma_3 & \sigma_4 & x^2 \\
\sigma_3 & \sigma_4 & \sigma_5 & x^3 \\
\end{pmatrix} = \begin{pmatrix}
3 & -6i & -6 & 1 \\
-6i & -6 & -9i & x \\
-6 & -9i & -54 & x^2 \\
-9i & -54 & 99i & x^3 \\
\end{pmatrix} = 135x^3 + 810ix^2 - 2025x - 2025i
\]

Thus

\[ D_0 = x^3 + 6ix^2 - 15x - 15i \]
5 Method of last minor

Let \( r = \frac{N}{D_1D_2D_3D_4} \) rational fraction and \( \theta \in \mathbb{C}(x) \) as : \( \theta' + \theta^2 = r \).

\( \theta \) solution of Eq (2) if and only if \( D_0 \) verify :

\[
BD_0'' + 2SBD_0' = RD_0
\]

(74)

We choose a complex number \( c \) not pole of \( r \) and we use the expression of polynomials following the powers of \( x - c \).

For the sake of simplicity, in the following we assume that \( c = 0 \). Constant coefficient constant of \( B \) is non-zero.

Denote by \( a_k, b_k \) and \( r_k \) coefficients of \( x^k \), in \( A, B \) and \( R \) respectively, \( k \in \mathbb{N} \).

If \( k \in \mathbb{Z} \) then :

\[
a_k = b_k = r_k = 0.
\]

5.1 Case \( d^o r < 0 \)

We put : \( S = \frac{A}{B} \)

If \( d^o B = 1 \) then \( r \) are pole unique, it is simple or double. If \( c \) simple pole of \( r \) then \( r \) of degree \( -1 \) and Eq (2) has no rational solution.

If \( c \) double pole of \( r \) then :

\[
D_0'' = -\frac{2(\alpha_c + \frac{1}{2})}{x - c}
\]

(75)

The solutions :

\[
D_0' = (-2\alpha_c)(x - c)^{-2\alpha_c - 1}
\]

(76)

\[
D_0 = (x - c)^{-2\alpha_c} + \beta
\]

(77)

where \( \beta \) non-zero constant .

We have, an infinity of the other rational solution when \(-2\alpha_c \in \mathbb{N}^*\), with the same parity of \( S \).

We assume : \( d^o B \geq 2 \) and we look \( D_0 \) polynomial of degree \( m \geq 1 \) verify Ed (72).

Proposition 24 : We assume : \( d^o r < 0 \), \( \begin{cases} 
\frac{d^o B}{d^o} \geq 2 \\
\frac{d^o}{d^o} = m \geq 1 \\
and \ D_0 \ verify \ Ed \ (72).
\end{cases} \)

1. We have:

\[
\begin{cases} 
\frac{d^o(BD_0'' + 2AD_0')}{d^o} \leq \frac{d^o B}{d^o} + m - 1 \\
\frac{d^o R}{d^o} \leq \frac{d^o B}{d^o} - 2
\end{cases}
\]

2. We have: \( r_{d^o - 2} = 2a_{d^o - 1}m + m(m - 1) \)

3. We have:

\[
\begin{cases} 
a_{d^o - 1} = \frac{1}{2} - m + \alpha_{\infty} \\
r_{d^o - 2} = m(2\alpha_{\infty} - m)
\end{cases}
\]

Proof.

1. \( d^o(BD_0'' + 2AD_0') \leq \sup(d^o(BD_0''), d^o(2AD_0')) \)

If \( d^o D_0 = 1 \) then :

\[ \sup(d^o(BD_0''), d^o(2AD_0')) = d^o(2AD_0') = d^o A \leq d^o B - 1 = d^o B + d^o D_0 - 2 \]
If \( d^oD_0 \geq 2 \) then:
\[
\begin{align*}
\frac{d^o(BD_0^\prime)}{d^o(2AD_0^\prime)} &= d^oB + m - 2 \\
\frac{d^o(2AD_0^\prime)}{d^oA + m - 1} &\leq d^oB + m - 2
\end{align*}
\]

2. We have: \( d^o(RD_0) \leq d^oB + m - 2 \) and coefficients of \( x^{d^oB+m-2} \) in Eq (72).

3. Accordingly, in view of: \( \lim_{x \to \infty} x^\theta = \alpha_\infty + \frac{1}{2} \).

We put:
\[
v(x) = -R \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^m \end{pmatrix} + 2A \begin{pmatrix} 0 & 1 \\ 2x & \ddots \\ & \ddots & \ddots \\ & & \ddots & 2x & \\ & & & mx^{m-1} & \end{pmatrix} + B \begin{pmatrix} 0 \\ 0 \\ \vdots \\ m(m-1)x^{m-2} \end{pmatrix}
\]

(78)

The \( k^{th} \) element of \( v(x) \) is:
\[
v_k(x) = B(k-1)(k-2)x^{k-3} + 2A(k-1)x^{k-2} - Rx^{k-1}
\]

(79)

\( v_k(x) \) of degree equal to or less than \( d^oB + k - 3 \) and \( x^{d^oB+k-3} \) of coefficient:
\[
(k-1)(k-2) + 2a_{d^oB-1}(k-1) - r_{d^oB-2} \]
\[
= (k-m-1)(\frac{r_{d^oB-2}}{m} + k-1) \]
\[
= (k-m-1)(2a_{d^oB-1} + m + k - 2)
\]

(80)

For \( k \leq m \), the coefficient of \( x^{d^oB+k-3} \) is zero if and only if:
\[
2a_{d^oB-1} = -(m + k - 2) \in \{-(m-1), -m, \ldots, -2(m-1)\}
\]

\( v_{m+1} \) of degree equal to or less than \( d^oB + m - 3 \).

For \( k \geq 3 \) the coefficient of more low degree of \( v_k \) is coefficient of: \( x^{k-3} \) equal \( b_0(k-1)(k-2) \).

Let \( V \) matrix of \( m+1 \) row and \( k^{th} \) row is row \( l_k \) of coefficients of \( v_k(x) \) in basis of \( \mathbb{C}_{d^oB+m-3}[X] \).

\( l_3, \ldots, l_{m+1} \) linearly independent system.

The Eq (72) give:
\[
D_0 = d_0 + d_1x + \ldots + x^m,
\]

(81)

obtained:
\[
d_0v_1(x) + \ldots + d_{m-1}v_{m-1}(x) + v_{m+1}(x) = 0
\]
\[
d_0l_1 + \ldots + d_{m-1}l_m + l_{m+1} = 0
\]

(82)

(83)
Thus, the matrix $V$ is rank $m - 1$, $m$ or $m + 1$. Accordingly, existence of $D_0$ correspondent linearly dependent in row $l_{m+1}$ of $l_1, \ldots, l_m$.

Let $c_1, \ldots, c_m$ roots of $D_0$ distinct.

We put for $j \in \mathbb{N}$: $\sigma_j = c_1^j + \ldots + c_m^j$ et $\Delta^2 = \begin{vmatrix} \sigma_0 & \sigma_1 & \ldots & \sigma_{m-1} \\ \sigma_1 & \sigma_2 & \ldots & \sigma_m \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m-1} & \sigma_m & \ldots & \sigma_{2m-2} \end{vmatrix}$

The $m$ column vector $\begin{pmatrix} \sigma_0 \\ \vdots \\ \sigma_m \end{pmatrix}$ of $m + 1$ column of are linearly independent.

The equation of hyperplane $H$ engendered by vectors $\begin{pmatrix} \sigma_0 \\ \vdots \\ \sigma_m \end{pmatrix}$ of $m + 1$ column of are linearly independent.

The coefficient of $x_{m+1}$ is $\Delta^2 \neq 0$ and $H$ as equation:

$$x_{m+1} = \lambda_1 x_1 + \ldots + \lambda_m x_m$$  \hspace{1cm} (85)

where $\lambda_1 \ldots \lambda_m$ scalar satisfying :

$$(\sigma_m, \ldots, \sigma_{2m-1}) = \lambda_1 (\sigma_0, \ldots, \sigma_{m-1}) + \ldots + \lambda_m (\sigma_{m-1}, \ldots, \sigma_{2m-2})$$  \hspace{1cm} (86)

The derivative of$\begin{vmatrix} \sigma_0 & \ldots & \sigma_{m-1} & 1 \\ \sigma_1 & \ldots & \sigma_m & x \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_m & \ldots & \sigma_{2m-1} & x^m \end{vmatrix}$ is derived last column. The Eq (72) equivalent to :

$$\begin{vmatrix} \sigma_0 & \ldots & \sigma_{m-1} & 1 \\ \sigma_1 & \ldots & \sigma_m & v_1(x) \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_m & \ldots & \sigma_{2m-1} & v_{m+1}(x) \end{vmatrix} = 0$$  \hspace{1cm} (87)

Thus, $\forall x$, $v(x) \in H$. Application of Taylor, this equivalent to columns of matrix $V$ in hyperplane $H$.

$$l_{m+1} = \lambda_1 l_1 + \ldots + \lambda_m l_m$$  \hspace{1cm} (88)
The polynomial $D_0$ partner to

$$
\begin{array}{c|c}
\sigma_0 & \sigma_{m-1} & 1 \\
\sigma_1 & \sigma_m & x \\
\sigma_2 & \sigma_{2m-1} & x^m \\
\vdots & \vdots & \vdots \\
\sigma_{m-1} & \sigma_{2m-2} & x^{m-1} \\
0 & \ldots & 0 \\
\sigma_1 & \sigma_m & x \\
\sigma_0 & \sigma_{m-1} & 1 \\
\end{array}
$$

Thus :

$$
D_0(x) = x^m - \lambda_1 - \lambda_2 x - \ldots - \lambda_m x^{m-1}
$$

For $k = 1, \ldots, m + 1$, we put:

$$v_k(x) = \rho_k(x) + x^{d^\nu B - 2} w_k
$$

where $d^\nu \rho_k(x) \leq d^\nu B - 3$.

The matrix $W$ partner to: \( \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} \) is triangular where $k^{th}$ entry diagonal equal $(k - 1 - m)(2a_{d^\nu B - 1} + m + k - 2)$. The coefficient is non-zero except possibly for a single value of $k$.

Thus, rank of \( \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} \) is $m$ or $m - 1$.

For $rank(V) = m$, the Eq (72) as solution equivalent to $l_1, \ldots, l_m$ linearly independent. Hence, after, we assume : $rank(V) = m - 1$. In that case $l_1$ and $l_2$ are linear combinations of $l_3, \ldots, l_{m+1}$. One of both combinations has to express $l_{m+1}$. Thus we have two situations.

**Situation 1** :

$l_2$ linear combination of $l_3, \ldots, l_{m+1}$ where coefficient of $l_{m+1}$ non-zero. By replacement of $l_{m+1}$, we obtain $l_1$ is linear combination of $l_2, \ldots, l_m$.

Thus, $w_1$ is linear combination of $w_2, \ldots, w_m$. ($w_2, \ldots, w_m$) libre system where $d^\nu w_k = k - 1$ for $k = 2, \ldots, m$. Thus : $w_1 = 0$.

If

$$
l_1 = \sum_{j=2}^{m} \alpha_j l_j
$$

where $\alpha_j$ constant, then :

$$
w_1 = \sum_{j=2}^{m} \alpha_j w_j
$$

As $w_1 = 0$, thus for all $j = 2, \ldots, m$, $\alpha_j = 0$ equivalent to $l_1 = 0$. Thus $R = 0$

**First way of having $D_0$**:

$$
\frac{D''_0}{D_0} = \frac{-2A}{B}
$$
\(-\frac{2A}{B}\) has to be the sum of simple elements of the shape: \(\frac{\mu_c}{x-c}\) where \(\mu_c \geq 1\).

Thus: \(D_4 = D_2 = D_1 = 1\) and \(D_3 = B\).

For all \(c\) root of \(D_3\), \(-2\alpha_c = \mu_c + 1\); where \(\alpha_c = \lim_{x \to c} (x - c)^2 r + \frac{1}{4}\)

Thus:

\[
\alpha_c < 0 \quad \text{and} \quad 4\left[\lim_{x \to c} (x - c)^2 r + \frac{1}{4}\right] = (\mu_c + 1)^2
\]

Thus:

\[
D_0' = m \prod_{c \in \text{Root}(D_3)} (x - c)^{\mu_c}
\]

\(D_0\) is primitive non-zero in roots of \(D_3\).

**Second way of having \(D_0\):**

\[
l_{m+1} = \sum_{j=2}^{m} \lambda_j l_j
\]

where \(\lambda_j\) constant \((j = 2, \ldots, m)\) The relations linearly dependent between \(l_{m+1}\) et \(l_1, \ldots, l_m\)

are:

\[
l_{m+1} = \lambda_1 + \sum_{j=2}^{m} \lambda_j l_j
\]

Thus we have an infinity of solutions:

\[
D_0 = x^m - \lambda - \sum_{j=2}^{m} \lambda_j x^{j-1}
\]

where \(\lambda\) arbitrarily constant.

**Situation 2:**

\(l_1\) is linear combination of \(l_3, \ldots, l_{m+1}\) where coefficient of \(l_{m+1}\) non-zero and \(l_2\) is linear combination of \(l_3, \ldots, l_m\). Thus, \(w_2\) is linear combination of \(w_3, \ldots, w_m\). Thus, \(w_2 = 0\). \(w_1, w_3, \ldots, w_m\) libre system.

If:

\[
l_2 = \sum_{j=3}^{m} \alpha_j l_j
\]

then:

\[
w_2 = \sum_{j=3}^{m} \alpha_j w_j
\]

As: \(w_2 = 0\). Thus, for all \(j = 3, \ldots, m\), \(\alpha_j = 0\) equivalent to \(l_2 = 0\) and \(2A - xR = 0\)

**First way of having \(D_0\):**

\[
\frac{2A}{x} D_0 = 2AD_0' + BD_0''
\]

\[
2A(D_0 - xD_0') = xBD_0''
\]

We put: \(Q = D_0 - xD_0'\). \(\frac{Q'}{Q} = -\frac{2A}{B}\).

\[
D_4 = D_2 = D_1 = 1 \quad \text{and} \quad Q = (1 - m) \prod_{c \in \text{Root}(D_3)} (x - c)^{\mu_c}
\]
Thus: $D_0 = Cx$ avec $C' = -\frac{Q}{x^2}$; $C$ rational function.
The coefficient of $x$ in $Q$ is zero. $Q'(0) = 0$ ($A(0) = 0$ ; $2A = xR$

**Second way of having $D_0$:**

$$l_{m+1} = \lambda_1 l_1 + \sum_{j=3}^{m} \lambda_j l_j$$ (105)

Thus :

$$D_0 = x^m - \sum_{j=1}^{m} \lambda_j x^{j-1}$$ (106)

$\lambda_2$ arbitrarily constant.

**5.2 Case $d^\nu r = 2\nu \geq 0$**

We put : $E = E(\theta)$ and $S = E + \frac{A}{B}$. We have:

$$\begin{cases}
d^\nu (BD_0'') \leq d^\nu B + m - 2 \\
d^\nu (EB + A)D_0' = d^\nu B + m - 1 + \nu
\end{cases}$$ (107)

Thus

$$d^\nu (RD_0) = d^\nu B + m - 1 + \nu$$ (108)

Thus, $d^\nu R = d^\nu B + \nu - 1$ and $r_{d^\nu B + \nu - 1} = 2\alpha m$ where $\alpha$ dominant coefficient of $E$.

We put :

$$v(x) = -R \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^m \end{pmatrix} + 2(EB + A) \begin{pmatrix} 0 \\ 1 \\ 2x \\ \vdots \\ mx^{m-1} \end{pmatrix} + B \begin{pmatrix} 0 \\ 1 \\ \vdots \\ m(m-1)x^{m-2} \end{pmatrix}$$ (109)

The $k^{th}$ element of $v(x)$ is :

$$v_k(x) = B(k - 1)(k - 2)x^{k-3} + 2(EB + A)(k - 1)x^{k-2} - Rx^{k-1}$$ (110)

$v_k(x)$ of degree equal to or less than $d^\nu B + \nu + k - 2$ where coefficient of $x^{d^\nu B + \nu + k - 2}$ equal $2\alpha(k - 1 - m)$.

For $k = 1, \ldots, m$, the $k^{th}$ element of $v(x)$ of degree: $d^\nu R + k - 1$.

The $(m + 1)^{th}$ element of $v(x)$ of degree equal to or less than $d^\nu R + m - 1$.

**Proposition 25** We put : $v_k(x) = \rho_k(x) + x^R w_k(x)$ where $d^\nu \rho_k(x) < d^\nu R$. 

---

1. \[
\begin{aligned}
\{ \ d^k w_k(x) &= k - 1 \quad \text{pour } k = 1, \ldots, m \\
\ d^m w_{m+1}(x) &\leq m - 1
\end{aligned}
\] (111)

2. \( (w_1, \ldots, w_m) \) libre system and
\[
w_{m+1}(x) = \lambda_1 w_1(x) + \ldots + \lambda_m w_m(x)
\] (112)

where \( \lambda_1, \ldots, \lambda_m \) constants

3. \[
D_0 = x^m - \lambda_1 - \lambda_2 x - \ldots - \lambda_{m-1} x^{m-1}
\] (113)
is solution if and only if
\[
l_{m+1} = \lambda_1 l_1 + \ldots + \lambda_m l_m
\] (114)

**Example 26** In this example we consider the Riccati differential equation (2) where:

\[
r = \frac{1}{(x+1)^4} - \frac{5}{(x+1)^3} + \frac{7}{4(x+1)^2} + \frac{1}{x+1} + x^2 + 2
\]

\( D_1 = D_2 = D_3 = 1, \ D_4 = (x+1)^2, \ d^\nu(r) = 2, \nu = 1. \) We have

\[
N_1 = 1 - 5(x+1)
\]

\[
F = x^2 + 2 + \frac{1 - 5(x+1)}{(x+1)^4} + \frac{1}{4(x+1)^2}
\]

**Study at (-1):** Laurent series development at \(-1:\)

\[
(F)^{\frac{1}{2}} = \varepsilon_{-1} \left( \frac{1}{(x+1)^2} - \frac{5}{2(x+1)} + \ldots \right); \varepsilon_{-1} = \pm 1
\]

**Study at infinity:** We have : \( \frac{E(r)}{x^2} = 1 + t^2 \) where \( t = \frac{1}{x}, \)

\[
(1 + 2t^2)^{\frac{1}{2}} = 1 + t^2 + o(t^2)
\]

Thus : \( s_{\nu+1} = 1 \) et \( E(\theta) = \alpha x \) where \( \alpha^2 = 1. \)

\[
\theta = \alpha x + \varepsilon_{-1} \left( \frac{1}{(x+1)^2} - \frac{5}{2(x+1)} \right) + \frac{1}{x+1} + \frac{D_0'}{D_0}
\]

\[
d^\nu D_0 = \alpha s_{\nu+1} - \nu \frac{5\varepsilon_{-1}}{2} - 1 = \alpha s_{\nu+1} + \frac{5\varepsilon_{-1}}{2} - \frac{3}{2}
\]

**Case :** \[
\begin{aligned}
\alpha &= -1 \\
\varepsilon_{-1} &= -1
\end{aligned}
\]

and \[
\begin{aligned}
\alpha &= 1 \\
\varepsilon_{-1} &= -1
\end{aligned}
\]

are to be rejected because we obtain negative values of \( D_0. \)

. If : \( \alpha = 1 \) and \( \varepsilon_{-1} = 1 \) then : \( d^\nu D_0 = 2 \)

. If : \( \alpha = -1 \) et \( \varepsilon_{-1} = 1 \) then : \( d^\nu D_0 = 0 \)
Case 1: $\alpha = 1$ and $\varepsilon_{-1} = 1$.

$$\theta = x + \frac{1}{(x+1)^2} - \frac{3}{2(x+1)} + \frac{D_0'}{D_0}$$

where $d^2D_0 = 2$. Research of coefficients of $D_0$.

$$S = x + \frac{1}{(x+1)^2} - \frac{3}{2(x+1)} = \frac{x^3 + 2x^2 - \frac{1}{2}}{(x+1)^2}$$

$$r - S' - S^2 = \frac{R}{B}, \text{ where: } R = 4x^2 + 4x \text{ and } B = (1 + x)^2.$$  

$$v(x) = -R \left( \begin{array}{c} 1 \\ x \\ x^2 \end{array} \right) + 2SB \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 2x \end{array} \right) + B \left( \begin{array}{c} -4x - 4x^2 \\ -1 - x - 2x^3 \\ 2 + 2x + 4x^3 \end{array} \right)$$

$$V = \left( \begin{array}{cccc} 0 & -4 & -4 & 0 \\ -1 & -1 & 0 & -2 \\ 2 & 2 & 0 & 4 \end{array} \right)$$

$$l_3 = -2l_2. \text{ Thus: } D_0 = x^2 + 2x.$$  

$$\theta = x + \frac{1}{(x+1)^2} - \frac{3}{2(x+1)} + \frac{2(x+1)}{x^2 + 2x} = x + \frac{1}{(x+1)^2} - \frac{3}{2(x+1)} + \frac{1}{x} + \frac{1}{x + 2}$$

Case 2: $\alpha = -1$ and $\varepsilon_{-1} = 1$. We obtain the rational fraction:

$$\theta = -x + \frac{1}{(x+1)^2} - \frac{3}{2(x+1)}$$

It cannot be solution because the sum of this fraction with the already found solution is not the logarithmic prime of a rational fraction.

Example 27: In this example we consider the Riccati differential equation (2) where:

$$r = \frac{1}{16} + \frac{1}{(x-1)^8} - \frac{4}{(x-1)^5} - \frac{29}{6(x-1)^4} - \frac{8}{9(x-1)^3} - \frac{64}{27(x-1)^2} - \frac{152}{18(x-1)} + \frac{30}{(x+2)^2} - \frac{10}{81(x+2)}$$

$$D_1 = D_2 = 1, D_3 = x + 2 \text{ and } D_4 = (x - 1)^4. \text{ We have: }$$

$$N_1 = (2418x - 2454)(x-1)^3 + (x+2)^2, \quad \delta = -\frac{1}{4(x+2)^2}$$

$$F = \frac{1}{16} + \frac{(2418x - 2454)(x-1)^3 + (x+2)^2}{(x-1)^8(x+2)^2} + \frac{1}{4(x+2)^2}$$

Laurent series development at 1:

$$(F)_{\frac{1}{2}} = \varepsilon_1 \left( \frac{1}{(x-1)^4} - \frac{2}{x-1} + ... \right); \varepsilon_1 = \pm 1$$

Laurent series development at 2:

$$(F)_{\frac{1}{2}} = \varepsilon_{-2} \left( \frac{11}{2(x+2)} + ... \right); \varepsilon_{-2} = \pm 1$$
We have:

\[
\frac{1}{2} \left( \frac{D_2 D_3 D_4}{D_2 D_3 D_4} \right)' = \frac{1}{2(x + 2)} + \frac{2}{x - 1}
\]

Thus:

\[
\theta = E(\theta) + \varepsilon_1 \left[ \frac{1}{(x - 1)^4} - \frac{2}{x - 1} \right] + \varepsilon_{-2} \left[ \frac{11}{2(x + 2)} \right] + \frac{1}{2(x + 2)} + \frac{2}{x - 1} + \frac{D_0'}{D_0}
\]

where

\[
E^2(\theta) = E(r) = \frac{1}{16}
\]

\[
d^o D_0 = -\frac{1}{E(\theta)} + 2\varepsilon_1 - \frac{11\varepsilon_{-2}}{2} - \frac{5}{2}
\]

Case \(\varepsilon_1 = 1\) and \(\varepsilon_{-2} = 1\) are to be rejected because we obtain negative values of \(D_0\).

**Case 1**: If \(\varepsilon_1 = 1\), \(\varepsilon_{-2} = -1\), \(E(\theta) = \frac{1}{4}\) then : \(d^o D_0 = 1\).

\[
\theta = \frac{1}{4} + \frac{1}{(x - 1)^4} - \frac{5}{x + 2} + \frac{D_0'}{D_0}
\]

Research of coefficients of \(D_0\).

\[
S = \frac{1}{4} + \frac{1}{(x - 1)^4} - \frac{5}{x + 2}
\]

\[
r - (S' + S^2) = \frac{R}{B} \text{ where: } R = \frac{1}{4} \left( x^4 - 10x^3 + 19x^2 - 18x + \frac{5}{2} \right), \quad B = (x - 1)^4(x + 2)
\]

\[
v(x) = -R \begin{pmatrix} 1 \\ x \end{pmatrix} + 2SB \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x^4 + 10x^3 - 19x^2 + 18x - \frac{5}{2} \\ -x^4 + 20x^3 - 5x^2 - 1 \end{pmatrix}
\]

\[
V = \begin{pmatrix} -\frac{5}{2} & 18 & -19 & 10 & -\frac{1}{2} \\ -5 & 36 & -38 & 20 & -5 \end{pmatrix}
\]

\(l_2 = 2l_1\). Thus : \(D_0 = x - 2\).

\[
\theta_1 = \frac{1}{4} + \frac{1}{(x - 1)^4} - \frac{5}{x + 2} + \frac{1}{x - 2}
\]

**Case 2**: If \(\varepsilon_1 = 1\), \(\varepsilon_{-2} = -1\), \(E(\theta) = -\frac{1}{4}\) then : \(d^o D_0 = 9\).

\[
\theta = -\frac{1}{4} + \frac{1}{(x - 1)^4} - \frac{5}{x + 2} + \frac{D_0'}{D_0}
\]

She cannot be solution because the sum of this fraction with the already found solution is not the logarithmic prime of a rational fraction.
Case 3: If $\varepsilon_1 = -1$, $\varepsilon_2 = -1$, $E(\theta) = -\frac{1}{4}$ then $d^0D_0 = 5$.

$$\theta_3 = -\frac{1}{4} - \frac{1}{(x-1)^4} - \frac{5}{x+2} + \frac{4}{x-1} + \frac{D_0}{D_0}$$

Research of coefficients of $D_0$.

$$S = -\frac{1}{4} - \frac{1}{(x-1)^4} - \frac{5}{x+2} + \frac{4}{x-1}$$

$$R = \frac{-2}{2}x^4 + 36x^3 - 137x^2 + 168x - 88, \quad B = (x-1)^4(x+2)$$

$$V = \begin{pmatrix} -37 & -168 & 137 & -36 & \frac{5}{2} & 0 & 0 & 0 & 0 \\ -32 & -\frac{151}{2} & -266 & 180 & -37 & 2 & 0 & 0 & 0 \\ 4 & -78 & -162 & -368 & 219 & -36 & \frac{3}{2} & 0 & 0 \\ 0 & 12 & -138 & -\frac{209}{2} & -474 & 254 & -33 & 1 & 0 \\ 0 & 0 & 24 & -212 & -95 & -584 & 285 & -28 & \frac{1}{2} \\ 0 & 0 & 0 & 40 & -300 & \frac{-139}{2} & -698 & 312 & -21 \end{pmatrix}$$

We consider $V_p$ the minor $6 \times 6$ obtained by column vector 1, 2, 6, 7, 8, 9. In $\mathbb{Z}/5\mathbb{Z}$:

$$V_p = \begin{pmatrix} -2 & 2 & 0 & 0 & 0 & 0 \\ -2 & 2 & 2 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 & 2 & -1 \end{pmatrix}$$

$$\det V_p = 1.$$ Accordingly, row 6 is not linear combination of other row and $D_0$ he does not have $D_0$. 

References


