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*Ordinary differential equations*

**Rational solutions of Riccati differential equation with coefficients rational**

Nadhem ECHI
Université de Tunis El Manar, B.P. 37,
1002-TUNIS Belvédère, Tunisie.
E-mail address: nadhemechi_fsg@yahoo.fr

## Abstract

This paper presents a simple and efficient method for determining the solution of Riccati differential equation with coefficients rational. In case the differential Galois group of the differential equation $(E_l) : y'' = ry$, $r \in \mathbb{C}(x)$ is reducible, we look for the rational solutions of Riccati differential equation $\theta' + \theta^2 = r$, by reducing the number of check to be made and by accelerating the search for the partial fraction decomposition of the solution reserved for the poles of $\theta$ which are false poles of $r$. This partial fraction decomposition of solution can be used to code $r$. The examples demonstrate the effectiveness of the method.

## 1 Introduction

The quadratic Riccati differential equation :

$$(E_R) : \sigma' = p_2\sigma^2 + p_1\sigma + p_0 \tag{1}$$

where $p_0$, $p_1$ and $p_2$ are in a differential field $\mathbb{K}$, $p_2 \neq 0$. The quadratic Riccati differential equation is first converted to a reduced Riccati differential equation :

$$(E_r) : \theta' + \theta^2 = r \tag{2}$$

where : $\theta = -p_2\sigma - \frac{1}{2}a$, with $a = \frac{p_2'}{p_2} + p_1$ and $r = \frac{1}{4}a^2 - \frac{1}{2}a' - p_2p_0$ .
Furthermore, we put: $\frac{y'}{y} = \theta$, reduced Riccati differential equation (2) is converted to a second-order linear ordinary differential equation :

$$(E_l) : y'' = ry \tag{3}$$

If we have a particular solution non-zero of $(E_l)$ then general solution is : $y = cu$ where $c' = \frac{\lambda}{u^2}$, $\lambda$ constant (see[ 6,9,14]).
In paper, we base ourselves mainly on the work of J.J. Kovačić [9] where differential Galois group of the differential equation $(E_l)$ is reducible and we take : $\mathbb{K} = \mathbb{C}(X)$.

In case where every solution of $(E_l)$ is Liouvillian corresponds to the case where reduced Riccati differential equation $(E_r)$ have algebraic solution over $\mathbb{K}$. The case where differential Galois group is reducible corresponds to the case where the Riccati differential equation $(E_r)$ have the rational solution $\frac{u'}{u}$ $u$ solution of $(E_l)$. The solution $u$ of $(E_l)$ is rational fraction if only if $\frac{u'}{u}$ the fraction of simples poles with the integers residues and negative degree.
The field $\mathbb{C}(X)[u]$ is differential extension of $\mathbb{C}(X)$ by exponential of an integral and if $v' = \frac{1}{u^2}$ then $(u, v)$ two solutions of $(E_l)$ linearly independent over field of constants $\mathbb{C}$. The ordinary extension $\mathbb{C}(X)[u, v]$ is differential extension of $\mathbb{C}(X)[u]$, by a integral. $\mathbb{C}(X)[u, v]$ is Picard-Vessiot extension of $\mathbb{C}(X)[u]$ for the differential equation$(E_l)$(see[8-9-10]). The existence of rational solution $\frac{u'}{u}$ of Riccati differential equation $(E_r)$ given all solutions of $(E_r)$ of course research primitive of $\frac{1}{u^2}$.

This paper presents a simple and efficient method for determining the solution of Riccati differential equation with coefficients rational. In case the differential Galois group of the differential equation $(E_l) : y'' = ry$, $r \in \mathbb{C}(x)$ is reducible, we look for the rational solutions of Riccati differential equation $\theta' + \theta^2 = r$, by reducing the number of check to be made and by accelerating the search for the partial fraction decomposition of the solution reserved for the poles of $\theta$ which are false poles of $r$. This partial fraction decomposition of solution can be used to code $r$. The examples demonstrate the effectiveness of the method.

## 2 Form of rational solution of equation : $(E_r)$

Let $r \in \mathbb{C}(x)$ $r \neq 0$ rational fraction and $\theta \in \mathbb{C}(x)$ the rational solution of Riccati differential equation $(E_r)$ : $\theta' + \theta^2 = r$.

### 2.1 Study in the pole $c$ of multiplicity $\nu$ of $\theta$

We put:

$$\theta = \frac{\tau}{(x-c)^\nu} \quad ; \ where \ \ \tau(c) \neq 0$$

We have :

$$r = \theta' + \theta^2 = (x-c)^{-2\nu}[(\tau - \frac{\nu}{2}(x-c)^{\nu-1})^2 - \frac{\nu^2}{4}(x-c)^{2\nu-2} + \tau'(x-c)^\nu]$$

Thus :

$$(\tau - \frac{\nu}{2}(x-c)^{\nu-1})^2 = (x-c)^{2\nu}r + (x-c)^\nu[\frac{\nu^2}{4}(x-c)^{\nu-2} - \tau']$$

1. <u>*Case 1 :*</u> $\nu \geq 2$
   The function $(x-c)^{2\nu}r$ define and equal $\tau(c)^2$ at $c$.
   Thus $c$ is pole of multiplicity $2\nu$ of $r$ where :

$$\lim_{x\to c}(x-c)^{2\nu}r = (\lim_{x\to c}(x-c)^\nu\theta)^2$$

2. <u>*Case 2*</u> : $\nu = 1$
   We have:
   $$(\tau - \frac{1}{2})^2 = (x-c)^2 r + \frac{1}{4} - (x-c)\tau'$$
   The function $(x-c)^2 r$ define and equal $\tau(c)(\tau(c)-1)$ at $c$.
   <u>*Situation 1*</u> : $\lim_{x\to c}(x-c)^2 r \neq 0, -\frac{1}{4}$
   $c$ is double pole of $r$ and the residue $\tau(c)$ of $\theta$ at $c$ have tow possibility values following :
   $$(\tau(c) - \frac{1}{2})^2 = \lim_{x\to c}(x-c)^2 r + \frac{1}{4}$$
   Thus, $c$ is double pole of $r$ and the residue of $\theta$ at simple pole $c$ equal:
   $$\tau(c) = \alpha_c + \frac{1}{2}$$
   where
   $$\alpha_c^2 = \lim_{x\to c}(x-c)^2 r + \frac{1}{4}$$
   <u>*Situation 2*</u> : $\lim_{x\to c}(x-c)^2 r = -\frac{1}{4}$
   $c$ is double pole of $r$ and the residue of $\theta$ at $c$ is $\frac{1}{2}$.
   <u>*Situation 3*</u> : $\lim_{x\to c}(x-c)^2 r = 0$
   $c$ is simple pole of $r$ or not pole of $r$ and the residue of $\theta$ at simple pole $c$ equal 1.

**Proposition 1** *Let $\theta \in \mathbb{C}(x)$ such as : $\theta' + \theta^2 = r$.*

1. *The fraction : $r = \frac{N}{D}$ with $N$ and $D$ polynomials relatively prime.*
   $$D = D_1 D_2^2 D_3^2 D_4^2 \quad (4)$$
   *where $D_1$, $D_2$, $D_3$ and $D_4$ polynomials relatively prime pair-wise. $D_1$, $D_2$ and $D_3$ which simples roots, $D_4$ without simple root.*
   $\forall c \in Root(D_2)$ $\lim_{x\to c}(x-c)^2 r = -\frac{1}{4}$, $\forall c \in Root(D_3)$ $\lim_{x\to c}(x-c)^2 r \neq -\frac{1}{4}$
2. (a) *Let $\nu \geq 2$. $c$ pole of multiplicity $\nu$ of $\theta \Leftrightarrow c \in Root(D_4)$*
   (b) *$c$ simple pole of $\theta$ with residue $\neq 1, \frac{1}{2} \Leftrightarrow c \in Root(D_3)$*
   *The residue of $\theta$ there $c$ equal $\alpha_c + \frac{1}{2}$ where*
   $$\alpha_c^2 = \lim_{x\to c}(x-c)^2 r + \frac{1}{4} \quad (5)$$
   (c) *$c$ simple pole of $\theta$ with residue $= \frac{1}{2} \Leftrightarrow c \in Roots(D_2)$*
   (d) *$c$ simple pole of $\theta$ with residue $= 1 \Leftrightarrow c \in Roots(D_1)$ or $c$ pole of $\theta$ and false pole of $r$*

**Corollary 2** *We assume that $r = \frac{N}{D}$ with $N$ and $D$ polynomials relatively prime, $D = D_1 D_2^2 D_3^2 D_4^2$ where $D_1$, $D_2$, $D_3$ and $D_4$ polynomials relatively prime pair-wise, $D_1$, $D_2$ and $D_3$ which simples roots, $D_4$ without simple root.*

$\forall c \in Root(D_2)\ \lim_{x\to c}(x-c)^2 r = -\frac{1}{4},\ \forall c \in Root(D_3)\ \lim_{x\to c}(x-c)^2 r \neq -\frac{1}{4}$
*A rational fraction* $\theta$ *Verify* $\theta' + \theta^2 = r$ *is the shape :*

$$\theta = E(\theta) + \sum_{c\in Roots(D_4)} \theta_c + \sum_{c\in Roots(D_3)} \frac{\alpha_c + \frac{1}{2}}{x-c} + \frac{1}{2}\frac{D_2'}{D_2} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \tag{6}$$

*with* $D_0$ *monic polynomial which simples roots and the roots are false poles of* $r$*.*

### 2.2 Study in the infinity

Let: $t \in \mathbb{C}(x);\ \ t \neq 0$ rational fraction such as : $\frac{dt}{dx} + t^2 = 0$.
**case 1 :** We assume that : $d^o(\theta) < 0$
We have : $d^o(\theta') < 0$ and $d^o(\theta^2) < 0$ thus : $d^o(r) < 0$.
We put:

$$\theta = t\sigma(t) \quad \sigma\ rational\ fraction\ defined\ at\ 0.$$

We have:

$$\sigma(0) = \lim_{x\to\infty} x\theta = sum\ the\ residues\ of\ \theta,$$

$$r = \theta' + \theta^2 = (\sigma^2 - \sigma)t^2 - \sigma' t^3 \ \ and \quad \lim_{x\to\infty} x^2 r = \sigma(0)^2 - \sigma(0)$$

Thus : $d^o r \leq\ -2$ and $(\sigma(0) - \frac{1}{2})^2 = \lim_{x\to\infty} x^2 r + \frac{1}{4}$
If : $\lim_{x\to\infty} x^2 r = -\frac{1}{4}$ then the sum of residues of $\theta$ : $\sigma(0) = \frac{1}{2}$
If : $\lim_{x\to\infty} x^2 r \neq -\frac{1}{4}$ then the sum of residues of $\theta$ : $\sigma(0) = \alpha_\infty + \frac{1}{2}$
where

$$\alpha_\infty^2 = \lim_{x\to\infty} x^2 r + \frac{1}{4} \tag{7}$$

**case 2 :** We assume that: $d^o(\theta) = 0$.
$E(\theta)$ constant $\neq 0$. We put :

$$\theta = E(\theta) + t\sigma(t); \qquad \sigma\ rational\ fraction\ defined\ at\ 0.$$

We have:

$$\sigma(0) = \lim_{x\to\infty} x(\theta - E(\theta)) \ = \ the\ sum\ of\ residues\ of\ \theta$$

and

$$\begin{aligned} r &= \theta' + \theta^2 \\ &= (E(\theta))^2 + 2E(\theta)t\sigma + (\sigma^2 - \sigma)t^2 - \sigma' t^3 \end{aligned}$$

So $E(r)$ constant equal $E(\theta)^2$ :

$$2E(\theta)[\ sum\ the\ residues\ of\ \theta] = \ sum\ the\ residues\ of\ r$$

**case 3 :** We assume that: $d^o(\theta) > 0$
We put :

$$\nu = d^o(\theta) \geq 1 \quad and \quad \theta = t^{-\nu}\sigma(t)$$

$\sigma$ rational fraction defined at 0 . The scalar $\sigma(0)$ is the dominant coefficient of $E(\theta)$. We have :

$$r = \theta' + \theta^2 = t^{-2\nu}[\sigma^2 + \nu t^{\nu+1}\sigma - t^{\nu+2}\sigma']$$

Thus:

$$t^{2\nu}r = \sigma(0)^2 + o(t)$$

So : $d^o r = 2\nu \;\; = \; 2d^o\theta$ and $\sigma^2(o)$ the dominant coefficient of $E(r)$.

**Proposition 3** *Let: $\theta \in \mathbb{C}(x)$ such as : $\theta' + \theta^2 = r$.*

1. *$d^o(\theta) < 0 \;\Leftrightarrow\; d^o(r) < 0$. Thus : $d^o(r) \le -2$ and :*

$$\textit{the sum of residues of } \theta = \begin{cases} \frac{1}{2} \quad if \lim_{x\to\infty} x^2 r = -\dfrac{1}{4} \\ \alpha_\infty + \frac{1}{2} \quad if \lim_{x\to\infty} x^2 r \neq -\dfrac{1}{4} \end{cases}$$

2. *$d^o(\theta) = 0 \;\Leftrightarrow d^o(r) = 0$. In the case : $E(\theta)$ is square root of $E(r)$ :*

$$\begin{aligned} 2E(\theta)(\textit{sum of residues of } \theta) &= \textit{sum of residues of } r \\ &= \lim_{x\to\infty} x(r - E(r)) \end{aligned}$$

3. *We have : $d^o(\theta) > 0 \;\Leftrightarrow\; d^o(r) > 0$. In the case :*
   - *(a) $d^o(r) = 2d^o(\theta)$*
   - *(b) The dominant coefficient of $E(\theta)$ is square root of $E(r)$*

### 2.3 Determination of $E(\theta)$; $d^o(r) = 2\nu > 0$

We assume that $r$ is a rational fraction of degree $2\nu > 0$ and $\theta \in \mathbb{C}(x)$ such as : $\theta' + \theta^2 = r$. Let $a$ the dominant coefficient of $E(r)$. Thus: $r \sim ax^{2\nu}$ if $x$ tend to $\infty$:

$$t^{2\nu}\frac{E(r)}{a} \;=\; 1 + a_1 t + \ldots + a_{2\nu}t^{2\nu}$$

The Taylor's expansion of order $\nu + 1$ at $0$ :

$$(t^{2\nu}\frac{E(r)}{a})^{\frac{1}{2}} = 1 + s_1 t + \ldots + s_{\nu+1}t^{\nu+1} + o(t^{\nu+1})$$

We have :

$$\begin{aligned} t^{2\nu}r &= t^{2\nu}E(r) + o(t^{2\nu}) \\ &= t^{2\nu}E(r) + o(t^{\nu+1}) \\ &= a[(\tfrac{t^{2\nu}E(r)}{a})^{\frac{1}{2}}]^2 + o(t^{\nu+1}) \end{aligned}$$

We have: $\theta \;=\; t^{-\nu}\sigma(t)$ with $\sigma$ rational fraction defined at $0$ , $\sigma(0)^2 = a$ and

$$\begin{aligned} (\sigma + \tfrac{\nu}{2}t^{\nu+1})^2 &= t^{2\nu}r + \tfrac{\nu^2}{4}t^{2\nu+2} + t^{\nu+2}\sigma' \\ &= a[\,1 + s_1 t + \ldots + s_{\nu+1}t^{\nu+1}]^2 + o(t^{\nu+1}) \end{aligned}$$

$$\sigma + \frac{\nu}{2}t^{\nu+1} \;=\; \sigma(0)[\,1 + s_1 t + \ldots + s_{\nu+1}t^{\nu+1}] + o(t^{\nu+1})$$

Thus :

$$\theta \;=\; t^{-\nu}\sigma \;=\; \sigma(0)[\,t^{-\nu} + s_1 t^{-(\nu-1)} + \ldots + s_{\nu+1}t] - \frac{\nu}{2}t + o(t)$$

Imply :

$$\begin{cases} E(\theta) \ = \ \sigma(0)[\ t^{-\nu} + s_1 t^{-(\nu-1)} + \ldots + s_\nu] \\ \\ \sigma(0)s_{\nu+1} - \frac{\nu}{2} \ = \ sum\ of\ residues\ of\ \theta \end{cases}$$

**Proposition 4** *Let $r$ is a rational fraction of degree $2\nu > 0$, $\theta \in \mathbb{C}(x)$ such as : $\theta' + \theta^2 = r$ and a the dominant coefficient of $E(r)$. If :*

$$(t^{2\nu}\frac{E(r)}{a})^{\frac{1}{2}} \ = \ 1 + s_1 t + \ldots + s_{\nu+1}t^{\nu+1} + o(t^{\nu+1}) \tag{8}$$

*Then :*

$$E(\theta) \ = \ \alpha[\ t^{-\nu} + s_1 t^{-(\nu-1)} + \ldots + s_\nu] \tag{9}$$

$$\alpha s_{\nu+1} - \frac{\nu}{2} \ = \ sum\ of\ residues\ of\ \theta \tag{10}$$

*where*

$$\alpha^2 = a \tag{11}$$

## 3 Determination of partial fraction decomposition

Let $r = \frac{N}{D}$ rational fraction with $N$ and $D$ polynomials relatively prime, $D = D_1 D_2^2 D_3^2 D_4^2$ where $D_1$, $D_2$, $D_3$ *et* $D_4$ polynomials relatively prime pair-wise, $D_1$, $D_2$ and $D_3$ which simples roots, $D_4$ without simple root.
$\forall c \in Root(D_2)\ \lim_{x\to c}(x-c)^2 r = -\frac{1}{4}$, $\forall c \in Root(D_3)\ \lim_{x\to c}(x-c)^2 r \neq -\frac{1}{4}$
Let $\theta \in \mathbb{C}(x)$ rational fraction Verify : $\theta' + \theta^2 = r$

### 3.1 Case $d^o D_3 = 0$ *and* $d^o D_4 = 0$

We have : $r = \frac{N}{D_1 D_2^2}$
This case corresponds to the fact that one pole $c$ of $r$ is or simple or double with:

$$\lim_{x\to c}(x-c)^2 r = -\frac{1}{4}$$

**Proposition 5** *We assume $d^o r < 0$ and $d^o D_3 = d^o D_4 = 0$. We have :*

$$\lim_{x\to\infty} x^2 r \ + \ \frac{1}{4} \ = \ (\frac{q}{2})^2$$

*with $q$ positive integer of parity against that of $d^o D_2$*

$$\theta = \frac{1}{2}\frac{D_2'}{D_2} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \tag{12}$$

*with $D_0$ polynomial of degree :*

$$d^o D_0 = \frac{1}{2}(q + 1 - d^o D_2) - d^o D_1 \tag{13}$$

**Proof.** We have :

$$\theta = \frac{1}{2}\frac{D_2'}{D_2} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}$$

Sum of residues of $\theta$ equal $\frac{1}{2}d^oD_2 + d^oD_1 + d^oD_0 = \alpha_\infty + \frac{1}{2}$
with $\alpha_\infty^2 = \lim_{x\to\infty} x^2r + \frac{1}{4}$. In particular : $\alpha_\infty = \frac{q}{2}$ *with* $q \in \mathbb{N}$
Remark : if $\lim_{x\to\infty} x^2r = -\frac{1}{4}$ then: $d^oD_2 = 1$, $d^oD_1 = d^oD_0 = 0$, $\theta = \frac{1}{2}\frac{1}{(x-c)}$ and $r = -\frac{1}{4(x-c)^2}$ ■

**Proposition 6** *: We assume* $d^or = 0$ *and* $d^oD_3 = d^oD_4 = 0$.

1. $E(\theta)$ *square root of* $E(r)$ *such as* $p = \frac{1}{E(\theta)}\lim_{x\to\infty} x[r - E(r)]$ *positive integer of same parity as* $d^oD_2$.
2. *We have :*

$$\theta = E(\theta) + \frac{1}{2}\frac{D_2'}{D_2} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \tag{14}$$

*with* $D_0$ *polynomial of degree:*

$$d^oD_0 = \frac{1}{2}p - d^oD_1 - \frac{1}{2}d^oD_2 \tag{15}$$

**Proof.** We have :

$$\theta = E(\theta) + \frac{1}{2}\frac{D_2'}{D_2} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}$$

We have:

$$2E(\theta) sum\ of\ residues(\theta) = sum\ of\ residues(r) = \lim_{x\to\infty} x[r - E(r)]$$

$$2E(\theta)[\frac{1}{2}d^oD_2 + d^oD_1 + d^oD_0] = \lim_{x\to\infty} x[r - E(r)]$$

Thus :

$$d^oD_2 + 2d^oD_1 + 2d^oD_0 = \frac{1}{E(\theta)}\lim_{x\to\infty} x[r - E(r)]$$

If r constant then $D_1 = D_2 = D_0 = 1$ and $\theta$ constant.
If $r$ non-constant then $r$ is not polynomial thus :

$$p = \frac{1}{E(\theta)}\lim_{x\to\infty} x[r - E(r)]$$

positive integer of same parity as $d^oD_2$. ■

**Proposition 7** *:*
*We assume* $d^or = 2\nu > 0$ *and* $d^oD_3 = d^oD_4 = 0$
*Let* $a$ *the dominant coefficient of* $E(r)$ *and we consider the Taylor's expansion at infinity :*

$$(t^{2\nu}\frac{E(r)}{a})^{\frac{1}{2}} = 1 + s_1t + \ldots + s_{\nu+1}t^{\nu+1} + o(t^{\nu+1}) \tag{16}$$

1.
$$4as_{\nu+1}^2 = p^2 \tag{17}$$
*where $p$ positive integer, $p \geq d^oD_2 + 2d^oD_1 + \nu$ and same parity of $d^oD_2 + 2d^oD_1 + \nu$.*

2. *We have :*
$$\theta = E(\theta) + \frac{1}{2}\frac{D_2'}{D_2} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \tag{18}$$
*with $D_0$ polynomial of degree :*
$$d^oD_0 = \frac{p - \nu - d^oD_2}{2} - d^oD_1 \tag{19}$$
*and $\alpha$ the dominant coefficient of $E(\theta)$ where :*
$$p = 2\alpha s_{\nu+1} \tag{20}$$

**Proof.** We have :
$$E(\theta) = \alpha[t^{-\nu} + s_1t^{-(\nu-1)} + .. + s_\nu]$$
$$\alpha s_{\nu+1} - \frac{\nu}{2} = sum\ of\ residues\ of \theta$$
$$\theta \ = \ E(\theta) + \frac{1}{2}\frac{D_2'}{D_2} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}$$
Thus:
$$\frac{1}{2}d^oD_2 + d^oD_1 + d^oD_0 = \alpha s_{\nu+1} - \frac{\nu}{2}$$
$\alpha s_{\nu+1} = \frac{p}{2}$ with $p$ positive integer. Thus: $p \geq d^oD_2 + 2d^oD_1 + \nu$ and same parity of $d^oD_2 + 2d^oD_1 + \nu$. ■

### 3.2 Case: $D_3 = X - c$ and $d^oD_4 = 0$

This case corresponds to the fact that a pole of $r$ is simple or double with a only double pole $c$ such as :
$$\lim_{x \to c}(x - c)^2 r \neq -\frac{1}{4} \tag{21}$$
$$r = \frac{N}{D_1D_2^2(x - c)^2} \tag{22}$$

**Proposition 8** *: Consider the Eq. (22) and let $\theta \in \mathbb{C}(x)$ rational fraction Verify : $\theta' + \theta^2 = r$ We assume : $d^or < 0$. Accordingly, in view of (5) and (7) we have :*
$$\theta = \frac{\alpha_c + \frac{1}{2}}{x - c} + \frac{1}{2}\frac{D_2'}{D_2} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \tag{23}$$
*with $D_0$ polynomial of degree :*
$$d^oD_0 \ = \ \lambda - \frac{1}{2}d^oD_2 - d^oD_1 \tag{24}$$
*where*
$$\lambda = \alpha_\infty - \alpha_c \tag{25}$$
*one half positive integer of same parity as $d^oD_2$.*

**Proof.** We have :

$$\lim_{x\to\infty} x\theta = \alpha_c + \frac{1}{2} + \frac{1}{2}d^oD_2 + d^oD_1 + d^oD_0$$

$$(\alpha_c + \frac{1}{2}d^oD_2 + d^oD_1 + d^oD_0)^2 = \lim_{x\to\infty} x^2 r + \frac{1}{4} = \alpha_\infty^2$$

Thus:

$$\frac{1}{2}d^oD_2 + d^oD_1 + d^oD_0 = \alpha_\infty - \ \alpha_c$$

■

**Proposition 9** *: Consider the Eq. (22) and let $\theta \in \mathbb{C}(x)$ rational fraction Verify : $\theta' + \theta^2 = r$ We assume : $d^o r = 0$. We have :*

$$\theta = E(\theta) + \frac{\alpha_c + \frac{1}{2}}{x-c} + \frac{1}{2}\frac{D_2'}{D_2} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \tag{26}$$

*with $E^2(\theta) = E(r)$ and $D_0$ polynomial of degree :*

$$d^oD_0 = \frac{1}{2}\lambda - \frac{1}{2}d^oD_2 - d^oD_1 - \frac{1}{2} \tag{27}$$

*where*

$$\lambda = \frac{1}{E(\theta)}\lim_{x\to\infty} x[r - E(r)] - 2\alpha_c \tag{28}$$

*$\lambda$ is positive integer of parity against that of $d^oD_2$.*

**Proof.** We have :

$$2E(\theta)[\alpha_c + \ \frac{1}{2} \ + \ \frac{1}{2}d^oD_2 + d^oD_1 + d^oD_0] \ = \ \lim_{x\to\infty} x[r - E(r)]$$

Thus

$$1 + d^oD_2 + 2d^oD_1 + 2d^oD_0 \ = \ \frac{1}{E(\theta)}\lim_{x\to\infty} x[r - E(r)] - 2\alpha_c$$

■

**Proposition 10** *: Consider the Eq. (22) and let $\theta \in \mathbb{C}(x)$ rational fraction Verify : $\theta' + \theta^2 = r$ We assume : $d^o r = 2\nu > 0$. Let $a$ the dominant coefficient of $E(r)$, $\alpha$ the dominant coefficient of $E(\theta)$, $t = \frac{1}{x}$ and Taylors expansion :*

$$(t^{2\nu}\frac{E(r)}{a})^{\frac{1}{2}} \ = \ 1 + s_1 t + \ldots + s_{\nu+1}t^{\nu+1} + o(t^{\nu+1})$$

*We have :*

$$\theta = E(\theta) + \frac{\alpha_c + \frac{1}{2}}{x-c} + \frac{1}{2}\frac{D_2'}{D_2} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \tag{29}$$

*$D_0$ polynomial of degree :*

$$d^oD_0 = \alpha s_{\nu+1} - \alpha_c - \frac{1 + \nu + d^oD_2}{2} - d^oD_1 \quad \textit{and} \quad \alpha^2 = a \tag{30}$$

**Proof.** We have :

$$E(\theta) = \alpha[t^{-\nu} + s_1 t^{-(\nu-1)} + .. + s_\nu]$$

and

$$\alpha s_{\nu+1} - \frac{\nu}{2} = sum\ of\ residues\ of\ \theta$$

Thus :

$$\theta = E(\theta) + \frac{\alpha_c + \frac{1}{2}}{x-c} + \frac{1}{2}\frac{D_2'}{D_2} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}$$

$$\alpha s_{\nu+1} - \frac{\nu}{2} = \alpha_c + \frac{1}{2} + \frac{1}{2} d^o D_2 + d^o D_1 + d^o D_0$$

Thus :

$$d^o D_0 = \alpha s_{\nu+1} - \alpha_c - \frac{1 + \nu + d^o D_2}{2} - d^o D_1$$

■

### 3.3 Case : $(d^o D_3 \neq 0$ and $d^o D_4 \neq 0)$ or $(d^o D_3 \geq 2$ and $d^o D_4 = 0)$

$D_1 D_2^2$ and $D_3 D_4$ polynomials relatively prime. Thus there are two only polynomials $N_1$ and $N_2$ such as :

$$\begin{cases} D_1 D_2^2 D_3^2 D_4^2 (r - E(r)) = N_1 (D_1 D_2^2) + N_2 (D_3 D_4) \\ d^o N_1 \quad < \quad d^o (D_3 D_4) \end{cases}$$

Thus

$$r = E(r) + \frac{N_1}{D_3^2 D_4^2} + \frac{N_2}{D_1 D_2^2 D_3 D_4} \tag{31}$$

We have :

$$d^o\left(\frac{N_1}{D_3^2 D_4^2}\right) = d^o\left(\frac{N_1}{D_3 D_4}\frac{1}{D_3 D_4}\right) < d^o\left(\frac{1}{D_3 D_4}\right) \leq -2 \tag{32}$$

Because $D_4$ does not have simple roots verify : $d^o D_4 = 0$ or $d^o D_4 \geq 2$. Thus: $d^o(D_3 D_4) \geq 2$. Thus :

$$\lim_{x \to \infty} x^2 \frac{N_1}{D_3^2 D_4^2} = 0 \tag{33}$$

$$d^o(r - E(r)) < 0 \tag{34}$$

$$d^o\left(\frac{N_2}{D_1 D_2^2 D_3 D_4}\right) < 0 \tag{35}$$

We consider the rational fraction :

$$F = E(r) + \frac{N_1}{D_3^2 D_4^2} - \delta \tag{36}$$

where

$$\delta = \frac{1}{4}\left[\frac{1}{k}\left(\frac{D_3'}{D_3}\right)^2 + \left(\frac{1}{k} + 1\right)\left(\frac{D_3'}{D_3}\right)'\right] \tag{37}$$

$$k = d^o D_3 \neq 0 \tag{38}$$

**Proposition 11** *: We assume* $\begin{cases} d^o D_3 \neq 0 \\ d^o D_4 \neq 0 \end{cases}$ *or* $\begin{cases} d^o D_3 \geq 2 \\ d^o D_4 = 0 \end{cases}$

1. *If :* $d^o r \geq 0$ *then :* $d^o F = d^o r$ *and* $E(F) = E(r)$

2. *If :* $d^o r < 0$ *then :* $d^o F = -2$ *where* $\lim_{x\to\infty} x^2 F = \dfrac{1}{4}$.

3. *For all* $c$ *root of* $D_3$ *we have:*

$$\lim_{x\to c}(x-c)^2 F = \lim_{x\to c}(x-c)^2 r + \frac{1}{4}$$

4. *For all* $c$ *root of* $D_4$ *of multiplicity* $\nu$ *we have:*

$$(x-c)^{2\nu} r = (x-c)^{2\nu} F + o((x-c)^{\nu-1})$$

**Proof.**

1. $r - F = \frac{N_2}{D_1 D_2^2 D_3 D_4} + \delta$ is from negative degree.

2. $\lim_{x\to\infty} x^2 F = \lim_{x\to\infty} x^2 \dfrac{N_1}{D_3^2 D_4^2} - \lim_{x\to\infty} x^2\delta = -\lim_{x\to\infty} x^2\delta = \dfrac{1}{4}$

3. Let $c$ root of $D_3$.

$$\begin{aligned}\lim_{x\to\infty}(x-c)^2(r-F) &= \lim_{x\to c}\frac{N_2}{D_1 D_2^2 D_4}\frac{(x-c)^2}{D_3} + \lim_{x\to c}(x-c)^2\delta \\ &= \lim_{x\to c}(x-c)^2\delta = -\frac{1}{4}\end{aligned}$$

4. Let $c$ root of $D_4$.

$$\begin{aligned}(x-c)^{2\nu}(r-F) &= \tfrac{N_2}{D_1 D_2^2 D_3}\tfrac{(x-c)^{2\nu}}{D_4} + (x-c)^{2\nu}\delta \\ &= o((x-c)^{\nu-1})\end{aligned}$$

■

**Lemma 12** *Let* $Z$ *be non-zero rational fraction .* $\Sigma$ *is a finished set such as :*

$$\Sigma \cap [Roots(Z) \cup poles(Z)] = \emptyset \tag{39}$$

1. *Exists* $O$ *open connected on which we have a square root holomorphic of* $Z$*, containing for every* $c \in \Sigma$ *a half-right closed by origin* $c$*.*

2. *If, besides,* $Z$ *is from even degree then can choose* $O$ *of complementary compact.*

**Proof.** We fix $c_0$ not an element of $\Sigma$. Let $\Sigma'$ a finished set containing roots, poles of $Z$ and $c_0$. They consider all rights linked to the different pairs of points of $\Sigma \cup \Sigma'$.They choose a point $w$ not being on these rights. For all $c \in \Sigma \cup \Sigma'$, which joins right $w$ in $c$ does not contain any other point of $\Sigma \cup \Sigma'$.
<u>**case 1** : $\mathbf{d^o Z}$ **even** :</u> Replacing $Z$ by rational fraction: $\frac{Z}{(x-c_0)^{d^o Z}}$.

Assume $d^oZ = 0$. We put: $K = \bigcup_{c\in\Sigma'} [w, c]$.$K$ is compact connected.
We put: $O = \mathbb{C} \setminus K$. $O$ is open at infinity such as for all $c \in \Sigma$ reaching right $w$ in $c$ private of $w$ is contained in $O$.
If $\gamma$ a shoelace of $O$ then $K$ is in one connected component of $\mathbb{C} \setminus \gamma$. Thus :

$$\frac{1}{2i\pi}\int_\gamma \frac{Z'}{Z}(x)dx = \pm\sum_{c\in\Sigma'} residue(\frac{Z'}{Z}, c) = \pm d^oZ \;=\; 0$$

Thus exist the primitive of $\frac{Z'}{Z}$ and the determination of logarithm of $Z$ and the square root of $Z$ in $O$.
**<u>case 2 : d^oZ odd</u>** : They use the previous case in replacing $Z$ by rational fraction $\frac{Z}{(x-c_0)}$. ■

**<u>Notation</u>:** We choose a square root of the polynomial of even degree : $N_1 + D_3^2 D_4^2[E(r) - \delta]$ on a connected open at infinity, roots of $D_3$ and root of $D_4$. We put :

$$(F)^{\frac{1}{2}} = \frac{1}{D_3D_4}(N_1 + D_3^2D_4^2[E(r) - \delta])^{\frac{1}{2}} \tag{40}$$

**Proposition 13** *: We assume* $\begin{cases} d^oD_3 \neq 0 \\ d^oD_4 \;\neq 0 \end{cases}$ *or* $\begin{cases} d^oD_3 \geq 2 \\ d^oD_4 = 0 \end{cases}$
*We have:*

$$\theta = E(\theta) + \sum_{c\in Root(D_3)\cup Root(D_4)} \varepsilon_c(partial\ fraction\ of (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2}\frac{(D_2D_3D_4)'}{D_2D_3D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \tag{41}$$

*where* $\varepsilon_c = \pm 1$

**Proof.** For $c$ root of multiplicity $\nu \geq 2$ of $D_4$

$$((x-c)^\nu\theta - \frac{\nu}{2}(x-c)^{\nu-1})^2 = ((x-c)^\nu(F)^{\frac{1}{2}})^2 + o((x-c)^{\nu-1})$$

Thus :

$$(x-c)^\nu\theta - \frac{\nu}{2}(x-c)^{\nu-1} = \varepsilon_c(x-c)^\nu(F)^{\frac{1}{2}} + o((x-c)^{\nu-1})$$

$$\theta - \frac{\nu}{2(x-c)} = \varepsilon_c(F)^{\frac{1}{2}} + o(\frac{1}{x-c})$$

Polar part of $\theta$, associated in root $c$ of $(D_4)$, minus $\frac{\nu}{2(x-c)}$, is in sign meadows that of $(F)^{\frac{1}{2}}$.
For $c$ root of $D_3$ :

$$[(residue\ \ of\ \ \theta\ at\ c) - \frac{1}{2}]^2 = \lim_{x\to c}((x-c)(F)^{\frac{1}{2}})^2$$

Thus:

$$[(residue\ \ of\ \theta\ at\ c) - \frac{1}{2}] = \varepsilon_c(residue\ \ of\ \ (F)^{\frac{1}{2}}\ at\ c)$$

Polar part of $\theta$, associated in root $c$ of $(D_3)$, minus $\frac{1}{2(x-c)}$, is in sign meadows that of $(F)^{\frac{1}{2}}$.
Thus :

$$\theta = E(\theta) + \sum_{c\in Root(D_3)\cup Root(D_4)} \varepsilon_c(partial\ fraction\ of (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2}\frac{(D_2D_3D_4)'}{D_2D_3D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}$$

■

**Proposition 14** *: We assume:* $d^o r < 0$ *and* $\begin{cases} d^o D_3 \neq 0 \\ d^o D_4 \neq 0 \end{cases}$ *or* $\begin{cases} d^o D_3 \geq 2 \\ d^o D_4 = 0 \end{cases}$

*We have:*

$$\theta = \sum_{c \in Root(D_3) \cup Root(D_4)} \varepsilon_c(partial\ fraction\ of (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2}\frac{(D_2D_3D_4)'}{D_2D_3D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \quad (42)$$

*where* $D_0$ *polynomial of degree :*

$$[\sum_{c \in Roots(D_3) \cup Roots(D_4)} \varepsilon_c\ (residue\ \ of\ ((F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2} d^o(D_2D_3D_4) + d^o D_1 + d^o D_0 - \frac{1}{2}]^2 = \lim_{x \to \infty} x^2 r + \frac{1}{4} \quad (43)$$

**Proof.** We have:

$$\lim_{x \to \infty} x\theta = \sum_{c \in Roots(D_3) \cup Roots(D_4)} \varepsilon_c(residue \quad of\ ((F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2} d^o(D_2D_3D_4) + d^o D_1 + d^o D_0$$

and $\lim_{x \to \infty} x^2 r + \frac{1}{4} = (\lim_{x \to \infty} x\theta - \frac{1}{2})^2$. Thus :

$$[\sum_{c \in Roots(D_3) \cup Zéros(D_4)} \varepsilon_c\ residue\ of\ (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2} d^o(D_2D_3D_4) + d^o D_1 + d^o D_0 - \frac{1}{2}]^2 = \lim_{x \to \infty} x^2 r + \frac{1}{4}$$

■

**Proposition 15** *: We assume* $d^o r = 0$*. We have:*

$$\theta = E(\theta) + \sum_{c \in Root(D_3) \cup Root(D_4)} \varepsilon_c(partial\ fraction\ of (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2}\frac{(D_2D_3D_4)'}{D_2D_3D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \quad (44)$$

*where* $E^2(\theta) = E(r)$ *and* $D_0$ *polynomial of degree :*

$$d^o D_0 = \frac{1}{2E(\theta)} \lim_{x \to \infty} x[r - E(r)] - [\sum_{c \in R(D_3) \cup R(D_4)} \varepsilon_c(residue\ of\ (F)^{\frac{1}{2}} at\ c) + \frac{1}{2} d^o(D_2D_3D_4) + d^o D_1] \quad (45)$$

**Proof.**

$$E^2(\theta) = E(r) \ \ and \ \ 2E(\theta) \sum_c \ residue\ of\ \theta\ at\ c = \sum_c \ residue\ of\ r\ at\ c$$

Thus:

$$2E(\theta)[\sum_{c \in R(D_3) \cup R(D_4)} \varepsilon_c(residue\ of\ (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2} d^o(D_2D_3D_4) + d^o D_1 + d^o D_0] = \lim_{x \to \infty} x(r - E(r))$$

■

**Proposition 16** : *We assume $d^o r = 2\nu > 0$. We have:*

$$\theta = E(\theta) + \sum_{c \in Root(D_3) \cup Root(D_4)} \varepsilon_c (partial\ fraction\ of (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2}\frac{(D_2D_3D_4)'}{D_2D_3D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \quad (46)$$

*where $D_0$ polynomial of degree :*

$$d^o D_0 = \alpha s_{\nu+1} - \frac{\nu}{2} - [\sum_{c \in Roots(D_3) \cup Roots(D_4)} \varepsilon_c (residue\ of\ (F)^{\frac{1}{2}} at\ c) + \frac{1}{2} d^o (D_2D_3D_4) + d^o D_1] \quad (47)$$

**Proof.** Let $a$ dominant coefficient of $E(r)$, and we have :
$\alpha s_{\nu+1} - \frac{\nu}{2} =$ sum of residues of $\theta$.

$$\theta = E(\theta) + \sum_{c \in Z\acute{e}ros(D_3) \cup Z\acute{e}ros(D_4)} \varepsilon_c (partial\ fraction\ of\ (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2}\frac{(D_2D_3D_4)'}{D_2D_3D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}$$

Thus :

$$\alpha s_{\nu+1} - \frac{\nu}{2} = [\sum_{c \in R(D_3) \cup R(D_4)} \varepsilon_c\ (residue\ of\ (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2} d^o (D_2D_3D_4) + d^o D_1 + d^o D_0]$$

Thus:

$$d^o D_0 = \alpha s_{\nu+1} - \frac{\nu}{2} - [\sum_{c \in R(D_3) \cup R(D_4)} \varepsilon_c\ (residue\ of\ (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2} d^o (D_2D_3D_4) + d^o D_1]$$

■

### 3.4 Case $d^o D_3 = 0$ and $d^o D_4 \neq 0$

$D_1D_2^2$ and $D_4$ polynomials relatively prime. Thus there are two only polynomials $N_1$ and $N_2$ such as :

$$\begin{cases} D_1D_2^2D_3^2D_4^2(r - E(r)) = N_1(D_1D_2^2) + N_2D_4 \\ d^o N_1 \quad < \quad d^o D_4 \end{cases}$$

We have :

$$r = E(r) + \frac{N_1}{D_4^2} + \frac{N_2}{D_1D_2^2D_4} \quad (48)$$

$$\lim_{x \to \infty} x^2 \frac{N_1}{D_4^2} = 0 \quad (49)$$

We consider the rational fraction :

$$F = E(r) + \frac{N_1}{D_4^2} - \frac{1}{4d^o D_4}(\frac{D_4'}{D_4})' \quad (50)$$

**Proposition 17** : *We assume* $\begin{cases} d^o D_3 = 0 \\ d^o D_4 \neq 0 \end{cases}$

1. *If : $d^o r \geq 0$ then : $d^o F = d^o r$ and $E(F) = E(r)$*

*2. If : $d^or < 0$ then : $d^oF = -2$ where $\lim_{x\to\infty} x^2F = \frac{1}{4}$.*

*3. For all $c$ root of $D_4$ of multiplicity $\nu$ we have:*

$$(x-c)^{2\nu}r = (x-c)^{2\nu}F + o((x-c)^{\nu-1})$$

**Proof.**

1. $r - F = \frac{N_2}{D_1D_2^2D_4} + \frac{1}{4d^oD_4}(\frac{D_4'}{D_4})'$ is from negative degree.

2.

$$\begin{aligned}\lim_{x\to\infty} x^2F &= \lim_{x\to\infty}\frac{N_1}{D_4^2} - \lim_{x\to\infty} x^2\frac{1}{4d^oD_4}(\frac{D_4'}{D_4})' \\ &= -\lim_{x\to\infty} x^2\frac{1}{4d^oD_4}(\frac{D_4'}{D_4})' = \frac{1}{4}\end{aligned}$$

3. Let $c$ root of $D_4$ :

$$\begin{aligned}(x-c)^{2\nu}(r-F) &= \tfrac{N_2}{D_1D_2^2}\tfrac{(x-c)^{2\nu}}{D_4} + (x-c)^{2\nu}\tfrac{1}{4d^oD_4}(\tfrac{D_4'}{D_4})' \\ &= o((x-c)^{\nu-1})\end{aligned}$$

■

**Notation:** We choose a square root of the polynomial of even degree : $N_1 + D_4^2[E(r) - \frac{1}{4d^oD_4}(\frac{D_4'}{D_4})']$ on a connected open at infinity, root of $D_4$. We put :

$$(F)^{\frac{1}{2}} = \frac{1}{D_4}(N_1 + D_4^2[E(r) - \frac{1}{4d^oD_4}(\frac{D_4'}{D_4})'])^{\frac{1}{2}} \quad (51)$$

**Proposition 18** *: We assume : $d^or < 0$. We have :*

$$\theta = \sum_{c\in Roots(D_4)} \varepsilon_c(partial\ \ fraction\ \ of\ \ (F)^{\frac{1}{2}}\ \ at\ \ c) + \frac{1}{2}\frac{(D_2D_4)'}{D_2D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \quad (52)$$

*where $\varepsilon_c = \pm 1$ and $D_0$ polynomial of degree :*

$$[\sum_{c\in Root(D_4)} \varepsilon_c\ (residue\ \ of\ \ ((F)^{\frac{1}{2}}\ \ at\ \ c) + \frac{1}{2}d^o(D_2D_4) + d^oD_1 + d^oD_0 - \frac{1}{2}]^2 = \lim_{x\to\infty} x^2r + \frac{1}{4} \quad (53)$$

**Proof.**

$$\lim_{x\to\infty} x\theta = \sum_{c\in Roots(D_4)} \varepsilon_c(residue\ \ of\ \ ((F)^{\frac{1}{2}}\ \ at\ \ c) + \frac{1}{2}d^o(D_2D_4) + d^oD_1 + d^oD_0$$

$\lim_{x\to\infty} x^2r + \frac{1}{4} = (\lim_{x\to\infty} x\theta - \frac{1}{2})^2$. Thus :

$$[\sum_{c\in Roots(D_4)} \varepsilon_c\ residue\ \ of\ \ (F)^{\frac{1}{2}}\ \ at\ \ c) + \frac{1}{2}d^o(D_2D_4) + d^oD_1 + d^oD_0 - \frac{1}{2}]^2 = \lim_{x\to\infty} x^2r + \frac{1}{4}$$

■

**Proposition 19** *: We assume $d^o r = 0$. We have :*

$$\theta = E(\theta) + \sum_{c \in Roots(D_4)} \varepsilon_c (partial\ fraction\ of\ (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2}\frac{(D_2 D_4)'}{D_2 D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \tag{54}$$

*where $\varepsilon_c = \pm 1$ and $D_0$ polynomial of degree :*

$$d^o D_0 = \frac{1}{2E(\theta)} \lim_{x \to \infty} x[r - E(r)] - [\sum_{c \in Roots(D_4)} \varepsilon_c\ (residue\ \ of\ \ (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2} d^o (D_2 D_4) + d^o D_1] \tag{55}$$

**Proof.**

$$E^2(\theta) = E(r)\ \ and\ \ 2E(\theta) \sum_c\ residue\ \ of\ \ \theta\ at\ c = \sum_c\ residue\ \ of\ \ r\ at\ c$$

Thus :

$$2E(\theta)[\sum_{c \in Roots(D_4)} \varepsilon_c\ (residue\ \ of\ \ (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2} d^o (D_2 D_4) + d^o D_1 + d^o D_0] = \lim_{x \to \infty} x(r - E(r))$$

■

**Proposition 20** *We assume $d^o r = 2\nu > 0$. We have :*

$$\theta = E(\theta) + \sum_{c \in Roots(D_4)} \varepsilon_c (partial\ fraction\ of\ (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2}\frac{(D_2 D_4)'}{D_2 D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0} \tag{56}$$

*where $\varepsilon_c = \pm 1$ and $D_0$ polynomial of degree :*

$$d^o D_0 = \alpha s_{\nu+1} - \frac{\nu}{2} - [\sum_{c \in Roots(D_4)} \varepsilon_c\ (residue\ \ of\ \ (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2} d^o (D_2 D_4) + d^o D_1] \tag{57}$$

**Proof.** Let $a$ dominant coefficient of $E(r)$. $E(\theta) = \alpha[t^{-\nu} + s_1 t^{-(\nu-1)} + .. + s_\nu]$. $\alpha s_{\nu+1} - \frac{\nu}{2} =$ *sum residue of* $\theta$, where $\alpha^2 = a$. Thus :

$$\theta = E(\theta) + \sum_{c \in Roots(D_4)} \varepsilon_c (partial\ fraction\ of\ (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2}\frac{(D_2 D_4)'}{D_2 D_4} + \frac{D_1'}{D_1} + \frac{D_0'}{D_0}$$

Thus:

$$\alpha s_{\nu+1} - \frac{\nu}{2} = [\sum_{c \in Roots(D_4)} \varepsilon_c\ (residue\ \ of\ \ (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2} d^o (D_2 D_4) + d^o D_1 + d^o D_0]$$

Thus :

$$d^o D_0 = \alpha s_{\nu+1} - \frac{\nu}{2} - [\sum_{c \in Roots(D_4)} \varepsilon_c\ (residue\ \ of\ \ (F)^{\frac{1}{2}}\ at\ c) + \frac{1}{2} d^o (D_2 D_4) + d^o D_1]$$

■

# 4 Recurrent method at infinity

## 4.1 Presentation of $D_0$ as determinant:

**Proposition 21** *: Let : $c_1, \ldots, c_m$ complexes constants . We put :*

$$\begin{aligned} P(x) &= (x-c_1)\ldots(x-c_m) \\ &= x^m - p_1x^{m-1} + p_2x^{m-2} - \ldots + (-1)^m p_m \end{aligned} \tag{58}$$

*For $j \in \mathbb{N}$, we put : $\sigma_j = c_1^j + \ldots + c_m^j$ and* $\Delta = \begin{vmatrix} 1 & c_1 & . & . & c_1^{m-1} \\ 1 & c_2 & . & . & c_2^{m-1} \\ . & & & & . \\ . & & & & . \\ 1 & c_m & . & . & c_m^{m-1} \end{vmatrix}$

*1.*

$$\Delta^2 = \begin{vmatrix} \sigma_0 & \sigma_1 & . & . & \sigma_{m-1} \\ \sigma_1 & \sigma_2 & . & . & \sigma_m \\ . & & & & . \\ . & & & & . \\ \sigma_{m-1} & \sigma_m & . & . & \sigma_{2m-2} \end{vmatrix} \tag{59}$$

*2.*

$$\Delta^2 P(x) = \begin{vmatrix} \sigma_0 & . & . & \sigma_{m-1} & 1 \\ . & & & . & x \\ . & & & . & . \\ . & & & . & . \\ \sigma_m & . & . & \sigma_{2m-1} & x^m \end{vmatrix} \tag{60}$$

**Proof.** $\Delta$, $\begin{vmatrix} \sigma_0 & \sigma_1 & . & . & \sigma_{m-1} \\ \sigma_1 & \sigma_2 & . & . & \sigma_m \\ . & & & & . \\ . & & & & . \\ \sigma_{m-1} & \sigma_m & . & . & \sigma_{2m-2} \end{vmatrix}$ polynomials at $c_1, \ldots, c_m$ with real coefficients. Thus, $\Delta^2 P(x)$ and $\begin{vmatrix} \sigma_0 & . & . & \sigma_{m-1} & 1 \\ . & & & . & x \\ . & & & . & . \\ . & & & . & . \\ \sigma_m & . & . & \sigma_{2m-1} & x^m \end{vmatrix}$ polynomials at $c_1, \ldots, c_m, x$ with real coefficients. Thus to have both identities we can assume $c_1, \ldots, c_m, x$ reals. Furthermore, we can content themselves with the open of Zariski $c_1, \ldots, c_m, x$ distinct real non-zero.
We have: $\forall j = 1, \ldots, m, \quad c_j^m = p_1c_j^{m-1} - p_2c_j^{m-2} - \ldots - (-1)^m p_m$.

We put:

$$\Delta(x) = \begin{vmatrix} 1 & c_1 & . & . & c_1^{m-1} & c_1^m \\ 1 & c_2 & . & . & c_2^{m-1} & c_2^m \\ . & & & & & . \\ . & & & & . & . \\ 1 & c_m & . & . & c_m^{m-1} & c_m^m \\ 1 & x & . & . & x^{m-1} & x^m \end{vmatrix}$$

$$\Delta(x) = \begin{vmatrix} 1 & c_1 & . & . & c_1^{m-1} & 0 \\ 1 & c_2 & . & . & c_2^{m-1} & 0 \\ . & & & & . & . \\ . & & & & . & . \\ 1 & c_m & . & . & c_m^{m-1} & 0 \\ 1 & x & . & . & x^{m-1} & P(x) \end{vmatrix} = \Delta P(x)$$

We put:

$$v_j \;=\; \begin{pmatrix} c_1^{j-1} \\ . \\ . \\ c_m^{j-1} \end{pmatrix}, \quad pour\ j = 1, \ldots, m$$

The scalar product:

$$< v_j,\, v_k > = \; c_1^{j+k-2} + \ldots + c_m^{j+k-2} \; = \; \sigma_{j+k-2}$$

Matrix of Gram of $v_1, \ldots, v_m$ is:

$$G = \begin{pmatrix} \sigma_0 & \sigma_1 & . & . & \sigma_{m-1} \\ \sigma_1 & \sigma_2 & . & . & \sigma_m \\ . & & & & . \\ . & & & & . \\ \sigma_{m-1} & \sigma_m & . & . & \sigma_{2m-2} \end{pmatrix}$$

Thus :

$$\Delta^2 = \begin{vmatrix} \sigma_0 & \sigma_1 & . & . & \sigma_{m-1} \\ \sigma_1 & \sigma_2 & . & . & \sigma_m \\ . & & & & . \\ . & & & & . \\ \sigma_{m-1} & \sigma_m & . & . & \sigma_{2m-2} \end{vmatrix}$$

We put : $v_j(x) = \begin{pmatrix} c_1^{j-1} \\ . \\ . \\ c_m^{j-1} \\ x^{j-1} \end{pmatrix}, \quad j = 1, \ldots, m+1.$ We notice:

$$< v_j(x),\, v_k(x) > = < v_j,\, v_k > + x^{j+k-2} \; = \; \sigma_{j+k-2} + x^{j+k-2}$$

The j-th column of matrix of Gram of matrix defines $\Delta(x)$ is :

$$\begin{pmatrix} \sigma_{j-1} \\ \sigma_j \\ . \\ . \\ \sigma_{j+m-1} \end{pmatrix} + x^{j-1} \begin{pmatrix} 1 \\ x \\ . \\ . \\ x^m \end{pmatrix}$$

Thus :

$$\begin{aligned} \Delta^2(x) &= \begin{vmatrix} \sigma_0 & \sigma_1 & . & . & \sigma_m \\ \sigma_1 & \sigma_2 & . & . & \sigma_{m+1} \\ . & & & & . \\ . & & & & . \\ \sigma_m & \sigma_{m+1} & . & . & \sigma_{2m} \end{vmatrix} + \begin{vmatrix} 1 & \sigma_1 & . & . & \sigma_m \\ x & \sigma_2 & . & . & \sigma_{m+1} \\ . & & & & . \\ . & & & & . \\ x^m & \sigma_{m+1} & . & . & \sigma_{2m} \end{vmatrix} + \ldots + \begin{vmatrix} \sigma_0 & . & \sigma_{m-1} & x^m \\ \sigma_1 & . & \sigma_m & x^{m+1} \\ . & & & . \\ . & & & . \\ \sigma_m & . & \sigma_{2m-1} & x^{2m} \end{vmatrix} \\ &= \sum_{i,j} x^{i+j-2} cofactor_{i,j}(M) \end{aligned}$$

where

$$M = \begin{pmatrix} \sigma_0 & \sigma_1 & . & . & \sigma_m \\ \sigma_1 & \sigma_2 & . & . & \sigma_{m+1} \\ . & & & & . \\ . & & & & . \\ \sigma_m & \sigma_{m+1} & . & . & \sigma_{2m} \end{pmatrix}$$

We obtain : $\Delta^2(x) = \Delta^2 P^2(x) = (1\ x \ldots x^m) Com(M) \begin{pmatrix} 1 \\ x \\ . \\ . \\ x^m \end{pmatrix}.$

The cofactor $(m+1, m+1)$ of $M$ is $\Delta^2$. The adjoint of $M$ is non-zero.
We prove that adjoint of $M$ of rank 1.
$\det(M) = 0$. Because : $\sigma_m = p_1\sigma_{m-1} - p_2\sigma_{m-2} - \ldots - (-1)^m p_m \sigma_0$
et $\forall k \geq m$; $\sigma_k = p_1\sigma_{k-1} - p_2\sigma_{k-2} - \ldots - (-1)^m p_m \sigma_{k-m}$
Thus, the (m+1)-th column of $M$ is is a linear combination of other column.

We consider the matrix: $E_{1,1} = \begin{pmatrix} 1 & 0 & . & 0 \\ 0 & & & . \\ . & & & . \\ 0 & . & . & 0 \end{pmatrix}.$

$$M + E_{1,1} = \begin{pmatrix} \sigma_0 + 1 & \sigma_1 & . & . & \sigma_m \\ \sigma_1 & \sigma_2 & . & . & \sigma_{m+1} \\ . & & & & . \\ . & & & & . \\ \sigma_m & \sigma_{m+1} & . & . & \sigma_{2m} \end{pmatrix} \quad and$$

$$\det(M+E_{1,1}) \;=\; \det(M)+\begin{vmatrix} 1 & \sigma_1 & . & . & \sigma_m \\ 0 & \sigma_2 & . & . & \sigma_{m+1} \\ . & & & & . \\ . & & & & . \\ 0 & \sigma_{m+1} & . & . & \sigma_{2m} \end{vmatrix} = \begin{vmatrix} \sigma_2 & \sigma_3 & . & . & \sigma_{m+1} \\ \sigma_3 & \sigma_4 & . & . & \sigma_{m+2} \\ . & & & & . \\ . & & & & . \\ \sigma_{m+1} & \sigma_{m+2} & . & . & \sigma_{2m} \end{vmatrix}$$

It is the determinant of Gram of $\begin{pmatrix} c_1 \\ . \\ . \\ c_m \end{pmatrix}, \ldots, \begin{pmatrix} c_1^m \\ . \\ . \\ c_m^m \end{pmatrix}$.

Thus:

$$\det(M+E_{1,1}) \;=\; \begin{vmatrix} c_1 & . & . & . & c_1^m \\ . & & . & . & . \\ . & & & & . \\ . & & & & . \\ c_m & . & . & . & c_m^m \end{vmatrix}^2$$

But:

$$\begin{vmatrix} c_1 & . & . & . & c_1^m \\ . & & . & . & . \\ . & & & & . \\ . & & & & . \\ c_m & . & . & . & c_m^m \end{vmatrix} = (-1)^{m-1}p_m \begin{vmatrix} c_1 & . & . & . & c_1^{m-1} & 1 \\ . & & . & . & . & 1 \\ . & & & & . & \\ . & & & & . & \\ c_m & . & . & . & c_m^{m-1} & 1 \end{vmatrix} = p_m\Delta$$

Thus : $\det(M+E_{1,1}) = p_m^2\Delta^2$. Or $p_m = c_1 ... c_m$ is scalar non-zero, $M+E_{1,1}$ is invertible matrix. $M$ symmetric matrix, $Com(M)$ symmetric matrix, we have :

$$MCom(M) = Com(M)M = \det(M)I = 0; \; I \; matrice \; identité$$

Thus:

$$(M+E_{1,1})Com(M) \;=\; E_{1,1}Com(M)$$

Or $M+E_{1,1}$ is invertible and $Com(M) \neq 0$, $Com(M)$ of rank 1.
Thus :

$$cofactor_{i,j}(M) \;=\; \frac{cofactor_{i,m+1}(M)}{cofactor_{m+1,m+1}(M)} cofactor_{m+1,j}(M)$$

For $i = 1, \ldots, m+1$, $j = 1, \ldots, m+1$, we put : $C_{i,j} = cofactor_{i,j}(M)$.

$$\frac{C_{i,j}}{\Delta^2} = \frac{C_{i,m+1}}{\Delta^2} \, \frac{C_{m+1,j}}{\Delta^2}$$

Thus:

$$\begin{aligned} P(x)^2 \;&=\; \sum_{1\le i,j\le m+1} \frac{C_{i,j}}{\Delta^2} \, x^{i+j-2} \\ &=\; \sum_{1\le i,j\le m+1} \frac{C_{i,m+1}}{\Delta^2} \, \frac{C_{j,m+1}}{\Delta^2} \, x^{i+j-2} \\ &=\; \Big(\sum_{1\le i\le m+1} \frac{C_{i,m+1}}{\Delta^2} \, x^{i-1}\Big)^2 \end{aligned}$$

Or $P$ is monic polynomial and : $\frac{C_{m+1,m+1}}{\Delta^2} = 1$

$$P(x) = \sum_i \frac{C_{i,m+1}}{\Delta^2}\ x^{i-1} = \frac{1}{\Delta^2}\begin{vmatrix} \sigma_0 & . & . & . & \sigma_{m-1} & 1 \\ \sigma_1 & . & . & . & \sigma_m & x \\ . & & & & . & . \\ . & & & & . & . \\ \sigma_m & . & . & . & \sigma_{2m-1} & x^m \end{vmatrix}$$

We obtain :

$$\Delta^2 P(x) = \begin{vmatrix} \sigma_0 & . & . & . & \sigma_{m-1} & 1 \\ \sigma_1 & . & . & . & \sigma_m & x \\ . & & & & . & . \\ . & & & & . & . \\ \sigma_m & . & . & . & \sigma_{2m-1} & x^m \end{vmatrix}$$

■

### 4.2 Taylor expansion of $\frac{P'}{P}$

**Proposition 22** *: Let $c_1, \ldots, c_m$ are pairwise distinct constant.*
*We put, for $j = 0, \ldots, m$: $\sigma_j = c_1^j + \ldots + c_m^j$ and $P(x) = (x - c_1)...(x - c_m)$*

*1.*

$$(\frac{P'}{P})' + (\frac{P'}{P})^2 = 2\sum_{i \neq j} \frac{1}{c_i - c_j}\ \ \frac{1}{x - c_i} \tag{61}$$

*2. We put : $t = \frac{1}{x}$. Taylor expansion on the neighborhood of infinity :*

*(a)*

$$(\frac{P'}{P})' + (\frac{P'}{P})^2 = \sum_{l=2}^{\infty}[(\sum_{\mu=0}^{l-2} \sigma_\mu \sigma_{l-2-\mu}) - (l-1)\sigma_{l-2}]t^l \tag{62}$$

*(b)*

$$x\frac{P'}{P} = \sum_{l=0}^{\infty} \sigma_l t^l \tag{63}$$

**Proof.** We have : $\frac{P'}{P} = \sum_{i=1}^{m} \frac{1}{x - c_i}$

1.

$$\begin{aligned} (\tfrac{P'}{P})' + (\tfrac{P'}{P})^2 &= \sum_{i \neq j} \frac{1}{x - c_i}\frac{1}{x - c_j} \\ &= \sum_{i \neq j}[\frac{\frac{1}{c_i - c_j}}{x - c_i} + \frac{\frac{1}{c_j - c_i}}{x - c_j}] \\ &= 2\sum_{i \neq j} \frac{1}{c_i - c_j}\frac{1}{x - c_i} \end{aligned}$$

2. We put: $t = \frac{1}{x}$. We have :

$$\begin{aligned}(\tfrac{P'}{P})' + (\tfrac{P'}{P})^2 &= 2t\sum_{i\neq j}\frac{1}{c_i - c_j}\frac{1}{1 - c_i t} \\ &= 2t\sum_{i\neq j}\frac{1}{c_i - c_j}\sum_{k=0}^{\infty}(c_i t)^k \\ &= \sum_{k=0}^{\infty}(2\sum_{i\neq j}\frac{c_i^k}{c_i - c_j})t^{k+1} \\ &= \sum_{k=1}^{\infty}\sum_{i\neq j}\frac{c_i^k - c_j^k}{c_i - c_j}t^{k+1}\end{aligned}$$

As :

$$\begin{aligned}\sum_{i\neq j}\frac{c_i^k - c_j^k}{c_i - c_j} &= \sum_{i\neq j}\sum_{\mu+\nu=k-1, 0\leq\mu\leq k-1} c_i^{\mu}c_j^{\nu} \\ &= \sum_{\mu+\nu=k-1,\ 0\leq\mu\leq k-1}[\sigma_\mu\sigma_\nu - \sum_{i=1}^{m}c_i^{\mu+\nu}] \\ &= [\sum_{\mu+\nu=k-1,\ 0\leq\mu\leq k-1}\sigma_\mu\sigma_\nu] - k\sigma_{k-1}\end{aligned}$$

Thus :

$$\begin{aligned}(\tfrac{P'}{P})' + (\tfrac{P'}{P})^2 &= \sum_{k=1}^{\infty}[(\sum_{\mu=0}^{k-1}\sigma_\mu\sigma_{k-1-\mu}) - k\sigma_{k-1}]t^l \\ &= \sum_{l=2}^{\infty}[(\sum_{\mu=0}^{l-2}\sigma_\mu\sigma_{l-2-\mu}) - (l-1)\sigma_{l-2}]t^l\end{aligned}$$

■

### 4.3 Research of $D_0$

Let $r = \frac{N}{D_1 D_2^2 D_3^2 D_4^2}$ rational fraction and $\theta \in \mathbb{C}(x)$ where : $\theta' + \theta^2 = r$.
We put :

$$\theta = S + \frac{D_0'}{D_0} \tag{64}$$

where $S = E(\theta) + \sum_{c\ poles\ of\ r}\theta_c$ and $D_0$ monic polynomial of degree $m$. We have :

$$\theta' + \theta^2 = S' + S^2 + (\frac{D_0'}{D_0})' + (\frac{D_0'}{D_0})^2 + 2S\frac{D_0'}{D_0} \tag{65}$$

$$\theta' + \theta^2 = r \Leftrightarrow (\frac{D_0'}{D_0})' + (\frac{D_0'}{D_0})^2 + 2S\frac{D_0'}{D_0} = \frac{R}{B} \tag{66}$$

with $B = D_1D_2D_3D_4$ and $R$ polynomials as: $r - S' - S^2 = \frac{R}{B}$.

**case 1:** $d^o r = 2\nu \geq 0$

We have: $d^o S = \nu$ and $d^o \frac{R}{B} = \nu - 1$. Taylors expansion of $\frac{1}{x^\nu}S$, $\frac{1}{x^{\nu-1}}\frac{R}{B}$ and $x\frac{D_0'}{D_0}$ at order $\mu$.
We obtain the Taylors expansion of $\frac{1}{x^{\nu-1}}[\frac{R}{B} - 2S\frac{D_0'}{D_0}]$ at order $\mu$.
The Taylors expansion equal the Taylors expansion of : $\frac{1}{x^{\nu-1}}[(\frac{D_0'}{D_0})' + (\frac{D_0'}{D_0})^2]$
Let $t = \frac{1}{x}$. In Taylors expansion of $\frac{1}{x^{\nu-1}}[(\frac{P'}{P})' + (\frac{P'}{P})^2]$ [see proposition 23] the coefficient of $t^{\nu+1}$ is : $\sigma_0^2 - \sigma_0 = m^2 - m$. For $l \geq 2$, coefficient of $t^{l+\nu-1}$ use : $\sigma_0, \ldots, \sigma_{l-2}$.
Besides, we have :

$$\begin{aligned} x\frac{D_0'}{D_0} &= \sum_{i=1}^{m}\frac{1}{1-c_it} \\ &= \sum_{i=1}^{m}\sum_{k=0}^{\infty} c_i^k t^k \\ &= \sum_{k=0}^{\infty}\sigma_k t^k \end{aligned} \tag{67}$$

Thus, Taylors expansion of :

$$\Phi = \frac{1}{x^{\nu-1}}[(\frac{D_0'}{D_0})' + (\frac{D_0'}{D_0})^2] - \frac{1}{x^{\nu-1}}[\frac{R}{B} - 2S\frac{D_0'}{D_0}] \tag{68}$$

and the coefficient of $t^k$ is:

$$-2\alpha\sigma_k + polynomial\ at\ \sigma_0, \ldots, \sigma_{k-1} \tag{69}$$

$\Phi = 0$ determine $\sigma_k$ by recurrence at $k = \nu$.

**case 2 :** $d^o r < 0$

The coefficient of $t^k$ in Taylors expansion of :

$$\Psi = x^2[(\frac{D_0'}{D_0})' + (\frac{D_0'}{D_0})^2] - x^2\frac{R}{B} + 2xS(x\frac{D_0'}{D_0}) \tag{70}$$

is :

$$2m\sigma_k - (k+1)\sigma_k + (2\lim_{x\to\infty} xS)\sigma_k + polynomial\ at\ \sigma_0, \ldots, \sigma_{k-1} \tag{71}$$

By recurrence $\sigma_k$ at condition :

$$k \neq 2\lim_{x\to\infty} xS + 2m - 1$$

We have : $\lim_{x\to\infty} xS + m = \alpha_\infty^2 + \dfrac{1}{2}$ with $\alpha_\infty^2 = \lim_{x\to\infty} x^2 r + \dfrac{1}{4}$.
For $k \neq 2\alpha_\infty$, to be $\sigma_k$ at function of $\sigma_0, \ldots, \sigma_{k-1}$.
If $2\alpha_\infty \in \mathbb{N}$ then coefficient of $t^{2\alpha_\infty}$ non-zero thus is no solution or is zero thus $\sigma_{2\alpha_\infty}$ is arbitrary and $\sigma_k$; $k > 2\alpha_\infty$ depend in a unique way.
$D_0$ is polynomial determined by $\sigma_0, \ldots, \sigma_{2m-1}$ by determinant formulae, the problem of non-unique suite $(\sigma_k)_{k\in\mathbb{N}}$ put if $2\alpha_\infty$ is positive integer equal to or less than $2m-1$, then $4\lim_{x\to\infty} x^2 r + 1$ is square of integer and :

$$\lim_{x\to\infty} x^2 r \leq m^2 - m \tag{72}$$

**Example 23** *In this example we consider the Riccati differential equation (2) where:*

$$r = -1 + \frac{z^2 - \frac{1}{4}}{x^2};\ z \in \mathbb{R}^* \tag{73}$$

*We have* $D_1 = 1,\ D_2 = 1,\ D_3 = x$ *and* $E^2(\theta) = -1$.
*We can assume* $E(\theta) = i$.

$$\lim_{x \to 0} x^2 r + \frac{1}{4} = z^2 = \alpha_0^2$$

$$\theta = i + \frac{\alpha_0 + \frac{1}{2}}{x} + \frac{D_0'}{D_0}$$

$2i(\alpha_0 + \frac{1}{2} + d^o D_0) = 0$. *Thus :* $m = d^o D_0 = -\alpha_0 - \frac{1}{2} \in \mathbb{N}$
*Thus :* $\alpha_0 = -m - \frac{1}{2}$, $r = -1 + \frac{(m+\frac{1}{2})^2 - \frac{1}{4}}{x^2} = -1 + \frac{m^2+m}{x^2}$ *and* $\theta = i - \frac{m}{x} + \frac{D_0'}{D_0}$.
$\theta$ *solution of Riccati equation if and only if* $D_0$ *verify :*

$$(\frac{D_0'}{D_0})' + (\frac{D_0'}{D_0})^2 + 2(i - \frac{m}{x})\frac{D_0'}{D_0} = 2i\frac{m}{x}$$

*Thus* $\Phi = 0$ *where*

$$\Phi = x[(\frac{D_0'}{D_0})' + (\frac{D_0'}{D_0})^2] + 2(i - \frac{m}{x})(x\frac{D_0'}{D_0}) - 2im$$

*Taylors expansion of* $\Phi = 0$ *at* $t = \frac{1}{x}$ *:*

$$\Phi = \sum_{k=1}^{\infty}[(\sum_{\mu=1}^{k-1} \sigma_\mu \sigma_{k-1-\mu}) - k\sigma_{k-1} - 2m\sigma_{k-1} + 2i\sigma_k]t^k$$

*As* $\sigma_0 = m$. *Thus:*

$$\Phi = 0 \Leftrightarrow \forall k \geq 1,\ (\sum_{\mu=0}^{k-1} \sigma_\mu \sigma_{k-1-\mu}) - k\sigma_{k-1} - 2m\sigma_{k-1} + 2i\sigma_k = 0$$

*For* $k = 1$, $\sigma_0^2 - \sigma_0 - 2m\sigma_0 + 2i\sigma_1 = 0$ *equivalent to* $2i\sigma_1 = m^2 + m$.

*For all :* $k \geq 2,\ 2i\sigma_k = k\sigma_{k-1} - \sum_{\mu=1}^{k-2} \sigma_\mu \sigma_{k-1-\mu}$

*For example* $m = 3$, *:* $\begin{cases} 2i\sigma_1 = 12 \\ 2i\sigma_2 = 2\sigma_1 \\ 2i\sigma_3 = 3\sigma_2 - \sigma_1^2 \\ 2i\sigma_4 = 4\sigma_3 - 2\sigma_1\sigma_2 \\ 2i\sigma_5 = 5\sigma_4 - 2\sigma_1\sigma_3 - \sigma_2^2 \end{cases}$

*Thus , the polynomial* $D_0$ *partner to :*

$$\begin{vmatrix} \sigma_0 & \sigma_1 & \sigma_2 & 1 \\ \sigma_1 & \sigma_2 & \sigma_3 & x \\ \sigma_2 & \sigma_3 & \sigma_4 & x^2 \\ \sigma_3 & \sigma_4 & \sigma_5 & x^3 \end{vmatrix} = \begin{vmatrix} 3 & -6i & -6 & 1 \\ -6i & -6 & -9i & x \\ -6 & -9i & -54 & x^2 \\ -9i & -54 & 99i & x^3 \end{vmatrix} = 135x^3 + 810ix^2 - 2025x - 2025i$$

*Thus*

$$D_0 = x^3 + 6ix^2 - 15x - 15i$$

# 5 Method of last minor

Let $r = \frac{N}{D_1 D_2^2 D_3^2 D_4^2}$ rational fraction and $\theta \in \mathbb{C}(x)$ as : $\theta' + \theta^2 = r$.
$\theta$ solution of Eq (2) if and only if $D_0$ verify :

$$BD_0'' + 2SBD_0' = RD_0 \tag{74}$$

We choose a complex number $c$ not pole of $r$ and we use the expression of polynomials following the powers of $x - c$.
For the sake of simplicity, in the following we assume that $c = 0$. Constant coefficient constant of $B$ is non-zero.
Denote by $a_k$, $b_k$ and $r_k$ coefficients of $x^k$, in $A$, $B$ and $R$ respectively, $k \in \mathbb{N}$.
If $k \in \mathbb{Z}$ then : $a_k = b_k = r_k = 0$ .

## 5.1 Case $d^o r < 0$

We put : $S = \frac{A}{B}$
If $d^o B = 1$ then $r$ are pole unique, it is simple or double. If $c$ simple pole of $r$ then $r$ of degree $-1$ and Eq (2) has no rational solution.
If $c$ double pole of $r$ then : $A = \alpha_c + \frac{1}{2}$, $r = \frac{\alpha_c^2 - \frac{1}{4}}{(x-c)^2}$ and $R = 0$.
Thus, Eq (72) :

$$\frac{D_0''}{D_0'} = \frac{-2(\alpha_c + \frac{1}{2})}{x - c} \tag{75}$$

The solutions :

$$D_0' = (-2\alpha_c)(x - c)^{-2\alpha_c - 1} \tag{76}$$

$$D_0 = (x - c)^{-2\alpha_c} + \beta \tag{77}$$

where $\beta$ non-zero constant .
We have, an infinity of the other rational solution when $-2\alpha_c \in \mathbb{N}^*$, with the same parity of $S$.
We assume : $d^o B \geq 2$ and we look $D_0$ polynomial of degree $m \geq 1$ verify Ed (72).

**Proposition 24** : *We assume :* $d^o r < 0, \begin{cases} d^o B \geq 2 \\ d^o = m \geq 1 \end{cases}$ *and* $D_0$ *verify Ed* (72).

1. *We have:* $\begin{cases} d^o(BD_0'' + 2AD_0') \leq d^o B + m - 2 \\ d^o R \leq d^o B - 2 \end{cases}$
2. *We have:* $r_{d^o B - 2} = 2a_{d^o B - 1} m + m(m - 1)$
3. *We have:* $\begin{cases} a_{d^o B - 1} = \frac{1}{2} - m + \alpha_\infty \\ r_{d^o B - 2} = m(2\alpha_\infty - m) \end{cases}$

**Proof.**

1. $d^o(BD_0'' + 2AD_0') \leq \sup(d^o(BD_0''), d^o(2AD_0'))$
   If $d^o D_0 = 1$ then :

$$\sup(d^o(BD_0''), d^o(2AD_0')) = d^o(2AD_0') = d^o A \leq d^o B - 1 = d^o B + d^o D_0 - 2$$

If $d^oD_0 \geq 2$ then : $\begin{cases} d^o(BD_0'') = d^oB + m - 2 \\ d^o(2AD_0') = d^oA + m - 1 \leq d^oB + m - 2 \end{cases}$

2. We have : $d^o(RD_0) \leq d^oB + m - 2$ and coefficients of $x^{d^oB+m-2}$ in Eq (72).
3. Accordingly, in view of : $\lim\limits_{x\to\infty} x\theta = \alpha_\infty + \frac{1}{2}$.

■

We put :

$$\begin{aligned} v(x) &= -R\begin{pmatrix} 1 \\ x \\ x^2 \\ . \\ . \\ x^m \end{pmatrix} + 2A\begin{pmatrix} 0\ 1 \\ 2x \\ . \\ . \\ mx^{m-1} \end{pmatrix} + B\begin{pmatrix} 0 \\ 0 \\ 2 \\ . \\ . \\ m(m-1)x^{m-2} \end{pmatrix} \\ &= \begin{pmatrix} v_1(x) \\ v_2(x) \\ v_3(x) \\ . \\ . \\ v_{m+1}(x) \end{pmatrix} \end{aligned} \tag{78}$$

The $k^{th}$ element of $v(x)$ is:

$$v_k(x) = B(k-1)(k-2)x^{k-3} + 2A(k-1)x^{k-2} - Rx^{k-1} \tag{79}$$

$v_k(x)$ of degree equal to or less than $d^oB + k - 3$ and $x^{d^oB+k-3}$ of coefficient:

$$\begin{aligned} & (k-1)(k-2) + 2a_{d^oB-1}(k-1) - r_{d^oB-2} \\ = \ & (k-m-1)(\tfrac{r_{d^oB-2}}{m} + k - 1) \\ = \ & (k-m-1)(2a_{d^oB-1} + m + k - 2) \end{aligned} \tag{80}$$

For $k \leq m$, the coefficient of $x^{d^oB+k-3}$ is zero if and only if :

$$2a_{d^oB-1} = -(m+k-2) \in \{-(m-1), -m, \ldots, -2(m-1)\}$$

$v_{m+1}$ of degree equal to or less than $d^oB + m - 3$.
For $k \geq 3$ the coefficient of more low degree of $v_k$ is coefficient of: $x^{k-3}$ equal $b_0(k-1)(k-2)$ .
Let $V$ matrix of $m+1$ row and $k^{th}$ row is row $l_k$ of coefficients of $v_k(x)$ in basis of $\mathbb{C}_{d^oB+m-3}[X]$.
$l_3, \ldots, l_{m+1}$ linearly independent system.
The Eq (72) give :

$$D_0 = d_0 + d_1x + \ldots + x^m, \tag{81}$$

obtained :

$$d_0v_1(x) + \ldots + d_{m-1}v_m(x) + v_{m+1}(x) = 0 \tag{82}$$

$$d_0l_1 + \ldots + d_{m-1}l_m + l_{m+1} = 0 \tag{83}$$

Thus, the matrix$V$ is rank $m-1$, $m$ or $m+1$ .Accordingly, existence of $D_0$ correspondent linearly dependent in row $l_{m+1}$ of $l_1,\dots,l_m$ .
Let $c_1,\dots,c_m$ roots of $D_0$ distinct.

We put for $j\in\mathbb{N}$ : $\sigma_j = c_1^j+\dots+c_m^j$ et $\Delta^2 = \begin{vmatrix} \sigma_0 & \sigma_1 & . & . & \sigma_{m-1} \\ \sigma_1 & \sigma_2 & . & . & \sigma_m \\ . & & & & . \\ . & & & & . \\ \sigma_{m-1} & \sigma_m & . & . & \sigma_{2m-2} \end{vmatrix}$

The $m$ column vector $\begin{pmatrix} \sigma_0 \\ . \\ . \\ \sigma_m \end{pmatrix} \dots\dots \begin{pmatrix} \sigma_{m-1} \\ . \\ . \\ \sigma_{2m-1} \end{pmatrix}$, of $m+1$ column of are linearly independent.

The equation of hyperplane $H$ engendered by vectors $\begin{pmatrix} \sigma_0 \\ . \\ . \\ \sigma_m \end{pmatrix} \dots \begin{pmatrix} \sigma_{m-1} \\ . \\ . \\ \sigma_{2m-1} \end{pmatrix}$ is:

$$\begin{vmatrix} \sigma_0 & . & . & . & \sigma_{m-1} & x_1 \\ \sigma_1 & . & . & . & \sigma_m & x_2 \\ . & & & & . & . \\ . & & & & . & . \\ \sigma_m & . & . & . & \sigma_{2m-1} & x_{m+1} \end{vmatrix} = 0 \tag{84}$$

The coefficient of $x_{m+1}$ is $\Delta^2\neq 0$ and $H$ as equation:

$$x_{m+1} = \lambda_1 x_1 + \dots + \lambda_m x_m \tag{85}$$

where $\lambda_1\dots\lambda_m$ scalar satisfying :

$$(\sigma_m,\dots,\sigma_{2m-1}) = \lambda_1(\sigma_0,\dots,\sigma_{m-1}) + \dots + \lambda_m(\sigma_{m-1},\dots,\sigma_{2m-2}) \tag{86}$$

The derivative of $\begin{vmatrix} \sigma_0 & . & . & . & \sigma_{m-1} & 1 \\ \sigma_1 & . & . & . & \sigma_m & x \\ . & & & & . & . \\ . & & & & . & . \\ \sigma_m & . & . & . & \sigma_{2m-1} & x^m \end{vmatrix}$ is derived last column . The Eq (72) equivalent to :

$$\begin{vmatrix} \sigma_0 & . & . & . & \sigma_{m-1} & 1 \\ \sigma_1 & . & . & . & \sigma_m & v_1(x) \\ . & & & & . & . \\ . & & & & . & . \\ \sigma_m & . & . & . & \sigma_{2m-1} & v_{m+1}(x) \end{vmatrix} = 0 \tag{87}$$

Thus, $\forall x,\ v(x)\in H$. Application of Taylor, this equivalent to columns of matrix $V$ in hyperplane $H$.

$$l_{m+1} = \lambda_1 l_1 + \dots + \lambda_m l_m \tag{88}$$

$$\begin{vmatrix} \sigma_0 & . & . & . & \sigma_{m-1} & 1 \\ \sigma_1 & . & . & . & \sigma_m & x \\ . & & & & . & . \\ . & & & & . & . \\ \sigma_m & . & . & . & \sigma_{2m-1} & x^m \end{vmatrix} = \begin{vmatrix} \sigma_0 & . & . & . & \sigma_{m-1} & 1 \\ \sigma_1 & . & . & . & \sigma_m & x \\ . & & & & . & . \\ \sigma_{m-1} & . & . & . & \sigma_{2m-2} & x^{m-1} \\ 0 & . & . & . & 0 & x^m - \lambda_1 - \lambda_2 x - \ldots - \lambda_m x^{m-1} \end{vmatrix} \quad (89)$$

$$= \Delta^2[x^m - \lambda_1 - \lambda_2 x - \ldots - \lambda_m x^{m-1}]$$

The polynomial $D_0$ partner to $\begin{vmatrix} \sigma_0 & . & . & . & \sigma_{m-1} & 1 \\ \sigma_1 & . & . & . & \sigma_m & x \\ . & & & & . & . \\ . & & & & . & . \\ \sigma_m & . & . & . & \sigma_{2m-1} & x^m \end{vmatrix}$.Thus :

$$D_0(x) = x^m - \lambda_1 - \lambda_2 x - \ldots - \lambda_m x^{m-1} \quad (90)$$

For $k = 1, \ldots, m+1$, we put:

$$v_k(x) = \rho_k(x) + x^{d^o B - 2} w_k \quad (91)$$

where $d^o \rho_k(x) \leq d^o B - 3$.

The matrix $W$ partner to: $\begin{pmatrix} w_1 \\ . \\ w_m \end{pmatrix}$ is triangulares where $k^{th}$ entry diagonal equal $(k-1-m)(2a_{d^o B-1} + m + k - 2)$. The coefficient is non-zero except possibly for a single value of $k$.

Thus, rank of $\begin{pmatrix} w_1 \\ . \\ w_m \end{pmatrix}$ is $m$ or $m-1$.

For $rank(V) = m$, the Eq (72) as solution equivalent to $l_1, \ldots, l_m$ linearly independent.
Hence, after, we assume : $rank(V) = m - 1$. In that case $l_1$ and $l_2$ are linear combinations of $l_3, \ldots, l_{m+1}$. One of both combinations has to express $l_{m+1}$. Thus we have two situations.

**Situation 1** :

$l_2$ linear combination of $l_3, \ldots, l_{m+1}$ where coefficient of $l_{m+1}$ non-zero. By replacement of $l_{m+1}$, we obtain $l_1$ is linear combination of $l_2, \ldots, l_m$.
Thus, $w_1$ is linear combination of $w_2, \ldots, w_m$. $(w_2, \ldots, w_m)$ libre system where $d^o w_k = k - 1$ for : $k = 2, \ldots, m$. Thus : $w_1 = 0$.
If

$$l_1 = \sum_{j=2}^{m} \alpha_j l_j \quad (92)$$

where $\alpha_j$ constant, then :

$$w_1 = \sum_{j=2}^{m} \alpha_j w_j \quad (93)$$

As $w_1 = 0$, thus for all $j = 2, \ldots, m$, $\alpha_j = 0$ equivalent to $l_1 = 0$. Thus $R = 0$

**First way of having $D_0$:**

$$\frac{D_0''}{D_0'} = \frac{-2A}{B} \quad (94)$$

$\frac{-2A}{B}$ has to be the sum of simple elements of the shape: $\frac{\mu_c}{x-c}$ where $\mu_c$ greater or equal than 1. Thus : $D_4 = D_2 = D_1 = 1$ and $D_3 = B$.
For all $c$ root of $D_3$, $-2\alpha_c = \mu_c + 1$; where $\alpha_c^2 = \lim_{x\to c}(x-c)^2 r + \frac{1}{4}$
Thus :

$$\alpha_c < 0 \quad and \qquad 4[\lim_{x\to c}(x-c)^2 r + \frac{1}{4}] = (\mu_c + 1)^2 \tag{95}$$

Thus :

$$D_0' = m \prod_{c\in Root(D_3)} (x-c)^{\mu_c} \tag{96}$$

$D_0$ is primitive non-zero in roots of $D_3$.

**Second way of having $D_0$:**

$$l_{m+1} \ = \ \sum_{j=2}^{m} \lambda_j l_j \tag{97}$$

where$\lambda_j$ constant $(j = 2, \ldots, m)$ The relations linearly dependent between $l_{m+1}$ et $l_1, \ldots, l_m$ are :

$$l_{m+1} = \lambda l_1 + \sum_{j=2}^{m} \lambda_j l_j \tag{98}$$

Thus we have an infinity of solutions :

$$D_0 = x^m - \lambda - \sum_{j=2}^{m} \lambda_j x^{j-1} \tag{99}$$

where $\lambda$ arbitrarily constant.
**<u>Situation 2</u>** :
$l_1$ is linear combination of $l_3, \ldots, l_{m+1}$ where coefficient of $l_{m+1}$ non-zero and $l_2$ is linear combination of $l_3, \ldots, l_m$. Thus, $w_2$ is linear combination of $w_3, \ldots, w_m$. Thus, $w_2 = 0$. $w_1, w_3, \ldots, w_m$ libre system .
If :

$$l_2 = \sum_{j=3}^{m} \alpha_j l_j \tag{100}$$

then :

$$w_2 \ = \sum_{j=3}^{m} \alpha_j w_j \tag{101}$$

As : $w_2 = 0$. Thus, for all $j = 3, \ldots, m$, $\quad \alpha_j = 0$ equivalent to $l_2 = 0$ and $2A - xR = 0$
**First way of having $D_0$:**

$$\frac{2A}{x} D_0 = 2AD_0' + BD_0'' \tag{102}$$

$$2A(D_0 - xD_0') = xBD_0'' \tag{103}$$

We put : $Q = D_0 - xD_0'$. $\frac{Q'}{Q} = -\frac{2A}{B}$.

$$D_4 = D_2 = D_1 = 1 \ \ and \ \ Q = (1-m) \prod_{c\in Root(D_3)} (x-c)^{\mu_c} \tag{104}$$

Thus: $D_0 = Cx$ *avec* $C' = -\frac{Q}{x^2}$; $C$ rational function.
The coefficient of $x$ in $Q$ is zero. $Q'(0) = 0$ ($A(0) = 0$ ; $2A = xR$)
**Second way of having $D_0$:**

$$l_{m+1} = \lambda_1 l_1 + \sum_{j=3}^{m} \lambda_j l_j \tag{105}$$

Thus :

$$D_0 = x^m - \sum_{j=1}^{m} \lambda_j x^{j-1} \tag{106}$$

$lambda_2$ arbitrarily constant.

## 5.2 Case $d^o r = 2\nu \geq 0$

We put : $E = E(\theta)$ and $S = E + \frac{A}{B}$. We have:

$$\begin{cases} d^o(BD_0'') \leq d^oB + m - 2 \\ d^o(EB + A)D_0' = d^oB + m - 1 + \nu \end{cases} \tag{107}$$

Thus

$$d^o(RD_0) = d^oB + m - 1 + \nu \tag{108}$$

Thus, $d^oR = d^oB + \nu - 1$ and $r_{d^oB+\nu-1} = 2\alpha m$ where $\alpha$ dominant coefficient of $E$.
We put :

$$\begin{aligned} v(x) &= -R\begin{pmatrix} 1 \\ x \\ x^2 \\ . \\ . \\ x^m \end{pmatrix} + 2(EB + A)\begin{pmatrix} 0 \\ 1 \\ 2x \\ . \\ . \\ mx^{m-1} \end{pmatrix} + B\begin{pmatrix} 0 \\ 0 \\ 2 \\ . \\ . \\ m(m-1)x^{m-2} \end{pmatrix} \\ &= \begin{pmatrix} v_1(x) \\ v_2(x) \\ v_3(x) \\ . \\ . \\ v_{m+1}(x) \end{pmatrix} \end{aligned} \tag{109}$$

The $k^{th}$ element of $v(x)$ is :

$$v_k(x) = B(k-1)(k-2)x^{k-3} + 2(EB + A)(k-1)x^{k-2} - Rx^{k-1} \tag{110}$$

$v_k(x)$ of degree equal to or less than $d^oB + \nu + k - 2$ where coefficient of $x^{d^oB+\nu+k-2}$ equal $2\alpha(k - 1 - m)$.
For $k = 1, \ldots, m$, the $k^{th}$ element of $v(x)$ of degree: $d^oR + k - 1$.
The $(m + 1)^{th}$ element of $v(x)$ of degree equal to or less than $d^oR + m - 1$.

**Proposition 25** *We put :* $v_k(x) = \rho_k(x) + x^{d^oR} w_k(x)$ *where* $d^o\rho_k(x) < d^oR$.

*1.*

$$\begin{cases} d^o w_k(x) = k-1 \quad pour\ k=1,\ldots,m \\ d^o w_{m+1}(x) \leq m-1 \end{cases} \tag{111}$$

*2.* $(w_1,\ldots,w_m)$ *libre system and*

$$w_{m+1}(x) = \lambda_1 w_1(x) + \ldots + \lambda_m w_m(x) \tag{112}$$

*where* $\lambda_1,\ldots,\lambda_m$ *constants*

*3.*

$$D_0 = x^m - \lambda_1 - \lambda_2 x - \ldots - \lambda_{m-1}x^{m-1} \tag{113}$$

*is solution if and only if*

$$l_{m+1} = \lambda_1 l_1 + \ldots + \lambda_m l_m \tag{114}$$

**Example 26** *In this example we consider the Riccati differential equation (2) where:*

$$r = \frac{1}{(x+1)^4} - \frac{5}{(x+1)^3} + \frac{7}{4(x+1)^2} + \frac{1}{x+1} + x^2 + 2$$

$D_1 = D_2 = D_3 = 1$, $D_4 = (x+1)^2$, $d^o(r) = 2$, $\nu = 1$. *We have*

$$N_1 = 1 - 5(x+1)$$

$$F = x^2 + 2 + \frac{1-5(x+1)}{(x+1)^4} + \frac{1}{4(x+1)^2}$$

**Study at (-1):** *Laurent series development at* $-1$ *:*

$$(F)^{\frac{1}{2}} = \varepsilon_{-1}\left(\frac{1}{(x+1)^2} - \frac{5}{2(x+1)} + \ldots\right);\ \varepsilon_{-1} = \pm 1$$

**Study at infinity:** *We have :* $\frac{E(r)}{x^2} = 1 + t^2$ *where* $t = \frac{1}{x}$,

$$(1+2t^2)^{\frac{1}{2}} = 1 + t^2 + o(t^2)$$

*Thus :* $s_{\nu+1} = 1$ *et* $E(\theta) = \alpha x$ *where* $\alpha^2 = 1$.

$$\theta = \alpha x + \varepsilon_{-1}\left(\frac{1}{(x+1)^2} - \frac{5}{2(x+1)}\right) + \frac{1}{(x+1)} + \frac{D_0'}{D_0}$$

$$d^o D_0 = \alpha s_{\nu+1} - \frac{\nu}{2} + \frac{5\varepsilon_{-1}}{2} - 1 = \alpha s_{\nu+1} + \frac{5\varepsilon_{-1}}{2} - \frac{3}{2}$$

*Case :* $\begin{cases} \alpha = -1 \\ \varepsilon_{-1} = -1 \end{cases}$ *and* $\begin{cases} \alpha = 1 \\ \varepsilon_{-1} = -1 \end{cases}$ *are to be rejected because we obtain negative values of* $D_0$.

. *If :* $\alpha = 1$ *and* $\varepsilon_{-1} = 1$ *then :* $d^o D_0 = 2$

. *If :* $\alpha = -1$ *et* $\varepsilon_{-1} = 1$ *then :* $d^o D_0 = 0$

*Case 1* : $\alpha = 1$ *and* $\varepsilon_{-1} = 1$.

$$\theta = x + \frac{1}{(x+1)^2} - \frac{3}{2(x+1)} + \frac{D_0'}{D_0}$$

*where* $d^oD_0 = 2$. *Research of coefficients of* $D_0$.

$$\begin{array}{rcl} S & = & x + \frac{1}{(x+1)^2} - \frac{3}{2(x+1)} \\ & = & \frac{x^3+2x^2-\frac{1}{2}x-\frac{1}{2}}{(x+1)^2} \end{array}$$

$r - S' - S^2 = \frac{R}{B}$, *where :* $R = 4x^2 + 4x$ *and* $B = (1+x)^2$.

$$v(x) = -R \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} + 2SB \begin{pmatrix} 0 \\ 1 \\ 2x \end{pmatrix} + B \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -4x - 4x^2 \\ -1 - x - 2x^3 \\ 2 + 2x + 4x^3 \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & -4 & -4 & 0 \\ -1 & -1 & 0 & -2 \\ 2 & 2 & 0 & 4 \end{pmatrix}$$

$l_3 = -2l_2$. *Thus :* $D_0 = x^2 + 2x$ .

$$\theta = x + \frac{1}{(x+1)^2} - \frac{3}{2(x+1)} + \frac{2(x+1)}{x^2+2x} = x + \frac{1}{(x+1)^2} - \frac{3}{2(x+1)} + \frac{1}{x} + \frac{1}{x+2}$$

*Case 2* : $\alpha = -1$ *and* $\varepsilon_{-1} = 1$. *We obtain the rational fraction :*

$$\theta = -x + \frac{1}{(x+1)^2} - \frac{3}{2(x+1)}$$

*It cannot be solution because the sum of this fraction with the already found solution is not the logarithmic prime of a rational fraction.*

**Example 27** *: In this example we consider the Riccati differential equation (2) where:*

$$r = \frac{1}{16} + \frac{1}{(x-1)^8} - \frac{4}{(x-1)^5} - \frac{29}{6(x-1)^4} - \frac{8}{9(x-1)^3} - \frac{64}{27(x-1)^2} - \frac{152}{18(x-1)} + \frac{30}{(x+2)^2} - \frac{10}{81(x+2)}$$

$D_1 = D_2 = 1$, $D_3 = x + 2$ *and* $D_4 = (x-1)^4$. *We have :*

$$N_1 = (2418x - 2454)(x-1)^3 + (x+2)^2, \quad \delta = -\frac{1}{4(x+2)^2}$$

$$F = \frac{1}{16} + \frac{(2418x - 2454)(x-1)^3 + (x+2)^2}{(x-1)^8(x+2)^2} + \frac{1}{4(x+2)^2}$$

*Laurent series development at* 1 *:*

$$(F)^{\frac{1}{2}} = \varepsilon_1\left(\frac{1}{(x-1)^4} - \frac{2}{x-1} + ...\right); \ \varepsilon_1 = \pm 1$$

*Laurent series development at* $-2$ *:*

$$(F)^{\frac{1}{2}} = \varepsilon_{-2}\left(\frac{11}{2(x+2)} + ...\right); \ \varepsilon_{-2} = \pm 1$$

*We have :*

$$\frac{1}{2}\frac{(D_2D_3D_4)'}{D_2D_3D_4} = \frac{1}{2(x+2)} + \frac{2}{x-1}$$

*Thus :*

$$\theta = E(\theta) + \varepsilon_1[\frac{1}{(x-1)^4} - \frac{2}{x-1}] + \varepsilon_{-2}[\frac{11}{2(x+2)}] + \frac{1}{2(x+2)} + \frac{2}{x-1} + \frac{D_0'}{D_0}$$

*where*

$$E^2(\theta) = E(r) = \frac{1}{16}$$

$$d^oD_0 = -\frac{1}{E(\theta)} + 2\varepsilon_1 - \frac{11\varepsilon_{-2}}{2} - \frac{5}{2}$$

*Case* $\begin{cases} \varepsilon_1 = 1 \\ \varepsilon_{-2} = 1 \end{cases}$ *and* $\begin{cases} \varepsilon_1 = -1 \\ \varepsilon_{-2} = 1 \end{cases}$ *are to be rejected because we obtain negative values of* $D_0$.

<u>*Case 1*</u> : *If* : $\varepsilon_1 = 1$, $\varepsilon_{-2} = -1$, $E(\theta) = \frac{1}{4}$ *then* : $d^oD_0 = 1$.

$$\theta = \frac{1}{4} + \frac{1}{(x-1)^4} - \frac{5}{x+2} + \frac{D_0'}{D_0}$$

*Research of coefficients of* $D_0$.

$$S = \frac{1}{4} + \frac{1}{(x-1)^4} - \frac{5}{x+2}$$

$r - (S' + S^2) = \frac{R}{B}$ *where* : $R = \frac{1}{2}x^4 - 10x^3 + 19x^2 - 18x + \frac{5}{2}$ , $B = (x-1)^4(x+2)$

$$v(x) = -R\begin{pmatrix} 1 \\ x \end{pmatrix} + 2SB\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x^4 + 10x^3 - 19x^2 + 18x - \frac{5}{2} \\ -x^4 + 20x^3 - 38x^2 + 36x - 1 \end{pmatrix}$$

$$V = \begin{pmatrix} -\frac{5}{2} & 18 & -19 & 10 & -\frac{1}{2} \\ -5 & 36 & -38 & 20 & -5 \end{pmatrix}$$

$l_2 = 2l_1$. *Thus* : $D_0 = x - 2$.

$$\theta_1 = \frac{1}{4} + \frac{1}{(x-1)^4} - \frac{5}{x+2} + \frac{1}{x-2}$$

<u>*Case 2*</u> : *If* : $\varepsilon_1 = 1$, $\varepsilon_{-2} = -1$, $E(\theta) = \frac{-1}{4}$ *then* : $d^oD_0 = 9$.

$$\theta = \frac{-1}{4} + \frac{1}{(x-1)^4} - \frac{5}{x+2} + \frac{D_0'}{D_0}$$

*She cannot be solution because the sum of this fraction with the already found solution is not the logarithmic prime of a rational fraction.*

*Case 3* : *If :* $\varepsilon_1 = -1$, $\varepsilon_{-2} = -1$, $E(\theta) = \frac{-1}{4}$ *then :* $d^o D_0 = 5$.

$$\theta_3 = \frac{-1}{4} - \frac{1}{(x-1)^4} - \frac{5}{x+2} + \frac{4}{x-1} + \frac{D_0'}{D_0}$$

*Research of coefficients of* $D_0$.

$$S = \frac{-1}{4} - \frac{1}{(x-1)^4} - \frac{5}{x+2} + \frac{4}{x-1}$$

$R = \frac{-2}{2}x^4 + 36x^3 - 137x^2 + 168x - 88$ , $B = (x-1)^4(x+2)$

$$V = \begin{pmatrix} -37 & -168 & 137 & -36 & \frac{5}{2} & 0 & 0 & 0 & 0 \\ -32 & \frac{-151}{2} & -266 & 180 & -37 & 2 & 0 & 0 & 0 \\ 4 & -78 & -162 & -368 & 219 & -36 & \frac{3}{2} & 0 & 0 \\ 0 & 12 & -138 & \frac{-209}{2} & -474 & 254 & -33 & 1 & 0 \\ 0 & 0 & 24 & -212 & -95 & -584 & 285 & -28 & \frac{1}{2} \\ 0 & 0 & 0 & 40 & -300 & \frac{-139}{2} & -698 & 312 & -21 \end{pmatrix}$$

*We consider* $V_p$ *the minor* $6 \times 6$ *obtained by column vector* 1, 2, 6, 7, 8, 9. *In* $\mathbb{Z}/5\mathbb{Z}$ *:*

$$V_p = \begin{pmatrix} -2 & 2 & 0 & 0 & 0 & 0 \\ -2 & 2 & 2 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 & 2 & -1 \end{pmatrix}$$

$\det V_p = 1$. *Accordingly, row* 6 *is not linear combination of other row and* $D_0$ *he does not have* $D_0$.

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