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Hydrodynamic equilibrium and stability for Particle's Energy-density Wave-packets: State and Revision

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Abstract

We consider the *3-dimensional* (3-D) model of the massive particles, represented as the rest-mass energy-density wave-packets, which is analog to the common physical objects which we experiment in our every-day life, if we consider a physical object with a mass m for example, as a matter/energy-density contained in this 3-D form, such that the integration of this energy-density contained in this 3-D volume is equal to the energy $E = mc^2$.
This is a complete revision and improvement of previous work [10] or the analysis of particle's internal dynamic during particle's accelerations and hence we use the three conservation laws for the compressive fluids: energy/matter conservation, momentum conservation and internal energy conservation laws. Consequently, we show the new method for computation of the hydrostatic equilibrium of a massive elementary particle during inertial propagation in a vacuum with a stable spherically symmetric rest-mass energy density distribution, such that with this stationary distribution the internal self-gravitational force is constant and equal in each point inside the particle. Then we show how the self-gravitational forces generated by the rest-mass energy-density of massive particles in 3-D provide the auto-stability process during the small perturbations, which cause particle's accelerations, and their return to the inertial propagation in the vacuum.

Keywords: Quantum Physics, Non point-like particles, Particle's internal dynamics

1 Introduction

The standard QM with the probabilistic wavefunctions and their statistical ensemble interpretation is based on the classical concept of a *point-like particle* and do not have the theory able to describe an individual particle with its trajectory and given momentum and energy in any fixed instance of time. Because of that as noted by Einstein it was an incomplete theory, differently from the classic mechanics which has both statistical theory (for example the thermodynamic of a gas) and theory for each individual object (Newton, Euler-Lagrange equations for the motion of an individual object). In the proposed completion of QM (provided recently in [10]) instead, an individual massive particle's wave-packet always occupies a nonzero 3-D volume. It holds also for bosons when they become unstable after an initial 'space explosion' and, consequently, assume the massive particle behavior and a finite but non-zero matter/energy-density volume in open 3-D space. Consequently, in this theory [10] for elementary particles based on energy-density wave-packets, the point-like particles are only the stable-state bosons when they propagate with speed of light in the vacuum, and with their energy-density distributed in higher compactified dimensions. In the Kaluza-Klein approach, such particles with different *quantized* charges can be simply obtained by addition of closed compactified dimensions where the matter can propagate in one or in opposite direction and hence producing the positive/negative quantized charge. Thus, as in the string theory, I assumed the existence of a number of closed dimensions as well [11], which differentiate the spin-zero neutral (basic feature) particles described by the complex scalar wave-packets Ψ in what follows.
It was shown [7, 8, 9, 10] that, generally, any massive particle can be defined in the Minkowski time-space by the complex wave-packet

$$\Psi = \Phi(t, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i\varphi_T} \tag{1}$$

where $\overrightarrow{\mathbf{r}} = q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3$ (for the 3-D Minkowski space orthonormal basis vectors $\mathbf{e}_j$, with $\mathbf{e}_j \cdot \mathbf{e}_j = -1$ for $1 \leq j \leq 3$ and $\mathbf{e}_0 \cdot \mathbf{e}_0 = 1$ for the time-coordinate $q_0 = ct$) composed by two sub components: by the shape $\Phi(t, \overrightarrow{\mathbf{r}})$ of particle's body that is a real function which defines the real *rest-mass energy-density* $\Phi_m \equiv \Psi\overline{\Psi} = \Phi^2(t, \overrightarrow{\mathbf{r}}) \geq 0$, and by the 'phase (pilot) wave' with phase φ_T, $\mathrm{e}^{-i\varphi_T} = \mathrm{e}^{-\frac{i}{\hbar}(\overrightarrow{\mathbf{p}}(\overrightarrow{\mathbf{r}_T} - \overrightarrow{\mathbf{r}_T}(0)) + Et)}$, which is a complex function defined only for the particle's barycenter $\overrightarrow{\mathbf{r}_T}(t) \equiv \frac{1}{\mathbf{1}_\Phi}\int \overrightarrow{\mathbf{r}}\Phi_m(t, \overrightarrow{\mathbf{r}})dV$, of the massive elementary particle with the *total energy* E and momentum $\overrightarrow{\mathbf{p}}$ which may change in time as well, and $\mathbf{1}_\Phi \equiv \int \Phi_m(t, \overrightarrow{\mathbf{r}})dV$ is the particle's invariant energy (equal to rest-mass energy m_0c^2 for massive particles and energy E_0 of a boson, measured in the frame in which massive source of this boson is in rest). Thus,

$$m_0 = \int m_0(t, \overrightarrow{\mathbf{r}})dV = \int \frac{\Phi_m(t, \overrightarrow{\mathbf{r}})}{c^2}dV \tag{2}$$

where $m_0(t, \overrightarrow{\mathbf{r}}) \equiv \frac{\Phi_m(t, \overrightarrow{\mathbf{r}})}{c^2}$ is the rest-mass density.
From the fact that a field is a quantity defined at every point $(t, \overrightarrow{\mathbf{r}})$ of the 4-D time-space manifold $\mathcal{M}$, such a quantity can be a complex number of the wave-packet $\Psi = \Phi(t, \overrightarrow{\mathbf{r}})\mathrm{e}^{-i(\overrightarrow{\mathbf{p}}(\overrightarrow{\mathbf{r}_T} - \overrightarrow{\mathbf{r}_T}(0)) + Et)/\hbar}$ or a real number of the energy-density $\Phi_m(t, \overrightarrow{\mathbf{r}}) = \Phi^2(t, \overrightarrow{\mathbf{r}}) = \overline{\Psi}\Psi$ (for a massive particle $\curlywedge\Phi_m(t, \overrightarrow{\mathbf{r}})$ is its matter-density, where $\curlywedge$ is the constant which transforms the rest-mass energy into the 'matter').
In any fixed instance of time t, this complex time-oscillation of the 'pilot-wave' exists only in a very limited space volume, in the points $\overrightarrow{\mathbf{r}}$ where particle's density $\Phi_m(t, \overrightarrow{\mathbf{r}}) > 0$, that is, only where is *present the matter/energy* of this particle (notice that the complex 'pilot wave' is defined only on the particle's barycenter). So, it is a local oscillation *embedded inside the matter* of the particle, and has no any propagation outside its matter, and hence *it is not a plain wave*. In fact, if a particle is in the rest state ($\overrightarrow{\mathbf{v}} = 0$) we have the time-invariant distribution Φ around its barycenter $\overrightarrow{\mathbf{r}_T}$, and its complex-oscillation is located in a very small volume around its barycenter, and does not propagate anywhere outside the matter/energy of this a particle, differently from the plain wave which propagates always in the whole space.
However, during acceleration generally each infinitesimal amount of energy-density $\Phi_m(t, \overrightarrow{\mathbf{r}})$ moves with different speed $\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}})$ w.r.t. the group velocity $\overrightarrow{\mathbf{v}}(t) = \frac{d}{dt}\overrightarrow{\mathbf{r}_T}(t) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$, with $v = \|\overrightarrow{\mathbf{v}}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ of particle's energy-density wave-packet and we can show that is satisfied the following relationship

$$\overrightarrow{\mathbf{v}}(t) = \frac{1}{\mathbf{1}_\Phi}\int \overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}})\Phi_m(t, \overrightarrow{\mathbf{r}})dV \tag{3}$$

so we can introduce the variation-velocity of the particle's matter flux $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) = \overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}}) - \overrightarrow{\mathbf{v}}(t)$ at each space-time point $(t, \overrightarrow{\mathbf{r}})$ inside particle's matter (where $\Phi_m(t, \overrightarrow{\mathbf{r}}) > 0$). As shown in [10] during an inertial propagation when the particle is in a hydrostatic equilibrium we have that Φ_m is spherically symmetric around particle's barycenter and with $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) = 0$ in every point inside particle's matter, so that every infinitesimal amount of Φ_m propagates with the constant wave-packet group velocity $\overrightarrow{\mathbf{v}}$. Only during the particle's

accelerations we have that $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) \neq 0$, so that particle's matter changes dynamically its shape in time.
In the assumption [10] of the topology of the matter of an elementary massive particle, the wave-packet do not undergo a spreading, also when it changes its matter density distribution (i.e., its energy-density Φ_m), and tends to its stable stationary spherically symmetric distribution during inertial propagation in the vacuum. That is, the matter has some internal self-gravitational autocohesive force analogously to the peace of fluid in the vacuum, so that at any instance of time, the 3-D space topology of particle's matter distribution, and consequently its compressible energy-density Φ_m is simply connected, closed, continuous and differentiable.
Analogously to the Euler first equation of fluid dynamics (continuity equation), which represents the conservation of mass, here we have the analog equation for the conservation of matter (that is of the particle's rest-mass energy):

$$\frac{\partial \Phi_m(t, \overrightarrow{\mathbf{r}})}{\partial t} + \nabla \cdot (\Phi_m(t, \overrightarrow{\mathbf{r}}) \overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}})) = 0 \tag{4}$$

In what follows $\nabla = \mathbf{e}_1 \frac{\partial}{\partial x} + \mathbf{e}_2 \frac{\partial}{\partial y} + \mathbf{e}_3 \frac{\partial}{\partial z}$ is the gradient (for $x \equiv q_1, y \equiv q_2$ and $z \equiv q_3$) so that the Laplacian is defined by $\triangle = -\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.
In Section 2.5 in [10], dedicated to the wave-packet stability, is considered the spherical expansion of the rest-mass energy density Φ_m (that is, of the rest-mass density $\frac{\Phi_m}{c^2}$), during the unstable particle's states where the variation-velocity $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})) \neq 0$. I tried to explain why in the stable particle's states, during an inertial propagation with the constant speed $\overrightarrow{\mathbf{v}}$ in the vacuum (sufficiently far from another particles), we have no internal motion of the rest-mass density of the particle, that is, we have that $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})) = 0$ in each point inside particles rest-mass density distributions (where $\overrightarrow{\mathbf{r}}$ is the vector from the barycenter to the observed point). Unfortunately, this Section 2.5 was written just when my proposal for the books [10, 11] was after so lot of time finally accepted by Nova Science to be published (after a lot of time that I tried to publish this completion of QM) and I did not have time to control and to stress, from different point of views, this section well and unfortunately I did not have anybody other (student or collaborator) to control it for me[1]. Consequently, immediately after the press of this first volume [10], I realized (by using the Gauss theorem) that was impossible to have equal to zero the self-gravity force inside

[1] I am sorry, for me in that moment, after so lot of time that I tried to publish the principal results of this new theory (and this Section 2.5 was only an addendum; all other part of this theory are independent of it), I preferred do not to lose this offered opportunity from Nova Science Publishers and to dedicate all my time to divide this theory in two thematic volumes and to obtain the printable version of them as soon as possible.

the spherically-symmetric particle's mass density during particle's hydrostatic equilibrium (during an inertial propagation of a massive particle), and that particle's stability must be explained differently.

This paper is in effect the revision of the errors in Section 2.5 in [10]. The explanation of particle's stability needs much more mathematical apparatus and another Euler's equation (or more complex Navier-Stokes momentum equation) derived from the Cauchy momentum equation. Both new equations introduce the common concepts of the pressure inside particle's matter opposed to particle's self-gravitational forces inside its matter, in order to guarantee the particle's hydrostatic equilibrium (analogously to the equation of hydrostatic equilibrium in the stars, for example).

In next section we will confront the conservation law (4) for an individual massive particle with the statistical Boltzmann equation to verify in which conditions the Boltzmann equation, which describes the statistical behaviour of a thermodynamical system not in state of equilibrium, can be reduced to the properties of the continuous and simple-connected particle's matter $\measuredangle\Phi_m$. After that, we will analyze the particle's stability by using the Cauchy stress 2-tensor (in his momentum equation) in the approximation of the Newton field-theoretic theory of gravitation applied to the particle's mass density.

2 Hydrostatic equilibrium of a massive particle

The duality relationship between the statistical Schrödinger equation (where complex wavefunction ψ represents the probabilistic density) appropriate for an ensemble of particles and the deterministic new equation for an individual elementary particle was discussed in details in [10]. Here we will discuss the relationship between the continuity equation (4) for an individual elementary particle and the Boltzmann equation for the fluids composed by the particles, with the probability density function $\psi(t, \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{p}})$,

$$\frac{\partial\psi}{\partial t} = (\frac{\partial\psi}{\partial t})_f + (\frac{\partial\psi}{\partial t})_d + (\frac{\partial\psi}{\partial t})_c \tag{5}$$

where $(\frac{\partial\psi}{\partial t})_f$ corresponds to the forces exerted on the particles in the fluid by an external influence (not by particles themeselves), $(\frac{\partial\psi}{\partial t})_d$ represents the diffusion of particles in this fluid, and $(\frac{\partial\psi}{\partial t})_c$ corresponds to the forces inside this fluid and acting between particles in collisions.

The Boltzman equation can be used for the fluids composed by the particles with positions $(t, \overrightarrow{\mathbf{r}})$ and momentum $\overrightarrow{\mathbf{p}}$, that is, for the thermodynamical fluid system not in state of equilibrium. It can be used to determine how physical quantities change, such as heat energy and momentum, when fluid is in transport, and hence can also derive other properties of the fluids such as viscosity, thermal conductivity, etc..

Let us consider this fluid composed by particles, each particle experiencing an external force field $\overrightarrow{f}(t, \overrightarrow{\mathbf{r}})$ not due to other particles in this fluid, and suppose that at time-instance t some number of particles have the position $\overrightarrow{\mathbf{r}} = q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3$ within an infinitesimal region $dV = dq_1dq_2dq_3$ and momentum $\overrightarrow{\mathbf{p}} = m\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}}) = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3$, with mass m and velocity $\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}})$, within $d^3\overrightarrow{\mathbf{p}} = dp_1dp_2dp_3 = m^3dw_1dw_2dw_3$. If the force $\overrightarrow{f}(t, \overrightarrow{\mathbf{r}})$ instantly acts on each particle, then at time $t+\delta t$ their position will be $\overrightarrow{\mathbf{r}} + \delta\overrightarrow{\mathbf{r}}$ and momentum $\overrightarrow{\mathbf{p}} + \delta\overrightarrow{\mathbf{p}} = \overrightarrow{\mathbf{p}} + \overrightarrow{f}\delta t = m(\overrightarrow{\mathbf{w}} + \delta\overrightarrow{\mathbf{w}})$. Hence, in the absence of collisions, it holds that

$\delta\Psi \equiv \psi(t+\delta t, \overrightarrow{\mathbf{r}} + \delta\overrightarrow{\mathbf{r}}), \overrightarrow{\mathbf{p}} + \delta\overrightarrow{\mathbf{p}})dVd^3\overrightarrow{\mathbf{p}} - \psi(t, \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{p}})dVd^3\overrightarrow{\mathbf{p}}$

is equal to zero, where we used (see the discussion under Liuville's theorem in my book [10]) that the phase-space volume $dVd^3\overrightarrow{\mathbf{p}}$ is constant. However, since collisions occur, the particle's density in the phase-space volume change, so

$\delta\psi dVd^3\overrightarrow{\mathbf{p}} = (\frac{\partial\psi}{\partial t})_c\delta t dVd^3\overrightarrow{\mathbf{p}}$,

where $\delta\psi$ is the total change of ψ and, in the case of the limit $\delta t \to 0$ and $\delta\psi \to 0$, from the result above we obtain

$$(\frac{\partial\psi}{\partial t})_c = lim_{\delta t\to 0}\frac{\delta\psi}{\delta t} = \frac{d\psi}{dt} \equiv \frac{\partial\psi}{\partial t} - \overrightarrow{\mathbf{w}}\nabla\psi - \frac{\overrightarrow{f}}{m}\nabla_{\mathbf{w}}\psi \tag{6}$$

where $\nabla_{\mathbf{w}}\psi \equiv \frac{\partial\psi}{\partial\overrightarrow{\mathbf{w}}} = \sum_{i=1}^{3}\mathbf{e}_i\frac{\partial\psi}{\partial w_i}$ and $\overrightarrow{f} = \frac{d\overrightarrow{\mathbf{p}}}{dt} = m\frac{d\overrightarrow{\mathbf{w}}}{dt}$. Note that in the case when $(\frac{\partial\psi}{\partial t})_c = 0$ the equation above is the Vlasov equation.

It is convenient to represent the statistical Boltzmann equation above, where ψ is a probability density (like the Schrödinger wavefunction in QM), in the following form:

$$\frac{\partial\psi}{\partial t} + \nabla\cdot(\overrightarrow{\mathbf{w}}\psi) = (\frac{\partial\psi}{\partial t})_c + \frac{\overrightarrow{f}}{m}\nabla_{\mathbf{w}}\psi + \psi(\nabla\cdot\overrightarrow{\mathbf{w}}) \tag{7}$$

It was proven only in 2010 that exact solutions to the Boltzmann equation are always mathematically well-behaved [6]. This means that if a system obeying the Boltzmann equation is perturbed, then it will return to equilibrium, rather than diverging to infinity or behaving otherwise. However, this existence proof is not helpful for solving the equation in realistic scenarios. Indeed, such statements only tell us whether the solution subject to specified conditions exist,

but not how to find them. In practice, numerical methods are used to find approximate solutions to the various forms of the Boltzmann equation.
Let us consider how we can apply this statistical equation (for a non-continuous flow composed by the fluid's molecules) to the continuous flow (simple connected) of the particle's matter density. In this case we replace the probabilistic density ψ in (7) by the rest-mass energy density Φ_m and the infinitesimal mass m by rest-mass density Φ_m/c^2 in a time-space point $(t, \overrightarrow{\mathbf{r}})$, and hence we obtain the conservation law (4) of internal dynamic assumption (Definition 5 in [10]), $0 = \frac{\partial \Phi_m}{\partial t} + \nabla \cdot (\overrightarrow{\mathbf{w}}\Phi_m) = (\frac{\partial \Phi_m}{\partial t})_c + \frac{\overrightarrow{f}}{\Phi_m/c^2}\nabla_{\mathbf{w}}\Phi_m + \Phi_m(\nabla \cdot \overrightarrow{\mathbf{w}})$,
for each point $(t, \overrightarrow{\mathbf{r}})$ where $\Phi_m(t, \overrightarrow{\mathbf{r}}) \neq 0$. Thus, we obtain for each massive particle that

$$\Phi_m(\nabla \cdot \overrightarrow{\mathbf{w}}) = -(\frac{\partial \Phi_m}{\partial t})_c - \frac{\overrightarrow{f}}{\Phi_m/c^2}\nabla_{\mathbf{w}}\Phi_m \qquad (8)$$

In the case of the hydrostatic equilibrium we have that the force $\overrightarrow{f} = 0$ and that we have no collisions inside particle's material body, that is $(\frac{\partial \Phi_m}{\partial t})_c = 0$, and hence must be $\Phi_m(\nabla \cdot \overrightarrow{\mathbf{w}}) = 0$, that is in any point of the particle's body where $\Phi_m(t, \overrightarrow{\mathbf{r}}) \neq 0$ must be $\nabla \cdot \overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}}) = 0$, that is, the fluid velocity $\overrightarrow{\mathbf{w}}$ must be constant vector (for any fixed time t) in each point inside particle's body, and that means that this constant velocity is just the group velocity of the particle during inertial propagation of the particle. Consequently, we obtained that the hydrostatic equilibrium of a massive particle holds only during an inertial propagation with constant velocity vector $\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}}) = \overrightarrow{\mathbf{v}}$ corresponding to the group velocity of the particle's wave-packet. In such a hydrostatic equilibrium the Boltzman expression (5) reduces to the simple expression $\frac{\partial \Phi_m}{\partial t} = (\frac{\partial \Phi_m}{\partial t})_f$, which represents the simple diffusion of the particle's rest-mass energy Φ_m density which, as we will show in what follows, becomes spherically symmetric w.r.t particle's barycenter.
For the coordinate system with the center in particle's barycenter, which will be used in the rest of this paper, this condition of an inertial propagation corresponds to the case when the variation-velocity $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})$ is equal to zero. That is, in this coordinate system during this hydrostatic equilibrium we have that the matter density of particle's body (thus also its rest-mass energy density Φ_m) is in rest and do not change during such an inertial propagation.
However, from the fact that, based on the General Relativity (GR), the rest-mass energy density generates a local gravitational field (that is, a local time-space curvature), this local gravitational force will involve each infinitesimal amount of its rest-mass density Φ_m/c^2. Thus, also during the stable static dis-

tribution (hydrostatic equilibrium) of the rest-mass density of a particle during an inertial propagation in the vacuum, inside particle's material body there exists the internal self-gravitational force. Let us show that it is different from zero also inside particle's mass density. It is enough to consider the non-relativistic case, when the constant velocity of particle satisfies $v << c$, so that there is not any external field which would generate the accelerations, and to consider that the local time-space of the vacuum in which propagates this particle is a flat Minkowski time-space. Thus, in such a situation, this isolated particle with rest-mass m_0 can be considered as a generator of the local Newtonian gravitation, with the coordinate center in particle's barycenter, described in what follows:

Newton's field-theoretic theory of gravitation: Newton's theory was presented as a two-body interaction theory, by importation of concepts and methods borrowed from electrostatics.
Let the density distribution of gravitating matter, relative to an inertial frame, be described by the function $\rho(\overrightarrow{\mathbf{r}})$, and let a test particle of mass ("gravitational charge") m reside momentarily at $\overrightarrow{\mathbf{r}}$, so that the force experienced by this test particle be $\overrightarrow{\mathbf{F}} = m\overrightarrow{\mathbf{g}}(\overrightarrow{\mathbf{r}})$, where the acceleration $\overrightarrow{\mathbf{g}}(\overrightarrow{\mathbf{r}})$ is the gravitational analog of an electrostatic field $\overrightarrow{\mathbf{E}}$. The force-law proposed by Newton is conservative, so $\nabla \times \overrightarrow{\mathbf{g}} = 0$, and hence we can introduce the gravitational potential ϕ so that

$$\overrightarrow{\mathbf{g}} = -\nabla\phi \tag{9}$$

Hence, in mimicry of the electrostatic equation $\nabla \cdot \overrightarrow{\mathbf{E}} = \rho$ (charge density regulates the divergence of the electric field), we obtain the differential for of the Gauss's law for gravity

$$\nabla \cdot \overrightarrow{\mathbf{g}} = -4\pi G\rho \tag{10}$$

where G is the gravitational constant and the minus sign reflects the fact that the gravitational interaction is attractive. Thus from (9) and (10) we obtain the gravitational Poisson equation:

$$\triangle\phi = -4\pi G\rho \tag{11}$$

If the mass-density ρ is completely inside a sphere of radius r then, by integrating, we obtain that the total mass interior to V is equal to $m_0 \equiv \int_V \rho dV = \int_V \frac{1}{4\pi G}\nabla \cdot \overrightarrow{\mathbf{g}}\, dV = \frac{1}{4\pi G}\oint_S \overrightarrow{\mathbf{g}}\, d\mathbf{S}$, i.e., it is equal to the gravitational influx trough the surface of this sphere $S = 4\pi r^2$. Taking that ρ is concentrated in the center of this sphere with radius r, we obtain $m_0 = \frac{1}{4\pi G} 4\pi r^2 g(r)$, that is

$$-\nabla\phi = \overrightarrow{\mathbf{g}}(r) = -\frac{Gm_0}{r^2}\mathbf{e}_r \tag{12}$$

where $\mathbf{e}_r$ is the unary radial vector and, hence, the solution of this differential equation defines the scalar gravitational field of an isolated point mass m_0:

$$\phi(r) = -\frac{Gm_0}{r} \tag{13}$$

For distributed mass-density ρ we have that

$$\phi(\overrightarrow{\mathbf{r}}) = -G\int \frac{\rho(\overrightarrow{\mathbf{r'}})}{|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r'}}|}dV \tag{14}$$

which gives back (13) in the point-like case when $\rho(\overrightarrow{\mathbf{r'}}) = m_0\delta(\overrightarrow{\mathbf{r'}})$.
The spherical symmetry of the gravitational potential makes each star's mass-density ideally spherically symmetric w.r.t. the center of coordinate system fixed in star's barycenter.
Let the mass of spherical shell of radius R and surface $S = 4\pi R^2$, with constant mass-density ρ, be $M = S\rho = 4\pi R^2\rho$. Then it is demonstrated that for any small test mass m, at *any* point inside this sphere, the net gravitational force on it is identically zero.
Physically, this is very important because any spherically symmetric mass distribution $\rho(r)$ outside the position of the test mass m can be build up as a series of such shells.
If a given test mass m is inside a spherically symmetric distribution of mass

$$\rho(r) = \begin{cases} \sigma(r) > 0, & \text{if } \ 0 \le r \le r_0; \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

at distance $r < r_0$ from the center of this distribution, that part of the mass outside its radius r does not contribute to the net force on it, *but only the mass* $m_r = \int_{V_r} \rho(r')dV \le m_0 \equiv \int \rho(r')dV$, *contained inside this sphere of radius* r with the volume $V_r = \frac{4\pi}{3}r^3$.
□
Let us apply this theory, instead to a star, to a massive elementary particle in its hydrostatic equilibrium (during inertial propagation of the particle) with $\rho = \Phi_m/c^2$ spherically symmetric around its barycenter (center of the spherical coordinate system in what follows). Thus, during the hydrostatic equilibrium (an inertial propagation) of a massive particle with spherically symmetric

energy-density $\Phi_m(r)$ inside the sphere with the radius r_0 (i.e., inside the particle's material body), we have the following self-gravitational force, for each $r \leq r_0$, generated by the particle's mass inside the volume $V = \frac{4\pi}{3}r^3$,

$$\overrightarrow{\mathbf{g}} = \mathbf{e}_r g(r) = -\mathbf{e}_r \frac{G}{r^2}\int_V \frac{\Phi_m}{c^2}dV = -\mathbf{e}_r \frac{4\pi G}{r^2}\int_0^r \frac{\Phi_m(s)}{c^2}s^2 ds \tag{16}$$

This self-gravitational force, directed into the barycenter of the particle, in order to avoid the generation of a black hole, must be balanced by the opposite force of the material fluid substance $\leftthreetimes\Phi_m$ of the particle's body. The force on the mass in an infinitesimal cylindric volume $drdS$ (with the base surface dS) $g(r)m = g(r)\frac{\Phi_m(r)}{c^2}drdS$ must be balanced by the pressure difference $P(r)dS - P(r+dr)dS = -\frac{dP}{dr}drdS$, so in the hydrostatic equilibrium we obtain that $g(r)\frac{\Phi_m(r)}{c^2} = \frac{dP}{dr}$, that is, the internal force $F(r)$ in the particle's body during its hydrostatic equilibrium is

$$F(r) = \frac{dP}{dr} = -\frac{4\pi G}{r^2}\frac{\Phi_m(r)}{c^4}\int_0^r \Phi_m(s)s^2 ds \tag{17}$$

This is in complete according with the fact that Einstein did not believe into the reduction of gravity to geometry [1] [2].

We need that the body of the particle $\leftthreetimes\Phi_m$ provides also the physical internal pressure $P(r)$ (which is a non-geometrical property) in order to guarantee the hydrostatic equilibrium of the massive particles. The hydrostatic equilibrium of a massive elementary particle demonstrated that the body of this particle $\leftthreetimes\Phi_m$ is a material substance, which is fluid and elastic, and which can not be reduced to the time-space geometry. This real physical material substance generates a curved time-space curvature inside and around it (the micro-island curvature), but this material substance can not be simply 'generated by time-space curvature'. That is, a massive elementary particle can not be reduced to the pure geometry: see also the debate of Einstein with Willem de Sitter who considered a solution of Einstein's GR in which there is no matter, as an exponentially expanding empty universe), as claimed by Emile Meyerson that Einstein's GR theory was the identification of matter with space (in his book La Deduction Relativiste, 1925).

[2] In effect, if we erroneously assume (as in my previous error in Section 2.5 in [10]) that during inertial propagation (hydrostatic equilibrium) of the particle, the self-gravitational force inside particle's body is zero then we would not need any internal pressure in the particle's body and hence would be possible to reduce ontologically the elementary particle into the time-space geometry (from the fact that the particle's stability could be completely explained by particle's self-gravity (time-space curvature)). Moreover, without internal pressure of particle's body it will not be possible to have the elastic Compton collisions of the particles.

Hence, based on the concept of the equilibrium inside particle's body, we expect that the internal force $F(r)$ during the hydrostatic equilibrium is constant in each point inside particle's body. So, we obtain the following hydrostatic equilibrium differential equation:

Lemma 1 *The hydrostatic equilibrium of a massive particle with spherically symmetric material body density, with the radius r_0 and energy density $\Phi_m(r) > 0$ for $0 < r \leq r_0$, has to satisfy the following first-order differential equation*

$$\frac{d\Phi_m}{dr} - \frac{2}{r}\Phi_m - \frac{4\pi G}{c^4 F}\Phi_m^3 = 0 \tag{18}$$

where the internal force $F = -\frac{4\pi G}{c^2}\frac{\Phi_m(r)}{r^2}\int_0^r \Phi_m(s)s^2 ds$ is a real constant. We obtain also the following second-order differential equation:

$$\frac{d^2\Phi_m}{dr^2} + (\frac{4}{r} - \frac{3}{\Phi_m}\frac{d\Phi_m}{dr})\frac{d\Phi_m}{dr} + 2\frac{\Phi_m}{r^2} = 0 \tag{19}$$

Proof: During the hydrostatic equilibrium of the massive elementary particle, we require that $\frac{dF(r)}{dr} = 0$ for the internal force $F(r)$ given by (17). So, by differentiation of equation (17) and by considering that $\frac{d}{dr}(\int_0^r \Phi_m(s)s^2 ds) = \Phi_m(r)r^2$, we obtain the equation $0 = r^2\Phi_m + (\frac{d\Phi_m}{dr} - \frac{2}{r}\Phi_m)\int_0^r \Phi_m(s)s^2 ds$, that is:

$$\int_0^r \Phi_m(s)s^2 ds = -r^2\Phi_m^2/(\frac{d\Phi_m}{dr} - \frac{2}{r}\Phi_m) \tag{20}$$

and hence by repeating the differentiation of both sides of this equation and by considering that $\frac{d}{dr}(\int_0^r \Phi_m(s)s^2 ds) = \Phi_m(r)r^2$, we obtain

$$\Phi_m(r)r^2 = -\frac{d}{dr}[r^2\Phi_m^2/(\frac{d\Phi_m}{dr} - \frac{2}{r}\Phi_m)] \tag{21}$$

So, by differentiation of the right-hand side of this equation, we obtain our second-order differential equation (19), which the density $\Phi_m(r)$ has to satisfy during the hydrostatic equilibrium. The equation (18) is obtained in an analog way, but by expressing the equation (17) in the form $\int_0^r \Phi_m(s)s^2 ds = -\frac{Fc^4}{4\pi G}\frac{r^2}{\Phi_m}$ and then by differentiating both sides on r and by considering that F is constant.

□

Perfect elasticity assumption: Although elasticity is most commonly associated with the mechanics of solid bodies or materials, even the early literature on classical thermodynamics defines and uses "elasticity of a fluid" in ways compatible with the broad definition provided in the Introduction above. Throughout

the theory of massive elementary particle [10] it is assumed that the particle's bodies undergoing the action of external forces are *perfectly elastic*, i.e., that they resume their initial form (in its hydrostatic equilibrium) completely after removal of forces.
The elastic body of an elementary particle has no any molecular internal structure and hence the particle's matter is *homogeneous* and and continuously distributed over its volume so that the smallest element cut from the body possesses the same specific physical properties as the whole body. That is, it is assumed that the particle's body is *isotropic*, i.e., that the elastic properties are the same in all spatial directions. Moreover, we have no any thermodynamic dissipation inside particle's body during its elastic deformations, that is, we have no thermal losses of particle's internal energy during such elastic deformations caused by the external forces: the particle's kinetic energy converted into its internal elastic potential energy V during elastic deformations have no any side-effects of thermal losses.
The essence of elasticity is the reversibility. Forces applied to an elastic material transfer energy into the material which, upon yielding that energy to its surroundings, can recover its original shape. Elastic energy occurs when objects are compressed and stretched, or generally deformed in any manner. For an massive elementary particle it corresponds to energy stored by changing the internal forces of particle's hydrostatic equilibrium based on particle's self-gravitational force.
Such properties of the particle's body differentiate it from the structural materials composed by the molecular structures as all material object used in our every-day practice.

3 Other two conservation laws in fluid dynamics of the elementary particles

Here we will consider another two conservation laws for the internal dynamic inside the material substance of an massive elementary particle, not considered previously in [10] (Section 2.5).
Hence, from the fact that inside the particle's material body we have a pressure (17) that makes the balance with internal self-gravitational force (16), our theory of massive elementary particles needs also the Cauchy conservation of momentum equation (here we will use the simpler convective form derived from

the second Newton's equation)[3], in the coordinate system with the center in the particle's barycenter so that matter density speed $\overrightarrow{\mathbf{w}}$ is equal to the that variation-velocity $\overrightarrow{\mathbf{u}} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$, describes the non-relativistic momentum transport in any *continuum* (the kinematics of materials modeled as a continuous mass Φ_m/c^2 of a massive elementary particle)rather than as discrete particles in a time-space point $(t, \overrightarrow{\mathbf{r}})$:

$$\frac{\Phi_m}{c^2}\frac{d\overrightarrow{\mathbf{u}}}{dt} = \nabla\cdot\sigma + \frac{\Phi_m}{c^2}\overrightarrow{\mathbf{g}} \tag{26}$$

where σ is the stress 2-tensor and both surface (such as viscous forces) and body forces are accounted for in one total force $\frac{\Phi_m}{c^2}\overrightarrow{\mathbf{g}}$ to an infinitesimally small point at $(t, \overrightarrow{\mathbf{r}})$ (for example,this total force may be expanded into an expression for the frictional and gravitational forces acting at a point in a flow).

This equation is appropriate for the Φ_m because Φ_m/c^2 is just a *continuous* rest-mass density of a massive elementary particle, and exists as a continuum (as specified by the topological properties in [10] of the massive elementary particles), meaning that the matter $\curlywedge\Phi_m$ of the massive elementary particle is continuously distributed in 3-D space (for any fixed instance of time t) and fils

[3]The derivation of another Cauchy equation comes from the momentum conservation law, when applied to a fixed control volume V bounded by surface S (with $\overrightarrow{dS} = \overrightarrow{\mathbf{n}}dS$ for the unary vector $\overrightarrow{\mathbf{n}}$ orthogonal to dS) inside the flow. Newton's second law states that during a short time interval dt, the impulse of a force $\overrightarrow{F}$ applied to the mass of the flow will produce the following momentum change in that affected mass:

$$\frac{d\overrightarrow{\mathbf{p}}}{dt} + \overrightarrow{\mathbf{p}}_{out} - \overrightarrow{\mathbf{p}}_{in} = \overrightarrow{F} \tag{22}$$

where for $\rho = \Phi_m/c^2$, $\overrightarrow{\mathbf{p}} = \int_V \rho\overrightarrow{\mathbf{p}}dV$, $\overrightarrow{F} = \int_V(\rho\overrightarrow{\mathbf{g}} + \nabla\cdot\sigma)dV$, where σ is 2-order tensor introduced in fhat follows, $\overrightarrow{\mathbf{p}}_{out}$ is added because mass leaving the control volume carriers away momentum provided by $\overrightarrow{F}$, which $\overrightarrow{\mathbf{p}}$ does not account for. The $\overrightarrow{\mathbf{p}}_{in}$ is subtracted because mass flowing into the control volume is incorrectly accounted in $\overrightarrow{\mathbf{p}}$ and hence must be discounted. Both terms are evaluated by a surface integral of the momentum flux over the entire boundary S, $\overrightarrow{\mathbf{p}}_{out} - \overrightarrow{\mathbf{p}}_{in} = \oint_S \rho(\overrightarrow{\mathbf{u}}\,\overrightarrow{\mathbf{n}})\overrightarrow{\mathbf{u}}dS = \int_V(\mathbf{e}_1\nabla\cdot(\rho u_1\overrightarrow{\mathbf{u}}) + \mathbf{e}_2\nabla\cdot(\rho u_2\overrightarrow{\mathbf{u}}) + \mathbf{e}_3\nabla\cdot(\rho u_3\overrightarrow{\mathbf{u}}))dV = \int_V \nabla\cdot(\rho\overrightarrow{\mathbf{u}}\otimes\overrightarrow{\mathbf{u}})dV$ where $\overrightarrow{\mathbf{u}}\otimes\overrightarrow{\mathbf{u}} = [u_1u_2u_3]^T\cdot[u_1u_2u_3]$ is the dyad of the velocity (second order tensor), so that we obtain the following integral momentum equation:

$$\frac{d}{dt}\int_V \rho\overrightarrow{\mathbf{u}}dV + \int_V \nabla\cdot(\rho\overrightarrow{\mathbf{u}}\otimes\overrightarrow{\mathbf{u}})dV = \overrightarrow{F} = \int_V(\nabla\cdot\sigma + \rho\overrightarrow{\mathbf{g}})dV \tag{23}$$

that is, $\int_V \frac{\partial}{\partial t}(\rho\overrightarrow{\mathbf{u}})dV + \int_V \nabla\cdot(\rho\overrightarrow{\mathbf{u}}\otimes\overrightarrow{\mathbf{u}}))dV = \overrightarrow{F} = \int_V(\nabla\cdot\sigma + \rho\overrightarrow{\mathbf{g}})dV$, which must be valid for any (also infinitesimal) control volume V, so that we obtain the second form of the Cauchy differential equation

$$\frac{\partial}{\partial t}(\rho\overrightarrow{\mathbf{u}}) + \nabla\cdot(\rho\overrightarrow{\mathbf{u}}\otimes\overrightarrow{\mathbf{u}}) = \nabla\cdot\sigma + \rho\overrightarrow{\mathbf{g}} \tag{24}$$

Which in our case where (from (27) and without viscosity) $\sigma = -P\cdot\mathbf{1}$ for the internal pressure P, we obtain

$$\frac{\partial}{\partial t}(\frac{\Phi_m}{c^2}\overrightarrow{\mathbf{u}}) + \nabla\cdot(\frac{\Phi_m}{c^2}\overrightarrow{\mathbf{u}}\otimes\overrightarrow{\mathbf{u}}) = \nabla\cdot\sigma + \frac{\Phi_m}{c^2}\overrightarrow{\mathbf{g}} = -\nabla P + \frac{\Phi_m}{c^2}\overrightarrow{\mathbf{g}} \tag{25}$$

the entire region of space it occupies (with a simply connected topology), so it can be continually subdivided into infinitesimal quantities.
In effect, the matter density of particle's body $\leftthreetimes\Phi_m$, which is a continuous material substance, must have the elastic property. That is, this body returns to its rest shape (hydrostatic equilibrium) after applied stresses are removed. For example, this property if a fundamental physical property of the material elementary particles during the collisions with the *Compton effects*.
Following the classical dynamics of Newton and Euler, the motion of particle's material body $\leftthreetimes\Phi_m$ is produced by the action of externally applied forces (during the interaction with the bosons of the fundamental field-forces, weak, strong and electromagnetic, or with the external gravitational force) which are assumed to be of the two kinds: surface forces $\overrightarrow{F}_S$ (during absorption of bosons or during the collisions with the Compton effects) and the body forces $\overrightarrow{F}_B$ (an external gravitational force or the forces generated during emissions of the bosons (as emission of the photon by an electron)).
Surface forces $\overrightarrow{F}_S$, expressed as a force per unit area, act on the bounding surface of the elementary particle's body, as a result of the mechanical interaction between the parts of the body to either side of the boundary surface (Euler-Cauchy stress principle), which uses the Cauchy stress tensor (second order)

$$\sigma \equiv \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \overrightarrow{T}^{\mathbf{e}_1} \\ \overrightarrow{T}^{\mathbf{e}_2} \\ \overrightarrow{T}^{\mathbf{e}_3} \end{pmatrix}$$

where $\overrightarrow{T}^{\mathbf{e}_j} = T_1^{\mathbf{e}_j}\mathbf{e}_1 + T_2^{\mathbf{e}_j}\mathbf{e}_2 + T_3^{\mathbf{e}_j}\mathbf{e}_3$, for $j = 1, 2, 3$, are stress vectors associated with each Cartesian coordinate unit vector $\mathbf{e}_i$. Hence, for any imaginary surface perpendicular to the unit vector $\overrightarrow{\mathbf{n}} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$, the stress vector $\overrightarrow{T}^{\mathbf{n}} = T_1^{\mathbf{n}}\mathbf{e}_1 + T_2^{\mathbf{n}}\mathbf{e}_2 + T_3^{\mathbf{n}}\mathbf{e}_3$, acting on the plane with the normal unit vector $\overrightarrow{\mathbf{n}}$, can be expressed by

$$\overrightarrow{T}^{\mathbf{n}} = \overrightarrow{\mathbf{n}} \cdot \sigma = [n_1 n_2 n_3] \cdot \sigma,$$

at each point $(t, \overrightarrow{\mathbf{r}})$.
The distribution of internal contact forces through the volume of particle's body (for example during the elastic collisions of elementary particles with Compton effects) is assumed to be continuous.
$\overrightarrow{T}^{\mathbf{n}}(t, \overrightarrow{\mathbf{r}})$ is the contact force density (or Cauchy traction field) that represents this distribution in a particle's matter density $\leftthreetimes\Phi_m$ at a given time t. It is not a vector field because $\overrightarrow{T}^{\mathbf{n}}(t, \overrightarrow{\mathbf{r}})$ depends also on the local orientation of the surface element dS as defined by its normal unit vector $\overrightarrow{\mathbf{n}}$. Any differential area dS with normal unit vector $\overrightarrow{\mathbf{n}}$ of a given internal surface S, bounding a

portion of the particle's body, experiences a contact force $\overrightarrow{F}_S$ arising from the contact between both portions of the body on each side of S, and is given by $d\overrightarrow{F}_S = \overrightarrow{T}^{\mathbf{n}}dS$.
In the case of the external gravitational (body) forces $\overrightarrow{F}_B$, the intensity of the force is proportional to the rest-mass density Φ_m/c^2 of the particle. In the case of the interactions with the bosons, during absorption/emission of the bosons, we have more complex events which accelerate/decelerate this massive elementary particle by temporary expansion/compression of its body shape (that is, of its matter distribution $\leftthreetimes\Phi_m(t, \overrightarrow{\mathbf{r}}))$.
The effect of stress in the particle's continuum flow of $\Phi_m(t, \overrightarrow{\mathbf{r}})$ is represented by the gradient of the internal pressure ∇P and divergence $\nabla\cdot\tau$ which describes viscous forces based on the 2-tensor τ, so that the stress tensor σ is given by

$$\sigma = -P\cdot\mathbf{1} + \tau \tag{27}$$

where $\mathbf{1}$ is the identity 3×3 matrix.

Conservation of the momentum law: Consequently, from (26) we obtain the following Cauchy momentum equation (in convective form derived from the second Newton's law) in a time-space point $(t, \overrightarrow{\mathbf{r}})$ inside particle's body where $\Phi_m(t, \overrightarrow{\mathbf{r}}) \neq 0$,

$$\frac{d\overrightarrow{\mathbf{u}}}{dt} = \frac{c^2}{\Phi_m}\nabla\cdot\tau - \frac{c^2}{\Phi_m}\nabla P + \overrightarrow{\mathbf{g}} \tag{28}$$

In what follows we will consider that there is no any significant viscosity inside particle's body, so that $\nabla\cdot\tau = 0$, and hence the equation above is equal to the Euler momentum equation (otherwise we would obtain the Navier-Stokes momentum equation).
Here we consider that $\overrightarrow{\mathbf{g}}$ is equal to the particle's self-gravitational internal force (by considering that locally it is much bigger than the external gravitational force). Obviously, the stress terms on the right-hand side of the equation above are yet unknown (they are the hidden variables of an elementary particle which can not be measured in every point $(t, \overrightarrow{\mathbf{r}})$ inside particle's body where $\Phi_m(t, \overrightarrow{\mathbf{r}}) \neq 0$), so that this equation can not be used to solve the dynamic phenomena of the perturbations of particle's hydrostatic equilibrium. However, they are useful in order to understand the internal dynamics of the particle's rest-mass energy flow during the accelerations of the particle (when particle is not more in the hydrostatic equilibrium because the vaiation-velocity $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})$ of the flow (in the coordinate system with the center in the particle's barycenter) in equation above is not more zero and depends on time as well). Moreover,

this equation will be used in next section for consideration of small perturbations of particle's hydrostatic equilibrium with small spherical expansions of particles body in connection with relatively small decelerations of the particle. In effect, a particle's body is considered stress-free if it is in the hydrostatic equilibrium (during inertial propagation of the particle), required to hold the fluid body together and to keep its spherically symmetric density shape Φ_m, in the absence of external interferences. Thus, in that equilibrium state, we have that $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) = 0$ in all points inside particles body, so that $\frac{d\overrightarrow{\mathbf{u}}}{dt} = 0$ and the equaiton above (in the absence of viscosity) reduces to the equation (17) of the pressure with spherically symmetric distribution during the hydrostatic equilibrium.
□
Let us consider now the enthalpy conservation and the specific internal energy of massive elementary particle. During the accelerations of an elementary particle (and its spherical expansion/compression), the internal pressure P which was static and spherically symmetric inside particle's body during previous inertial propagation (with the hydrostatic equilibrium), balansing the opposite self-gravitational force, now changes in time during spherical expansion (when $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) \neq 0$) of particle's matter density $\leftthreetimes\Phi_m$, so that $\frac{dP}{dt} \neq 0$.
Thus, we have to consider the conservation of energy in such internally unstable dynamics. It is well known that the general conservation of energy for the fluids in ordinary classical mechanics is given by the following equation

$$\rho\frac{dh}{dt} = \frac{dP}{dt} + \nabla \cdot (k\nabla T) + \phi \tag{29}$$

where h is a specific entalpy, k is the thermal conductivity of the fluid, T is temperature and ϕ is the viscous dissipation function.
In our case of the elementary particle's matter density fluid $\leftthreetimes\Phi_m$ with $\rho = \Phi_m/c^2$, we have no any kind of internal thermodynamic phenomena and viscosity dissipation, so that this general equation above reduces to

$$\frac{dh}{dt} - \frac{c^2}{\Phi_m}\frac{dP}{dt} = 0 \tag{30}$$

where the specific enthalpy is given by $h = e + \frac{P}{\rho} = e + \frac{c^2 P}{\Phi_m}$, for any point $(t, \overrightarrow{\mathbf{r}})$ inside particle's body where $\Phi_m(t, \overrightarrow{\mathbf{r}}) \neq 0$, where e is the specific *internal energy* density. So, we obtain
$\frac{de}{dt} = \frac{dh}{dt} - \frac{d}{dt}(\frac{c^2 P}{\Phi_m}) = (\frac{dh}{dt} - \frac{c^2}{\Phi_m}\frac{dP}{dt}) - \frac{c^2 P}{\Phi_m^2}\frac{d\Phi_m}{dt}$,
and hence from (30) and the rest-mass energy conservation (4), that is from

$\frac{d\Phi_m}{dt} = \frac{\partial \Phi_m}{\partial t} - \overrightarrow{\mathbf{u}}\nabla\Phi_m = -\nabla \cdot (\Phi_m \overrightarrow{\mathbf{u}}) - \overrightarrow{\mathbf{u}}\nabla\Phi_m = -\Phi_m(\nabla \cdot \overrightarrow{\mathbf{u}})$, we obtain **Conservation of internal energy law**:

$$\frac{de}{dt} = -\frac{c^2 P}{\Phi_m}\nabla \cdot \overrightarrow{\mathbf{u}} \tag{31}$$

which is the 'third Euler's equation' (obtained a century later).
Note that the specific internal energy e keeps account of the gains and losses of energy of the system that are due to changes in its internal state. The internal energy of a given state of a massive elementary particle cannot be directly measured.
However, it is fundamental concept which explains the principles of 'internal energy V', introduced [10] in the Definition 5 (internal dynamic assumption) and described in Section 2.6 in [10], dedicated to phenomena of 'virtual particles' (which does not satisfy the energy relationship $E^2 = m_0^2c^4 + c^2p^2$ for the massive particle with the total energy E and momentum p), to the phenomena of massive bosons and to the physical explanation of Higgs mechanism. In all these more complex internal dynamic phenomena of the massive particles, it is valid the energy equation $(E + V)^2 = m_0^2c^4 + c^2p^2$, where E is the measurable total energy of the particle (which during 3-D space breaking of an inertial propagation of the particle (considered as a closed system) can *remain constant*; but this breaking of 3-D symmetry will produce the changing of particle's body shape [10]). V is a potential internal energy (based on this specific internal energy e in the equation above) used, for example, during the spatial expansion of the particle's body (during strong excitations) when has to be spent some energy against the autocohesive self-gravitational force inside particle's body which dynamically changes particle's shape (with the density flow velocity $\overrightarrow{\mathbf{u}}$) and its internal pressure P.

4 Stability of massive particles: perturbations of their hydrostatic equilibrium

In the case of the particle's accelerations, if it is generated by an external gravitational force, we have that the particle is moving in a curved time-space which changes this stable rest-mass density distribution of particle's hydrostatic equilibrium . Consequently, an infinitesimal particle's matter/energy density at a point $(t, \overrightarrow{\mathbf{r}})$ begins to move, so that we obtain that the variation-velocity in this point is different from zero, i.e., $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) \neq 0$, and hence

$\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}}) \equiv \overrightarrow{\mathbf{v}}(t) + \overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})$ becomes different from the (group) velocity of the particle $\overrightarrow{\mathbf{v}}(t)$. The same fact verifies if the particle propagates in some external (electromagnetic, weak or strong) field, when the acceleration is the consequence of the interactions of the particle with the bosons of this field. Again this resulting acceleration changes particle's body-shape and internal rest-mass density distribution by introducing the variation-velocity $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) \neq 0$ different from zero and hence, from the conservation of the momentum law (28) in the absence of the viscosity, we obtain

$$\frac{d\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})}{dt} = -\frac{c^2}{\Phi_m(t, \overrightarrow{\mathbf{r}})}\nabla P(t, \overrightarrow{\mathbf{r}}) + \overrightarrow{\mathbf{g}}(t, \overrightarrow{\mathbf{r}}) \tag{32}$$

where $\overrightarrow{\mathbf{g}}(t, \overrightarrow{\mathbf{r}})$ is the internal autocohesive self-gravitational acceleration and $\nabla P(t, \overrightarrow{\mathbf{r}})$ the gradient of internal pressure inside particle's body where $\Phi_m(t, \overrightarrow{\mathbf{r}}) \neq 0$.

Active principle of particle's auto-stability: The self-gravitational forces, generated by the rest-mass energy density Φ_m, constitute also the physical principle of the auto-stability of the particles. After any acceleration, when a particle again begins to propagate in the vacuum without external fields, the self-gravitational internal forces reestablish again the stable and stationary rest-mass density of the hydrostatic equilibrium with $\overrightarrow{\mathbf{u}} = 0$ and hence the equation (32) reduces to

$$\nabla P(r) = \frac{\Phi_m(r)}{c^2}\overrightarrow{\mathbf{g}}(r) \tag{33}$$

for the self-gravitational acceleration vector $\overrightarrow{\mathbf{g}}(r)$ given previously by equation (16), so that from the equation above, we obtain again the hydrostatic pressure equation (17). This fact is obtained naturally: if there are external forces that accelerate the particle, they generate an internal movement with velocity $\overrightarrow{\mathbf{u}} \neq 0$ of its rest-mass density satisfying the equation (32), as a reaction to these forces; if external forces are eliminated, then residual internal self-gravitational forces will produce again the movement of particle's rest-mass density up to the moment when the obtained new density distribution becomes equal to that of hydrostatic equilibrium. This auto-equilibrium is realized in a very short interval of time, when the particle passes from an acceleration into an inertial propagation.
This fact demonstrated that the gravitational force is fundamental force in any elementary particle, and the self-gravitational force of the particle generates a curved time-space region around each particle (so called micro-island time-space curvature). It is considered in details in [11] (Chapter 1), dedicated to

non-separability of the Quantum Mechanics theory from the General Relativity.

Another important property of the massive particle's auto-stability must be the property which does not permit that the reduction of the radius r_0 of the particle's density Φ_m becomes less than the Schwarzschild radius $r_s = \frac{2Gm_0}{c^2}$ and hence does not permit a creation of a black hole (otherwise such a micro black hole would grow, by Compton interactions with other particles, and the whole Universe would become composed by only the black holes). Consequently, we can assume that there exists a natural limit to the particle's density compressibility which guarantee that a particle can not be transformed into a black hole, and we can optionally extend the Topology Assumptions with the following compressibility assumption:

Definition 1 Limit Compressibility Assumption*:*
For any massive particle with density distribution Φ_m, at each point $(t, \overrightarrow{\boldsymbol{r}})$ where $\Phi_m(t, \overrightarrow{\boldsymbol{r}}) > 0$, it must hold that

$$\Phi_m(t, \overrightarrow{\boldsymbol{r}}) \leq \Phi_\infty \tag{34}$$

where the finite *positive real constant Φ_∞ is the maximal possible density.*

The consequence of such a limited compression of the particle's density is that:

- At any finite 3-D region where Φ_m reaches this maximal possible density Φ_∞, the internal self-gravitational force (oriented toward particle's barycenter) in this region can not move any infinitesimal amount of this limit-compressed density in this region;

- During any inertial propagation of the particle, for which we obtain a spherically symmetric density distribution of $\Phi_m > 0$, with a barycenter in the center of a sphere with radius r_0, we have that the density Φ_m is equal to this limit-compressed density Φ_∞ at least with an infinitesimally small radius $r_m << r_0$. This nucleus is generated during transition from accelerated particle into the auto-stable state of inertial particle's propagation. It is based on the fact that the variation speed $\overrightarrow{\mathbf{u}}$ of the particle's density during the acceleration will be oriented toward the particle's barycenter during reestablishment of the particle's auto-stability during inertial propagation. The only contra-force to this density flow toward the particle's barycenter (of self-gravitational forces) is the gradient-pressure and this resistance caused by maximal possible density.

Based on this optional compressibility density principle above, during the hydrostatic equilibrium we have that

$$\Phi_m(t,r,\varphi,\theta) \equiv \begin{cases} \sigma(r), & \text{if } \; r_m \le r \le r_0(m_0); \\ \Phi_\infty & 0 \le r \le r_m \end{cases} \tag{35}$$

where $\sigma(r)$ is a continuous decreasing energy-density function such that $\sigma(r_m) = \Phi_\infty$. Obviously, we can have also that $r_m = 0$ if during the compression of the particle (in very strong interaction dynamics) its radius r_0 remains always greater than the Schwarzschild radius r_s, so that matter/energy density inside the particle's body never reaches this maximum limit Φ_∞.
Let us denote by $R_0 \equiv \sqrt[3]{\frac{3m_0c^2}{4\pi\Phi_\infty}} > 0$, the minimal possible radius r_0 of the particle when its whole density is maximal, so that in such a case $r_0 = r_m = R_0 > 0$. The stability of elementary particles requires that R_0 be greater than the Schwarrzchild's radius $r_s = \frac{2m_0G}{c^2}$, in order to avoid particle's transformation into a black-hole, and hence this principle creates the following upper limit value for the maximal possible density Φ_∞,

$$\Phi_\infty < \frac{3c^4}{32\pi G^3}\frac{1}{m_0^2} \tag{36}$$

which must be satisfied for all elementary particles, thus also for that which has the maximal value of its rest-mass m_0.
It is easy to verify that this "non-ideal" compressibility energy-density distribution in (35) satisfies the continuity equation (37). In fact, also for $r \le r_m$ where Φ_m is constant and equal to maximal density Φ_∞, we have that $\frac{\partial\Phi_m}{\partial t} = \frac{\partial\Phi_\infty}{\partial t} = 0$ and for constant energy-density speed $\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}}) = \overrightarrow{\mathbf{v}}$ (during inertial propagation with the constant particle's speed $\overrightarrow{\mathbf{v}}$), $\nabla\cdot(\Phi_m\overrightarrow{\mathbf{v}}) = \Phi_\infty\nabla\cdot\overrightarrow{\mathbf{v}} = \Phi_\infty\cdot 0 = 0$, so that $\frac{\partial\Phi_m}{\partial t} = -\nabla\cdot(\Phi_m\overrightarrow{\mathbf{w}})$, as expected.
Thus, the differential equations (38) and (39) in next section can be considered only for the region $r_m \le r \le r_0$ where Φ_m depends on r.
That is, if in the hydrostatic equilibrium the particle has this "nucleus", inside it we have a linearly decreasing gravitational force oriented to the barycenter, which can not move the energy-density because it has the maximal possible and uniform density and can not be compressed more, and the energy-density also in this part of particle is in stationary state. Consequently, in the whole particle's energy-density distribution we have the stationary condition that the variation speed $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) = 0$. During the inertial propagation in a fixed referential system in which this particle propagates with the constant (group) speed $\overrightarrow{\mathbf{v}}$,

each infinitesimal amount of particle's energy-density propagates with the same velocity $\overrightarrow{\mathbf{v}}$ along rectilinear particle's trajectory (in a locally flat Minkowski time-space).
The perturbations of massive particles can be extremely strong, like in the case of the sharp 3-D space symmetry breaking during an inertial propagation (as in the case of the double-slit experiments for the electrons [10] with a rapid transversal (cylindric) explosion of particle's body in a wide disk-like shape orthogonal to the direction of particle's trajectory and parallel to the very large massive obstacle in front of this particle), or relatively small perturbations during non-relativistic interactions with external fields (the bosons) of external gravitational forces. During extremely strong excitations, the particle's body can become very large for an extremely short interval of time with very low matter/energy density and without any "nucleus" composed by maximal density around particle's barycenter. However, after such a shot-time 3-D space explosion of the particle's body, when again particle begins to propagate in the ordinary vacuum 3-D space symmetry without any external solicitation, the self-gravity will invert the direction of the flow velocity $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})$ toward particle's barycenter, so we will have the strong compression forces inside particle's body, and the hypothesis in Definition 1 would guarantee that the particle would not be transformed into a black-hole, by constitution (at least a minimal) "nucleus" around particle's barycenter. In that cases it is reasonable to assume that inside an infinitesimaly small sphere with radius $r_m << r_0$ around particle's barycenter we do not have any movement of particle's density but only in the rest of the particle's volume.
In next section we will consider this case of the small ordinary perturbations (with consecutive de/accelerations) which ideally generate a spherically symmetric expansions/compressions of particle's body w.r.t. its spherically symmetric hydrostatic equilibrium body shape.

5 Simple example model of auto-stability for small spherical perturbations

Let us consider now the auto-stability of a massive elementary particle around its hydrostatic equilibrium. Let us consider an inertial propagation of a particle, during which it is in the hydrostatic equilibrium with a spherically symmetric energy density Φ_m, assuming that the center of the spherical coordinate system (with coordinates $\overrightarrow{\mathbf{r}} \equiv (r, \varphi, \theta)$) is in particle's barycenter, up to the time in-

stance $t = 0$, and that for $t > 0$ particle is influenced by a small perturbation (with acceleratin/deceleration) in a short interval of time. Hence we have that $\Phi_m(t, r)$ is a spherically symmetric for all $t \leq 0$, that is, that $\Phi_m(0, r) \neq 0$ for $r \leq r_0$ represents the initial hydrostatic equilibrium of the particle's material body with radius r_0 and velocity $\overrightarrow{\mathbf{u}} = 0$ in all point inside particle's body.
Thus, let us consider the solution of the differential equation (4) for the rest-mass energy density Φ_m (by considering that in this coordinate system the group velocity $\overrightarrow{\mathbf{v}}(t) = 0$ so that $\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}}) = \overrightarrow{\mathbf{v}}(t) + \overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) = \overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) = u_r \mathbf{e}_r + u_\varphi \mathbf{e}_\varphi + u_\theta \mathbf{e}_\theta$), for $t > 0$, during this small perturbation which produces a relatively small spherical expansion/compression of the particle's body.
In this simple idealized model, we consider the case when the particle is accelerated but during this acceleration preserves its 3-D spherical symmetry of its stationary inertial-propagation state (hydrostatic equilibrium) with $|\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})| > 0$ only for $r \geq r_m$ where r_m is an infinitesimal radius inside the body in which this perturbation does not change the density (we do not require that the density inside this infinitesimally small sphere has the constant value or that is equal to Φ_∞), i.e., $0 \approx r_m << r_0$. So, the equation (4), rewritten in spherical coordinates (r, φ, θ), becomes equal to

$$-\frac{\partial \Phi_m}{\partial t} = \nabla \cdot (\Phi_m \overrightarrow{\mathbf{w}}) = \frac{1}{r^2 \sin\theta}(\sin\theta \frac{\partial}{\partial r}(r^2 u_r \Phi_m) + r\frac{\partial (u_\varphi \Phi_m)}{\partial \varphi} + \frac{\partial (u_\theta \Phi_m \sin\theta}{\partial \theta}) \tag{37}$$

which, in this case of the small spherical expansion with $u_\varphi = u_\theta = 0$ and $u_r = u(t, r)$, reduces to the simple equation

$$-\frac{\partial \Phi_m(t, r)}{\partial t} = \frac{1}{r^2}(\frac{\partial}{\partial r}(r^2 u(t, r)\Phi_m(t, r)) = \frac{1}{r^2}(2ru\Phi_m + r^2\frac{\partial u}{\partial r}\Phi_m + r^2 u\frac{\partial \Phi_m}{\partial r}) \tag{38}$$

considered for $r \geq r_m > 0$ and $t > 0$ (because otherwise both sides of this equation are trivially equal to zero).
Let us seek a solution for such a spherical expansion/compression of the particle's body for $t \geq 0$, by separating the variables, $\Phi_m(t, r) = T(t)R(r)$, where $T(t)$ is a dimensionless function while $R(r)$ is a 3-D energy density function $[Joule/cm^3]$ and $u(t, r) = u_T(t)u_R(r)$ where $u_T(t) \geq 0$ is a function in $[sec^{-1}]$ and $u_R(r)$ is a 1-D function $[cm]$(which express a distance). The functions $T(t)$ is the time evolution of the rest-mass energy density and hence we fix its initial value at $t = 0$ by $T(0) = 1$. We fix also $u_T(0) = 0$, with $u_T(t) > 0$ for $0 < t < \triangle t$ where $\triangle t$ is the short time-interval during which an external force generates this perturbation.
Notice that $u_T(t)$ *is not* the time evolution of the radial velocity, and also $u_R(r)$

is not a velocity; only their product represents the radial velocity. Thus, at $t = 0$ we have still the hydrostatic equilibrium with $\Phi_m(0, r) = T(0)R(r) = R(r)$ if $r \le r_0$; 0 otherwise, and the energy flux velocity $u(0, r) = u_T(0)u_R(r) = 0$ (because from above $u_T(0) = 0$). Consequently, the equation (38) reduces into the following differential equation, for $r > r_m > 0$ (note that differently from $\Phi_m(0, r)$, for $R(r)$ we allow any finite big value for r) and $t > 0$,

$$\frac{1}{u_T(t)T(t)}\frac{\partial T(t)}{\partial t} = k = -\frac{u_R(r)}{R(r)}\frac{\partial R}{\partial r} - \frac{2}{r}u_R(r) - \frac{\partial u_R(r)}{\partial r} \tag{39}$$

where k is a *dimensionless* constant real value (because left-hand side depends only on the free time-variable t and right-hand side depends only on the free variable r (radial coordinate); all functions different from $u_R(r)$ are positive, while the radial velocity component $u_R(r) \ge 0$ for the spherical *expansion* and $u_R(r) \le 0$ for the spherical *compression* of the particle's energy-density distribution inside a sphere of time-dependent radius $r_0(t)$ during particle's accelerations.

In what follows we consider that the radial velocity $u(t, r) = u_T(t)u_R(r)$ is determined by the total external force which causes this perturbation, so that other dynamic variables can be expressed by using $u_T(t)$ and $u_R(r)$.

From the left-hand side of the equation above, we obtain the following solution for the time-dependent component of the energy-density distribution $T(t) > 0$ for any $t \ge 0$:

$$T(t) = \mathrm{e}^{k\int_0^t u_T(s)ds} \tag{40}$$

Thus, from the fact that all functions at the right-hand side of this equation are positive and $t \ge 0$, in order to obtain that the $T(t)$ component of the energy density diminishes with time during particle's spherical expansions (because the total rest-mass energy is invariant), we conclude that:

- During the spherical expansion, when $u_R(r) \ge 0$, we must have that $k < 0$ (energy density diminishes with time in any point inside the particle's energy-density sphere with radius $r_0(t)$ that increases with time);
- During the spherical compression, when $u_R(r) \le 0$, we must have that $k > 0$ (energy density increases with time in any point inside the particle's energy-density sphere with radius $r_0(t)$ which decreases with time).

Remark: the value $|k|$ is proportional to the external force which causes this spherical perturbation from the particle's hydrostatic equilibrium. In effect, a bigger value of $|k|$ will produce stronger changes of $T(t)$ which represents

the time-evolution of the particle's rest-mass energy density caused by applied external force field (gravitational or other fundamental quantum field forces mediated by the bosons). Consequently, it is a fundamental parameter which represents the strength of this external force, and will be considered in what follows as a given (non derived) parameter in all equations.
□
From the right-hand side of equation (39), we obtain the following dependance of the energy-density distribution component $R(r) \geq 0$ on the radial velocity component $u_R(r)$, for $0 \approx r_m \leq r$ (for which $u_R(r) \neq 0$) :

$$\frac{1}{R(r)}\frac{\partial R(r)}{\partial r} = -(\frac{k}{u_R(r)} + \frac{2}{r} + \frac{1}{u_R(r)}\frac{\partial u_R(r)}{\partial r}) \tag{41}$$

Notice that for $0 \leq r < r_m$ the rest-mass energy density $\Phi_m(t,r)$ does not change in time and remains equal to that in the hydrostatic equilibrium.

Lemma 2 *From the fact that* $|\frac{1}{u_R(r)}|$ *is a finite value for all* $r \geq r_m$*, the general solution of differential equation (41), for* $r \geq r_m$*, is*

$$R(r) = \frac{k_2}{r^2 u_R(r)} \mathrm{e}^{-k\int_{r_m}^{r}\frac{1}{u_R(s)}ds} \tag{42}$$

where $k_2 \neq 0$ *is a (positive or negative) real constant (in* $[Joule]$*) such that* $\frac{k_2}{u_R(r)} > 0$*. The radial component of the flow velocity* $u_R(r)$ *for* $r_m \leq r$*, depends on the rest-mass density* $\Phi_m(0,r)$ *of the particle in the hydrostatic equilibrium as follows:*

$$\frac{1}{u_R(r)}(\frac{\partial u_R}{\partial r} + k) = -(\frac{2}{r} + \frac{1}{\Phi_m(0,r)}\frac{\partial \Phi_m(0,r)}{\partial r}) \tag{43}$$

Proof: It is easy to verify that (42) is the solution of the differential equation (41) where partial derivatives can be substituted by total (material) derivatives because their arguments are the functions of only one variable r, and the fact that $\frac{d}{dr}\mathrm{e}^{-k\int_{r_m}^{r}\frac{1}{u_R(s)}ds} = \mathrm{e}^{-k\int_{r_m}^{r}\frac{1}{u_R(s)}ds}(\frac{d}{dr}(-k\int_{r_m}^{r}\frac{1}{u_R(s)}ds) = -\frac{k}{u_R(r)}\mathrm{e}^{-k\int_{r_m}^{r}\frac{1}{u_R(s)}ds}$ holds because the function $\frac{1}{u_R(s)}$ in the integral is finite in the range of the integration. From the fact that the value $\mathrm{e}^{-k\int_{r_m}^{r}\frac{1}{u_R(s)}ds}$ is dimensionless, $r^2 u_R(r)$ is a 3-D volume and $R(r)$ is an 3-D energy density, we obtain that the real constant k_2 has the unit of energy (in $[Joule]$).
Let us denote shortly $\mathrm{e}^{k\int_{r_m}^{r}\frac{1}{u_R(s)}ds}$ by the function $f(r)$, so that $\frac{1}{f(r)}\frac{\partial f}{\partial r} = \frac{k}{u_R(r)}$. Hence, in the hydrostatic equilibrium when $R(r) = \Phi_m(0,r)$, for $r_m \leq r \leq r_0$,

where r_0 is the radius of the particle in the hydrostatic equilibrium, and from (42), we obtain (a) $u_R(r)f(r) = \frac{k_2}{r^2\Phi_m(0,r)}$, and by differentiation $\frac{\partial}{\partial r}\frac{k_2}{r^2\Phi_m(0,r)} = \frac{\partial u_R(r)f(r)}{\partial r} = \frac{\partial u_R}{\partial r}f(r) + u_R(r)\frac{\partial f}{\partial r} = (\frac{\partial u_R}{\partial r} + k)f(r)$, and by division of this equation by equation (a), we obtain
$\frac{1}{u_R(r)}(\frac{\partial u_R}{\partial r} + k) = \frac{\partial}{\partial r}\frac{k_2}{r^2\Phi_m(0,r)}(\frac{k_2}{r^2\Phi_m(0,r)})^{-1} = -(\frac{2}{r} + \frac{1}{\Phi_m(0,r)}\frac{\partial\Phi_m(0,r)}{\partial r})$,
that is, the equation (43).
□

Example 1 *The equation (43) gives us the possibility to obtain the radius-dependent component $u_R(r)$ of the radial velocity $u(t,r)$ if we know the hydrostatic equilibrium solution of the particle's rest-mass energy density $\Phi_m(0,r)$ as, for example, in these three cases of the spherical expansion(when $u_R(r) > 0, k_2 > 0$ and $k < 0$):*

1. *For the energy density $\Phi_m(0,r) = \frac{K}{r}$ where, from $m_0c^2 = \int \Phi_m dV = 4\pi\int_0^{r_0}\Phi_m r^2 dr$, we obtain $K = \frac{2m_0c^2}{4\pi r_0^2}$, and hence*
$u_R(r)\mathrm{e}^{k\int_{r_m}^{r}\frac{1}{u_R(s)}ds} = \frac{k_2}{K}\frac{1}{r}$, *i.e.,* $-k\int_{r_m}^{r}\frac{1}{u_R(s)}ds = ln(\frac{K}{k_2}u_R(r)r)$,
so, by differentiation on r of both sides of the last equation, we obtain $\frac{-k}{u_R} = \frac{1}{ru_R}\frac{d}{dr}(ru_R)$, i.e., the differential equation $u_R(r) + r\frac{du_R}{dr} = -kr$, with the linear solution
$u_R(r) = -\frac{k}{2}r > 0$,
and, by substitution in one of the integral equation above, we obtain $r_m^2 = -\frac{2k}{kK} > 0$, that is, $r_m = r_0(-\frac{4\pi}{k}\frac{k_2}{m_0c^2})^{1/2}$. So, from the fact that k_2 is an infinitesimal rest-mass energy in an infinitesimal region inside particle's body, we have that $\frac{k_2}{m_0c^2} \approx 0$, so that $r_m << r_0$ is an infinitesimal radius as assumed.

2. *For the distribution $\Phi_m(0,r) = \frac{K}{r^2}$ where, from $m_0c^2 = \int \Phi_m dV = 4\pi\int_0^{r_0}\Phi_m r^2 dr$, we obtain $K = \frac{m_0c^2}{4\pi r_0}$, and hence*
$u_R(r)\mathrm{e}^{k\int_{r_m}^{r}\frac{1}{u_R(s)}ds} = \frac{k_2}{K}$, *i.e.,* $-k\int_{r_m}^{r}\frac{1}{u_R(s)}ds = ln(\frac{K}{k_2}u_R(r))$,
so, by differentiation on r of both sides of the last equation, we obtain $\frac{du_R}{dr} = -k$ with the linear solution
$u_R(r) = -kr > 0$, *for* $r \geq r_m = -\frac{k_2}{kK} = -\frac{4\pi}{k}\frac{k_2}{m_0c^2}r_0 > 0$.
So, from the fact that k_2 is an infinitesimal rest-mass energy in an infinitesimal region inside particle's body, we have that $\frac{k_2}{m_0c^2} \approx 0$, so that $r_m << r_0$ is an infinitesimal radius as assumed.

In effect, we can generalize this example as follows:

Corollary 1 *The linear solutions for $u_R(r)$ are possible if the hydrostatic equilibrium is of the form $\Phi_m(0,r) = \frac{K}{r^n}$ for any real n such that $0 \le n < 3$, with $K = \frac{(3-n)m_0c^2}{4\pi r_0^{3-n}}$.*
In all these cases, the linear solution is $u_R(r) = \frac{k}{n-3}r$ with $r_m = r_0(\frac{4\pi}{|k|}\frac{k_2}{m_0c^2})^{1/(3-n)}$ so that $0 < r_m << r_0$.

Proof: If we substitute this form of $\Phi_m(0,r)$ into (43), we obtain the differential equation $(n-2)u_R(r) - r\frac{du_R}{dr} = kr$, with the solution $u_R(r) = \frac{k}{n-3}r$. The limit for the real value n is $n < 3$, because u_R and k must have the opposite signs. Moreover, we can not permit that $n < 0$ because, in that case, we would obtain that in the barycenter (for $r = 0$) we have the zero energy-density, in contrast with the particle's topological properties.
□
It is easy to verify that the two examples above are the particular cases of this corollary for $n = 1.0$ and $n = 2.0$.
Based on this corollary and the possible model of $\Phi_m(0,r)$ in the hydrostatic equilibrium, let us try to find what can be the true candidate for $0 \le n < 3$ of the hydrostatic equilibrium. In effect, based on the equations (16) and (17) for the hydrostatic equilibrium, the dependences of the self-gravitational acceleration $g(0,r)$, internal pressure $P(0,g)$ and internal self gravitational force $\overrightarrow{F} = F\mathbf{e}_r = \frac{\Phi_m}{c^2}\overrightarrow{\mathbf{g}} = \nabla P$, are the following:
$g = -\frac{4\pi GK}{c^2(3-n)}r^{1-n}$, $F = \frac{dP}{dr} = -\frac{4\pi GK^2}{c^4(3-n)}r^{1-2n}$ and $P(r) = P_0 - A(n)r^{2-2n} > 0$ if $n \ne 1$ (notice that for $n = 1$ we would have $P = P_0 - \frac{2\pi GK^2}{c^4}ln(r))$ where $A(n) = \frac{2\pi GK^2}{c^4(3-n)(1-n)} = \frac{m_0^2c^2G}{8\pi r_0^{6-2n}}\frac{3-n}{1-n}$ and P_0 is a constant pressure in particle's barycenter (where $r = 0$) such that guarantees that $P(r)$ be always a positive value inside the particle's body. The dependencies on r can be shown in the following table:

n	Self-gravitational g	Pressure P	Internal force F
0	$-r$	$P_0 - A(0)r^2$	$-r$
0.1	$-r^{0.9}$	$P_0 - A(0.1)r^{1.8}$	$-r^{0.8}$
0.9	$-r^{0.1}$	$P_0 - A(0.9)r^{0.2}$	$-1/r^{0.8}$
1.1	$-1/r^{0.1}$	$P_0 + A(1.1)/r^{0.2}$	$-1/r^{1.2}$
1.5	$-1/\sqrt{r}$	$P_0 + A(1.5)/r$	$-1/r^2$
2.0	$-1/r$	$P_0 + (A(2)/r^2$	$-1/r^3$
2.5	$-1/r^{1.5}$	$P_0 + A(2.5)1/r^3$	$-1/r^4$
2.9	$-1/r^{1.9}$	$P_0 + A(2.9)/r^{3.8}$	$-1/r^{4.8}$

From the fact that the pressure must be positive (against the self-gravitational force) during the hydrostatic equilibrium, we can chose enough big constant pressure P_0 such that the internal pressure on the particle's surface is enough in order to guarantee its compactness during the *elastic collisions* between massive particle with Compton effects.

The first comment is that, for $1 < n < 3$, we have the singularities with very rapid changing of internal force for $r \mapsto 0$, which is not a property of equilibrium. Moreover, we have that on the particle's surface the internal pressure is *minimal* instead of to be *maximal* (what is necessary in order to have the elastic Compton effects during weak collisions between massive particles - in that case the strong internal pressure on the particle's boundary (surface) is more resistant to the deformations during the Compton-effect collisions, and permits the fusion between two massive particles during the collisions only in high-energy impacts.

Consequently, it seems reasonable to seek for the ideal model in the range $0 \leq n < 1$; in effect, the best candidate for the hydrostatic equilibrium is that one for which we have the *equilibrium of the internal force* (see the Lemma 1)in each point inside particle's body and hence the most simple (which is usually also reasonable from the physicist's point of view that the good solutions in nature are frequently the most simple solutions) model for the density of the particle's body-matter $\leftthreetimes\Phi_m$ in particle's hydrostatic equilibrium seems to be the following:[4]

[4]Notice that it is different from $\Phi_m(0, r) = \frac{K}{r^2}$, assumed in [10], Section 2.5, derived from the erroneous definition of the self-gravitational internal force in the hydrostatic equilibrium of the massive particle.

Corollary 2 *The solution for the hydrostatic equilibrium for $r_m \leq r \leq r_0$ is given by*

$$\Phi_m(0,r) = \frac{K}{\sqrt{r}} \tag{44}$$

with the linearly deceasing internal pressure $P(r) = P_0 - \frac{8\pi G K^2}{5c^4} r > 0$, where, from the Corollary 1, $K = \frac{2.5 m_0 c^2}{4\pi r_0^{2.5}}$ and $r_m = r_0(\frac{4\pi}{|k|}\frac{k_2}{m_0 c^2})^{1/(2.5)}$, so that $0 < r_m << r_0$.

Proof: If we replace the model $\Phi_m(0,r) = \frac{K}{r^n}$ into the second order differential equation in Lemma 1, we obtain the following equation:
$0 = n(n+1)r^{-n-2} + (\frac{4}{r} - 3r^n(-nr^{-n-1})(-nr^{-n-1}) + 2r^{-n-2}$,
that is, the quadratic equation
$-2n^2 - 3n + 2 + 0$,
which has only one acceptable solution $n = 1/2$ (other one is negative value -2 which physically is not acceptable because we would have maximal particle's density on its boundary surface and zero in the particle's barycenter).
Let us show that this is *the* solution. In effect. the first-order differential equation (18) can be rewritten by substitution $h(r) = (\frac{1}{\Phi_m(0,r)})^2$, for $0 \leq r \leq r_0$ where $\Phi_m(0,r) > 0$, in the following form

$$\frac{dh(r)}{dr} + \frac{4}{r}h(r) = -\frac{8\pi G}{Fc^4} \tag{45}$$

where the right-hand side is a constant value. From the first and second functional term on the left-hand side, we deduce that $h(r)$ must be a polyoma on r, that is, for some positive integer $n \geq 1$, we seek the polynomial solution $h(r) = \sum_{i=1}^n a_i r^i$, so by substitution in (45), we obtain the equation
$\sum_{i=1}^n a_i(i+4)r^{i-1} = -\frac{8\pi G}{Fc^4}$
with the unique solution $a_1 = -\frac{8\pi G}{5Fc^4}$ and $a_i = 0$ for $i \geq 2$. So, we obtain the unique solution $(\frac{1}{\Phi_m(0,r)})^2 = h(r) = -\frac{8\pi G}{5Fc^4}r$ with $F < 0$, that is, $\Phi_m(0,r) = \frac{K}{\sqrt{r}}$ where $K^2 = -\frac{5Fc_4}{8\pi G}$, that is, $F = -\frac{8\pi G K}{5c^4} < 0$. It is easy to verify that really this constant value F is obtained from the computation of the internal force $F = \frac{\Phi_m(0,r)}{c^2}g(r) = \frac{\Phi_m(0,r)}{c^2}(-\frac{4\pi G}{r^2}\int_0^r \frac{\Phi_m(0,s)}{c^2}s^2 ds)$ for $\Phi_m(0,r) = \frac{K}{\sqrt{r}}$.
□
Remark: From Corollary 2, we obtain that $R(r) = \frac{K}{r^n}$ for $n = 1/2$ (because $\Phi_m(0,r) = T(0)R(r) = 1 \cdot R(r) = R(r)$) so, from the Corollary 1, we obtain that $u_R(r) = -\frac{2k}{5}r$ for a given perturbation with the strength fixed by the value of k. The fact that the radial velocity component $u_R(r)$ is incrementing with r is compatible with the necessities of particle's "radial explosions" in extremely

excitations during extremely short time-intervals required by the theory in [10]. □

Let us consider now the dynamic changes of this hydrostatic equilibrium density during the small spherical perturbations:

Proposition 1 *The changing of the particle's rest-mass energy density during a small spherical perturbations for $t > 0$ and $r_m \leq r \leq r_0(t)$ is given by*

$$\Phi_m(t,r) = \frac{k_2}{r^2 u_R(r)} \mathrm{e}^{k(\int_0^t u_T(s)ds - \int_{r_m}^r \frac{1}{u_R(s)}ds)} \tag{46}$$

where the constants k is a given parameter (negative for particle's expansion) representing the external force which causes the perturbation from the hydrostatic equilibrium of a massive particle with rest-mass m_0 and with radius r_0, while k_2 is a derived real constant with the same sign as $u_R(r)$

$$k_2 = \frac{m_0c^2}{4\pi}(\int_0^{r_0} \frac{1}{u_R(r)} \mathrm{e}^{-k\int_{r_m}^r \frac{1}{u_R(s)}ds} dr)^{-1} = -\frac{km_0c^2}{4\pi}(\frac{r_m}{r_0})^{5/2} = -\frac{2kr_m^{5/2}}{5}K \tag{47}$$

The changing of the radius $r_0(t)$, with $r_0(0) = r_0$, of the particle's body $\leftthreetimes\Phi_m(t,r)$, for $t > 0$ is given by

$$\frac{dr_0(t)}{dt} = -\frac{km_0c^2}{4\pi k_2} u_T(t) u_R(r_0(t)) \mathrm{e}^{-k(\int_0^t u_T(s)ds - \int_{r_m}^{r_0(t)} \frac{1}{u_R(s)}ds)} \tag{48}$$

or by a pure second-order differential equation:

$$\frac{d^2r_0(t)}{dt^2} = (\frac{1}{u_T(t)}\frac{du_T(t)}{dt} - ku_T(t))\frac{dr_0(t)}{dt} + \frac{1}{u_R(r_0(t))}(k + \frac{du_R(r)}{dr})(\frac{dr_0(t)}{dt})^2 \tag{49}$$

Proof: From the definition $\Phi_m(t,r) = T(t)R(r)$ and hence from (40) and (42), we obtain the solution during spherical perturbation for $t > 0$ (for a small interval of time) given by (46).

The constant k_2 can be derived by using the equation (46) for the computation of the particle's rest-mass energy:
$m_0c^2 = \int \Phi_m(t,r)dV = 4\pi \int_0^{r_0(t)} \Phi_m r^2 dr = 4\pi T(t) \int_0^{r_0(t)} R(r)r^2 dr$
$= k_2(4\pi T(t) \int_0^{r_0} \frac{1}{u_R(r)} \mathrm{e}^{-k\int_{rm}^r \frac{1}{u_R(s)}ds} dr)$, so that
$k_2 = \frac{m_0c^2}{4\pi T(t)}(\int_0^{r_0(t)} \frac{1}{u_R(r)} \mathrm{e}^{-k\int_{rm}^r \frac{1}{u_R(s)}ds} dr)^{-1}$,
and for $t = 0$, from $T(0) = 1$ and $r_0(0) = r_0$, we obtain the first result in (47), and by substitution $u_R(r) = -\frac{2k}{5}r$, we obtain the second result in (47), while third result is obtained from the fact that $K = \frac{2.5m_0c^2}{4\pi r_0^{5/2}}$.

Notice that, from the first line of the derivation above, we obtain that the time-evolution of the particle's density is given by

$$\int_0^{r_0(t)} R(r)r^2 dr = \frac{m_0 c^2}{4\pi T(t)} \tag{50}$$

By differentiation of both sides of this equation above and from Leibniz integral rule we have that the right-hand side is equal to $\frac{d}{dt}\int_0^{r_0(t)} R(r)r^2 dr = R(r_0(t))r_0^2(t)\frac{dr_0(t)}{dt}$ while the left-hand side is $\frac{d}{dt}\frac{m_0c^2}{4\pi T(t)} = -\frac{m_0c^2}{4\pi T^2(t)}\frac{dT}{dt}$ and by substitution of $k = \frac{1}{u_t(t)T(t)}\frac{dT}{dt}$ we obtain that
$\frac{dr_0(t)}{dt} = -\frac{km_0c^2}{4\pi R(r_0(t))r_0^2(t)}\frac{u_T(t)}{T(t)} = -\frac{km_0c^2}{4\pi k_2}u_R(r_0(t))\frac{u_T(t)}{T(t)}\mathrm{e}^{k\int_{r_m}^{r_0(t)}\frac{ds}{u_R(s)}}$,
and hence by substitution of $T(t)$, we obtain the equation (48). The equation above can be rewriten in this form
$k\int_{r_m}^{r_0(t)}\frac{ds}{u_R(s)} = ln(\frac{-4\pi k_2}{km_0c^2}\frac{T(t)}{u_T u_R(r_0(t))}\frac{dr_0}{dt})$, so by derivation $\frac{d}{dt}$, by using Leibniz integral rule for the left-hand side), we obtain
$k\frac{1}{u_R(r_0(t))}\frac{dr_0}{dt}$
$= \frac{d}{dt}ln(\frac{-4\pi k_2}{km_0c^2}\frac{T(t)}{u_T(t)u_R(r_0(t))}\frac{dr_0}{dt})$
$= \frac{d}{dt}ln(\frac{T(t)}{u_T(t)u_R(r_0(t))}\frac{dr_0}{dt})$
$= \frac{u_T(t)u_R(r_0(t))}{T(t)\frac{dr_0}{dt}}\frac{d}{dt}(\frac{T(t)}{u_T(t)u_R(r_0(t))}\frac{dr_0}{dt})$
$= \frac{1}{T(t)}\frac{dT}{dt} - \frac{1}{u_T(t)}\frac{du_T}{dt} - \frac{1}{u_R(r_0(t)))}\frac{du_R(r_0(t))}{dt} + \frac{1}{\frac{dr_0}{dt}}\frac{d^2r_0}{dt^2}$
$= \frac{1}{T(t)}\frac{dT}{dt} - \frac{1}{u_T(t)}\frac{du_T}{dt} - \frac{1}{u_R(r_0(t)))}\frac{du_R(r)}{dr}\frac{dr_0}{dt} + \frac{1}{\frac{dr_0}{dt}}\frac{d^2r_0}{dt^2}$,
and hence, by using $ku_T(t) = \frac{1}{T(t)}\frac{dT}{dt}$ from (39), we obtain the second-order differential equation (49).
□
It is easy to verify that (46) reduces into $\Phi_m(t,r) = T(t)\frac{K}{r^n}$ for the linear expansion velocity $u_R(r) = -\frac{2}{5}kr$ obtained from Corollary 1 for $n = 1/2$ of the hydrostatic-equilibrium model.
Hence, for the computation of the time evolution of the particle's body radius $r_0(t)$, during its spherical expansion/compression, we can use the ordinary second-order differential equation (49). However, we can use also the integral-differential equation (48). Let us consider it for an infinitesimal but finite time $t = \delta t$ of perturbation from particle's hydrostatic equilibrium. Then
$\frac{dr_0}{dt} \approx \frac{r_0(\delta t)-r_0}{\delta t}$ and for $T(\delta t) = \mathrm{e}^{-k\int_0^{\delta t}u_T(s)ds} \approx T(0) = 1$, from (48) we obtain:
$r_0(\delta t) = r_0 - \frac{km_0c^2}{4\pi k_2}u_R(r_0)\mathrm{e}^{k\int_{r_m}^{r_0}\frac{1}{u_R(s)}ds)}u_T(\delta t)\delta t = r_0 - \frac{km_0c^2}{4\pi}\frac{u_T(\delta t)\delta t}{r_0^2 R(0,r)}$
$= r_0 - \frac{km_0c^2}{4\pi}\frac{u_T(\delta t)\delta t}{r_0^2\Phi_m(0,r)}$, where $\Phi_m(0,r_0)$ is the hydrostatic equilibrium density on

the surface of particle's body.
Consequently, any short interval of time t can be divided in a number of very small intervals δt such that $t = N\delta t$ for $N >> 1$, and we can use the following recursive algorithm to compute the dynamic changing of the particle's radius $r(t)$, from initial hydrostatic-equilibrium radius r_0, during these spherical perturbations:

Time	radius
0	$r_0(0) = r_0$
δt	$r_0(\delta t) = r_0(0) - \frac{km_0c^2}{4\pi}\frac{u_T(\delta t)\delta t}{r_0^2\Phi_m(0,r)}$
...	...
$t-\delta t$	$r_0(t-\delta t) = r_0(t-2\delta t) - ...$
t	$r_0(t) = r_0(t-\delta t) - \frac{km_0c^2}{4\pi k_2}u_R(r_0(t-\delta t))\cdot$
	$\cdot\, \mathrm{e}^{-k(\int_0^{t=\delta t} u_T(s)ds - \int_{rm}^{r_0(t-\delta t)}\frac{1}{u_R(s)}ds)}u_T(t-\delta t)\delta t$

However, if we assume the hydrostatic-equilibrium model (44) and hence $R(r) = T(0)R(r) = \Phi_m(0,r) = \frac{K}{\sqrt{r}} = \frac{2.5m_0c^2}{4\pi r_0^{5/2}}r^{-1/2}$ then we can use directly the equation (50) in order to compute $r_0(t)$ as follows
$\frac{m_0c^2}{4\pi T(t)} = \int_0^{r_0(t)} R(r)r^2dr = \frac{2.5m_0c^2}{4\pi r_0^{5/2}}\int_0^{r_0(t)} r^{3/2}dr = \frac{2.5m_0c^2}{4\pi r_0^{5/2}}\frac{2}{5}r_0^{5/2}(t) = \frac{m_0c^2}{4\pi}(\frac{r_0(t)}{r_0})^{5/2}$,
and we obtain the simple solution

$$r_0(t) = r_0T(t)^{-2/5} = r_0(\mathrm{e}^{-k\int_0^t u_T(s)ds})^{2/5} \tag{51}$$

where $k < 0$ during the spherical expansion and $k > 0$ during the spherical compression. In effect, it is easy to verify that (51) is the solution of the second-order differential equation (49) when $u_R(r) = -\frac{2}{5}kr$ (with the energy-density speed $u(t,r) = u_T(t)u_R(r) = -\frac{2}{5}ku_T(t)r$) is the linear solution in Corollary 1 for $n = 1/2$ of the hydrostatic-equilibrium model.
So, from (51), the speed of particles expansion is
$\frac{dr_0}{dt} = -\frac{2}{5}r_0T(t)^{-7/5}\frac{dT}{dt} = -\frac{2}{5}r_0ku_T(t)T(t)^{-2/5} = -\frac{2}{5}ku_T(t)u_Rr_0(t) = u(t,r_0(t))$,
i.e., exactly the speed $u(t,r_0(t)) = u_T(t)u_R(r_0(t))$ of the energy-density of the particle's body surface. It is proportional to the strength of the external force (expressed by k) and increments linearly with the particle's dynamic radius $r_0(t)$ as expected. If we replace $u_r(r)$ by $-\frac{2k}{5}r$ into (48) then it is easy to verify that this equation reduces to the simple equation $\frac{dr_0}{dt} = u(t,r_0(t))$ above (obtained in a different way from (51) previously). It is also easy to compute from (51) the acceleration $\frac{d^2r_0(t)}{dt^2}$ of the particle' body expansion during an expansive perturbation, and to show that when we replace $u_r(r)$ by $-\frac{2k}{5}r$ into

(49) then this equation reduces to the simple equation $\frac{d^2 r_0(t)}{dt^2} = \frac{d}{dt}u(t, r_0(t)) = u_R(r_0(t))\frac{du_T}{dt} + u_R(r_0(t))\frac{du_R}{dr}u_T^2(t)$.

Example 2 *Let us consider a single interaction of a massive particle with a single boson of some external field. We consider that such an absorbtion of this boson by the direct collision with this massive particle happens in an extremely short interval of time $\triangle t$ so that the function $u_T(t) = 1$ for $0 < t \leq \triangle t$ and zero for $t \leq 0$ and for $t > \triangle t$. In this case we obtain that the time-evolution of the particle's density for $0 < t \leq \triangle t$ is a simple exponential function $T(t) = \mathrm{e}^{kt}$ and hence we have an exponential decreasing (for $k < 0$ during a small spherical expansion) of the particles density from its hydrostatic equilibrium,*
$\Phi_m(t, r) = T(t)R(r) = \frac{K}{\sqrt{r}}\mathrm{e}^{kt}$,
while the radius of the particle expands exponentially in time $r_0(t) = r_0\mathrm{e}^{-\frac{2}{5}kt}$ and the speed of spherical expansion of the particle's body increments exponentially in time
$\frac{dr_0}{dt} = -\frac{2}{5}r_0 k\mathrm{e}^{-\frac{2}{5}kt}$
for this extremely short interval of time $\triangle t$.
Obviously, such an impact of the boson will temporarily accelerate/decelerate this massive particle so that after this impact the particle will continue an inertial propagation (by returning into its hydrostatic equilibrium again) but with a different velocity vector of propagation. We recall [10] that, before the direct collision (and successive absorption) of a massless boson with this massive particle, also this boson transforms from the point-like particle into a massive boson with a finite 3-D region of its energy-density.
□

Based on (51), we are able to represent all time-dependent components of radial expansion in dependence only on the dynamic radial expansion/compression $r_0(t)$ of the particle:

Corollary 3 *The time-dependent component of particle's spherical perturbation are given by*

$$T(t) = \left(\frac{r_0}{r_0(t)}\right)^{5/2}, \quad \textit{and} \qquad u_T(t) = -\frac{5}{2k}\frac{1}{r_0(t)}\frac{dr_0}{dt} \tag{52}$$

Proof: The first equation is obtained directly from (51), while the second from equation (39), i.e., from $u_T(t) = \frac{1}{kT(t)}\frac{dT}{dt}$. Notice that if we represent the normalized form of $u_T(t) = -\frac{5}{2k}\frac{1}{r_0(t)/r_0}\frac{dr_0/r_0}{dt}$ and substitute it in (40) we obtain the

first equation of this corollary as well.
□
Consequently, we have that the particle's density time-evolution and radial speed of spherical extension/compression are given by

$$\Phi_m(t,r) = (\frac{r_0}{r_0(t)})^{5/2}\frac{K}{\sqrt{r}}, \quad and \quad u(t,r) = u_T(t)u_R(r) = \frac{r}{r_0(t)}\frac{dr_0}{dt} \tag{53}$$

so that the matter/energy density on the particle's spherical surface (when $r = r_0(t)$) is equal to

$$\Phi_m(t, r_0(t)) = \frac{5}{8\pi}\frac{m_0c^2}{r_0^3(t)} = \frac{5}{6}\frac{m_0c^2}{V_t} \tag{54}$$

where V_t is the volume of the particle at the time t and hence the particle's density on its surface is $\frac{5}{6}$ of the medium density.
Now we will consider internal dynamics connected with evolution of the internal pressure P during spherical perturbations:

Lemma 3 *In the absence of the viscosity, during the small spherical perturbations also the internal pressure $P(t,r)$ is a spherically symmetric with its gradient inside particle's body $r_m \leq r \leq r_0(t)$,*

$$-\frac{\partial P(t,r)}{\partial r} = \frac{k_2T(t)}{c^2r^2u_R(r)}\mathrm{e}^{-k\int_{r_m}^{r}\frac{1}{u_R(s)}ds)}\cdot$$
$$[\ u_R(r)\frac{du_T(t)}{dt} + u_T(t)^2u_R(r)\frac{du_R(r)}{dr} + \frac{4\pi Gk_2T(t)}{r^2}\int_0^r\frac{1}{u_R(s)}\mathrm{e}^{-k\int_{r_m}^{s}\frac{1}{u_R(q)}dq}ds\ ] \tag{55}$$

where $r_0(t)$ is the radius of particle's body in the time instance $t > 0$, and time evolution $T(t) = \mathrm{e}^{k(\int_0^t u_T(s)ds}$.

Proof: From (32) we obtain

$$-\nabla P(t,r) = \frac{\Phi_m(t,r)}{c^2}(\frac{d\overrightarrow{\mathbf{u}}(t,r)}{dt} - \overrightarrow{\mathbf{g}}(t,r)) \tag{56}$$

So by substitution that $-\nabla P(t,r) = -\frac{\partial P(t,r)}{\partial r}\mathbf{e}_r$, $\frac{d\overrightarrow{\mathbf{u}}(t,r)}{dt} = \frac{\partial\overrightarrow{\mathbf{u}}(t,r)}{\partial t} - (\overrightarrow{\mathbf{u}}(t,r)\cdot\nabla)\overrightarrow{\mathbf{u}}(t,r) = (u_R(r)\frac{\partial u_T(t)}{\partial t} + u_T(t)^2u_R(r)\frac{\partial u_R}{\partial r})\mathbf{e}_r$, $\frac{\Phi_m(t,r)}{c^2} = \frac{T(t)R(r)}{c^2}$ and $\overrightarrow{\mathbf{g}}(t,r)) = -\frac{G}{r^2}m = -\frac{4\pi G}{r^2}\int_0^r\Phi_m(t,s)s^2ds = -\frac{4\pi GT(t)}{r^2}\int_0^r R(s)s^2ds$, by substitution of $R(r)$ given by (42), we obtain the equation (55).
□

Hence, for the hydrostatic equilibrium model given by Corollary 2, when $R(r) = \frac{K}{\sqrt{r}}$ and $u_R(r) = -\frac{2}{5}kr$, we obtain that the internal force (*pressure-gradient force* per unit of area) inside particle's body is

$$F(t,r) = -\frac{\partial P(t,r)}{\partial r} = -\frac{\partial P(0,r)}{\partial r}T^2(t) - \frac{2kK}{5c^2}T(t)(\frac{du_T(t)}{dt} - \frac{2k}{5}u_T^2(t))\sqrt{r} \quad > 0 \tag{57}$$

where $-\frac{\partial P(0,r)}{\partial r} = \frac{8\pi G K^2}{5c^4} > 0$ is the constant internal force inside the particle's body during its hydrostatic equilibrium.

If we substitute $t = 0$ in (57) we obtain the solution of the hydrostatic equilibrium when the internal force inside particle's body $\overrightarrow{F} = -\frac{\partial P(0,r)}{\partial r}\mathbf{e}_r > 0$ is balanced by the self-gravitational force oriented into particle's barycenter. However, for $t > 0$ in the case of spherical expansion (when $k < 0$) we have also the second term created by the external force, which caused this perturbation and generated an internal density speed $u(t,r) = u_T(t)u_R(r) = -\frac{2}{5}kru_T(t) > 0$ with the acceleration $\frac{d}{dt}u(t,r) = \frac{du_T(t)}{dt}u_R(r) = -\frac{2}{5}kr\frac{du_T(t)}{dt}$ proportional to $\frac{du_T(t)}{dt} \neq 0$. Thus, now the internal force $\overrightarrow{F}(t,r)$ given by (57) is not more constant in each point of particle's body but changes proportionally to $\sqrt{r}$ (in each fixed time-instance $t > 0$ if $\frac{du_T(t)}{dt} \neq 0$). That is, the major changes of the internal force we have at the parts of particle's body that are more far from particle's barycenter, as expected.

Let us consider a complete cycle of particle's auto-equilibrium dynamics when a particle passes from an inertial propagation with a stationary distribution $\Phi_m = \frac{K}{\sqrt{r}}$ to a small acceleration and consecutive energy-density spherical expansion, and then, after some interval of time, again returns into the inertial stationary propagation:

- **Expansion process**: It happens when, at $t = 0$, a particle, from its initial inertial stationary propagation with energy-density distribution $\Phi_m(0,r) = \frac{K}{\sqrt{r}}$ in the sphere with initial radius r_0, starts to be accelerated (by the direct action of an external field) and begins the process of particle's expansion with the velocity $\overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{u}} = u(t,r)\mathbf{e}_r = u_T(t)u_R(r)\mathbf{e}_r = -\frac{2}{5}kr\mathbf{e}_r$ (with $k < 0$).

  During this particle's acceleration, i.e., the time interval $0 < t < \triangle t$, this stationary energy density changes as a result of the spherical expansion velocity $u(t,r) = u_T(t)u_R(r)$. In this case, at the end of this expansion, we obtain that the density of the particle's body diminished and becomes equal to

$$\Phi_m(\triangle t,r)=T(\triangle t)\frac{K}{\sqrt{r}}=\frac{1}{\mathrm{e}^{|k|\int_0^{\triangle t}u_T(s)ds}}\frac{K}{\sqrt{r}}<\frac{K}{\sqrt{r}} \tag{58}$$

for $r_m < r \leq r_0(\triangle t) = r_0T(\triangle t)^{-2/5} = r_0\mathrm{e}^{\frac{2}{5}|k|\int_0^{\triangle t}u_T(s)ds}$ and hence with $r_0(\triangle t) > r_0$.

During this period of the expansion (when $k < 0$) the second term on the right-side of the internal force

$$F(t,r)=-\frac{\partial P(t,r)}{\partial r}=-\frac{\partial P(0,r)}{\partial r}T^2(t)-\frac{2kK}{5c^2}T(t)\left(\frac{du_T(t)}{dt}-\frac{2k}{5}u_T^2(t)\right)\sqrt{r}\;>0 \tag{59}$$

is positive as well. However, at the end of this time interval, i.e., at $t = \triangle t$, when the external force disappears, also the expansion acceleration disappears, that is $\frac{du_T(t)}{dt}|_{t=\triangle t} = 0$. So, in this instance of time the internal force becomes $\overrightarrow{F}(\triangle t,r) = -\frac{\partial P(\triangle t,r)}{\partial r} = -\frac{\partial P(0,r)}{\partial r}T^2(\triangle t)$ much more smaller than at an instance of time before $\triangle t$, so that the self-gravitational force becomes bigger than this internal pressure and hence begins the process of particles compression (the speed $u(t,r) = -\frac{2}{5}kru_T(t)$ for $t > \triangle t$ becomes negative (now with $k > 0$)).

- **Self-compression process**: Let us now consider what happens for $t > \triangle t$ when the external field (which caused this short time spherical perturbation) is equal to zero, so that the particle has to propagate (after a very short time during which this particle reaches again its hydrostatic equilibrium) again with a new constant velocity in the vacuum. The energy-density distribution of the particle at $t = \triangle t$ is that given by (58). Consequently, we have that during the expansion for $0 < t < \triangle t$, $T(t) = \mathrm{e}^{k\int_0^t u_T(s)ds} = \mathrm{e}^{-|k|\int_0^t u_T(s)ds} < 1$, while now for $t > \triangle t$, $T(t) = \mathrm{e}^{k\int_0^t u_T(s)ds} = \mathrm{e}^{|k|\int_0^t u_T(s)ds} > 1$, and hence now $\Phi_m(t,r) = T(t)R(r)$ is growing with time, that is, the density of particle now increases during this compression phase. We obtained, from (51), the maximal spherical extension of particle's body $r_0(\triangle t) = r_0T(\triangle t)^{-2/5}$. However, now for $t > \triangle t$, $T(t) > 1$, and hence $r_0(t)$ now decreases confirming that we have the compression of the particle's body with the velocity $\frac{dr_0}{dt} = u(t,r_0(t)) = u_T(t)u_R(r_0(t)) = u_T(t)(-\frac{2}{5}kr_0(t)) = -\frac{2}{5}|k|u_T(t)r_0(t) < 0$, because now for $t > \triangle t$ we have $k > 0$ and hence the radial velocity now is negative (oriented toward the particle's barycenter) as expected.

So, we have now an inverse process, that is, a compression of the particle's energy-density up to the radius r_0 when the distribution of the energy-density again becomes time-invariant $\Phi_m(t,r) = \frac{K}{\sqrt{r}}$. Thus, we obtain again a stationary particle's state in which internal forces become constant in each point inside particle's body, and hence internal energy-flow velocity $\overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{u}} = u(t,r)\mathbf{e}_r$ becomes zero again. Thus, we return again to particle's hydrostatic equilibrium[5], as it was in the initial moment $t = 0$ before the particle's expansion caused by an acceleration, however now with a different (but constant) speed of this particle.

Consequently, any particle's acceleration changes its internal energy-density distribution which, as a side effect, generates the dynamic changes of internal forces in the particle's body and creates the internal density-flux velocity $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})$ (which is zero in the hydrostatic equilibrium during an inertial propagation, making equal internal force in every point inside particle's body). The self-gravitational forces, in the absence of the external fields (that caused particle's acceleration), now generate again the stationary (stable) energy-density distribution $\frac{K}{\sqrt{r}}$ in which there is a perfect equilibrium of particle's internal forces (equal in each point inside particle's body). Consequently, the auto-stability of any elementary massive particle is explained by this internal self-gravitational process.

Notice that we can not obtain a non-banal equation for radial density speed $u(t,r) = u_R(r)u_T(t) = -\frac{2}{5}kru_T(t)$, that is, of its temporal evolution $u_T(t)$, from the continuity equation (rest-mass energy conservation) (37), because it reduces to the equation (39), i.e., to $ku_T(t) = \frac{1}{T(t)}\frac{dT(t)}{dt}$ (because $T(t) = \mathrm{e}^{k\int_0^t u_T(s)ds}$ so that the previous equation reduces to banal equation $u_T(t) = u_T(t)$). However, it is possible to obtain this non-banal equation from the momentum conservation law (56), that is from its reduction into (57), as follows:

Corollary 4 *Based on the momentum conservation law (56) during particle's spherical perturbation, we can derive the following equation for the time-evolution of the particle's internal pressure $P(t,r)$, by knowing its hydrostatic-equilibrium pressure $P(0,r) = P_0 - \frac{8\pi GK^2}{5c^4}r$ given by Corollary 2, and the time-*

[5] It will be necessary also to investigate the damping ratio: the eventual oscillatory behavior when the particle after acceleration returns into its inertial propagation with internal hydrostatic equilibrium. Does we have an exponential decrease as a function of time (analog to Landau damping, for example) of such spherical particle's density oscillatory waves has to be investigated.

evolution of the speed $u_T(t)$,

$$P(t,r) = \frac{4kK}{15c^2}T(t)(\frac{du_T(t)}{dt} - \frac{2k}{5}u_T^2(t))r^{3/2} + P(0,r)T^2(t) \tag{60}$$

where $T(t) = \mathrm{e}^{k\int_0^t u_T(s)ds}$. *This equation is quadratic equation for* $T(t)$ *with the solution* $T(t) = (-b(\frac{du_T(t)}{dt} - \frac{2k}{5}u_T^2(t)) + \sqrt{b^2(\frac{du_T(t)}{dt} - \frac{2k}{5}u_T^2(t))^2 - 4ac})/2a$ *where* $a = P(0,r) > 0$, $c = -P(t,r) < 0$ *and* $b = \frac{4kK}{15c^2}r^{3/2}$. *So, we obtain the following second-order differential equation for* $u_T(t)$*:*

$$u_T(t) = \frac{1}{k}\frac{d}{dt}ln((-b(\frac{du_T(t)}{dt} - \frac{2k}{5}u_T^2(t)) + \sqrt{b^2(\frac{du_T(t)}{dt} - \frac{2k}{5}u_T^2(t))^2 - 4ac})/2a) \tag{61}$$

Moreover, from the fact that $u_T(0) = 0$*, at the very beginning of the perturbation, for an infinitesimal but finite* δt*, we obtain the initial value for* u_T *by*

$$u_T(\delta t) \approx \frac{4\pi r_0^2}{m_0 k}\delta t^2 \frac{\partial^2 P(t,r)}{\partial t \partial r}|_{r=r_0,t=\delta t} \quad > 0 \tag{62}$$

expressed by the time-changing of the internal pressure-gradient force $-\frac{\partial P(t,r)}{\partial r}$ *on the particle's surface.*

Proof: Let us seek a solution of (57) of the form $P(t,r) = P(0,r)T^2(t)+f(t,r)$, so that $f(0,r) = 0$, and hence by substitution of it in (57), we obtain the equation

$$\frac{1}{\sqrt{r}}\frac{\partial f(t,r)}{\partial r} = \frac{2kK}{5c^2}T(t)\frac{du_T}{dt} \tag{63}$$

and hence, from the fact that the right-hand side of this equation does not depend on r, also the left-hand side must be so, and this can be done by setting $f(t,r) = h(t)r^{3/2}$, so that we obtain (by substitution in (63)),

$$h(t) = \frac{4kK}{15c^2}T(t)\frac{du_T}{dt} \tag{64}$$

Thus, we obtained the solution (60). Note that at $t = 0$, $0 = f(t,r) = h(t)r^{3/2}$ and hence it must be $h(0) = 0$. Thus, from $T(0) = 1$ and from the equation above, we obtain that $\frac{du_T(t)}{dt}|_{t=0} = 0$.

The equation (60) can be rewritten as the quadratic equation $aT^2(t)+b(\frac{du_T(t)}{dt} - \frac{2k}{5}u_T^2(t))T(t)+c = 0$ and from the fact that $T(t) = \mathrm{e}^{k\int_)^t u_T(s)ds}$ is always positive, it must be $b^2-4ac > 0$ because $-4ac > 0$. So, we take the real positive solution $\mathrm{e}^{k\int_0^t u_T(s)ds} = T(t) = (-b(\frac{du_T(t)}{dt} - \frac{2k}{5}u_T^2(t))+\sqrt{b^2(\frac{du_T(t)}{dt} - \frac{2k}{5}u_T^2(t))^2 - 4ac})/2a > 0$

and hence from this equation we obtain
$\int_0^t u_T(s)ds = \frac{1}{k} lnT(t)$
$= \frac{1}{k} ln((-b(\frac{du_T(t)}{dt} - \frac{2k}{5}u_T^2(t)) + \sqrt{b^2(\frac{du_T(t)}{dt} - \frac{2k}{5}u_T^2(t))^2 - 4ac})/2a)$.
Then, by differentiation on t, we obtain from Leibniz integral rule (from the fact that $u_T(0) = 0$) that $\frac{d}{dt}\int_0^t u_T(s)ds = u_T(t)$ and hence the second-order differential equation of $u_T(t)$, given by equation (61).
From (57), for an infinitesimal amount of time δt after beginning of the spherical perturbation ($t = 0$), we obtain that for $T(\delta t) \approx T(0) = 1$,

$$\frac{du_T(t)}{dt}|_{t=\delta t} \approx (\frac{\partial P(\delta t, r)}{\partial r} - \frac{\partial P(0,r)}{\partial r})\frac{5c^2}{2kK}r^{-1/2} \tag{65}$$

and, from the fact that the left-hand side is $\frac{du_T(t)}{dt}|_{t=\delta t} \approx \frac{u_T(\delta t)-u_T(0)}{\delta t} = \frac{u_T(\delta t)}{\delta t} >> -\frac{2k}{5}u_T^2(\delta t)$, by multiplying both sides of the equation above by δt and from the fact that $(\frac{\partial P(\delta t,r)}{\partial r} - \frac{\partial P(0,r)}{\partial r}) \approx \delta t\frac{\partial}{\partial t}(\frac{\partial P(t,r)}{\partial r})|_{t=\delta t} = \delta t\frac{\partial^2 P(t,r)}{\partial t\partial r}|_{t=\delta t}$, and the fact that the right-hand side of (65) does not depend on r (because the left-hand side depends only on t) so that we can fix $r = r_0$ and substitute K by $\frac{5m_0c^2}{8\pi r_0^{5/2}}$, we obtain the equation (62).
□
Notice that we can use the same method for derivation of $u_T(\delta t)$ from equation (60) instead, so that we obtain also

$$u_T(\delta t) \approx \frac{6\pi r_0}{m_0 k}\delta t^2\frac{\partial P(t, r_0)}{\partial t}|_{t=\delta t} \quad > 0 \tag{66}$$

which demonstrates how the time-evolution of the particle's density at the very beginning of a perturbation depends on the time-changing of the pressure on the surface of particle's body.

6 Conclusion: General case of returning to hydrostatic equilibrium

In the vacuum and in the absence of any external field (as, for example, during the self-compression process described in the previous section), the unique forces that determine the internal energy-density flow are that which are generated by self-gravitational forces. Hence, in such cases it is valid the following corollary:

Corollary 5 *The self-gravitational force inside the particle's rest-mass energy during the self-compression in absence of any other external field (or force),* *for a massive particle with a speed* $\overrightarrow{\boldsymbol{v}}(t)$ *in a locally flat Minkowski time space around it (observer's reference frame), generates the following variation-velocity of particle's energy-density at the position* $\overrightarrow{\boldsymbol{r}} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3$,

$$\overrightarrow{\boldsymbol{u}}(t, \overrightarrow{\boldsymbol{r}}) = \overrightarrow{\boldsymbol{u}}(0, \overrightarrow{\boldsymbol{r}}) - \int_0^t dt'(\frac{c^2\nabla P(t', \overrightarrow{\boldsymbol{r}})}{\Phi_m(t', \overrightarrow{\boldsymbol{r}})} + \frac{G}{c^2}\int \frac{(\overrightarrow{\boldsymbol{r}} - \overrightarrow{\boldsymbol{r}}')\Phi_m(t', \overrightarrow{\boldsymbol{r}}')}{\|\overrightarrow{\boldsymbol{r}} - \overrightarrow{\boldsymbol{r}}'\|^3}dV') \tag{67}$$

where $dV' = dx'dy'dz'$ *and* $\overrightarrow{\boldsymbol{r}}' = x'\boldsymbol{e}_1 + y'\boldsymbol{e}_2 + z'\boldsymbol{e}_3$ *is the position inside the particle's body with energy-density* $\Phi_m(t', \overrightarrow{\boldsymbol{r}}')$.
Consequently, the energy-density Φ_m *of this particle must satisfy the following integral equation, during the time evolution* $t \geq 0$,

$$\int \Phi_m(t, \overrightarrow{\boldsymbol{r}})[\overrightarrow{\boldsymbol{u}}(0, \overrightarrow{\boldsymbol{r}}) - \int_0^t dt'(\frac{c^2\nabla P(t', \overrightarrow{\boldsymbol{r}})}{\Phi_m(t', \overrightarrow{\boldsymbol{r}})} + \frac{G}{c^2}\int \frac{(\overrightarrow{\boldsymbol{r}} - \overrightarrow{\boldsymbol{r}}')\Phi_m(t', \overrightarrow{\boldsymbol{r}}')}{\|\overrightarrow{\boldsymbol{r}} - \overrightarrow{\boldsymbol{r}}'\|^3}dV')]dV = 0 \tag{68}$$

where $dV = dxdydz$.

Proof: From the general equation (32),

$$\frac{d\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})}{dt} = -\frac{c^2}{\Phi_m(t, \overrightarrow{\mathbf{r}})}\nabla P(t, \overrightarrow{\mathbf{r}}) + \overrightarrow{\mathbf{g}}(t, \overrightarrow{\mathbf{r}}) \tag{69}$$

where $\overrightarrow{\mathbf{g}}(t, \overrightarrow{\mathbf{r}}) = -\frac{G}{c^2}\int \frac{(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}')\Phi_m(t, \overrightarrow{\mathbf{r}}')}{\|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\|^3}dV'$ is the internal autocohesive self-gravitational acceleration, by integration, we obtain (67).
The equation (68) is derived from [10] (from equation (1.53) in Corolary 1).
□
Notice that if we assume that $t = 0$ is the beginning of particle's perturbation, so that for $t \leq 0$ it was in the hydrostatic equilibrium during an inertial propagation (with a constant (group) velocity), then in (67) we have that $\overrightarrow{\mathbf{u}}(0, \overrightarrow{\mathbf{r}}) = 0$. So, we obtain that during the whole perturbation (with the expansion/compression) up to the next hydrostatic equilibrium at t_f when again $\overrightarrow{\mathbf{u}}(t_f, \overrightarrow{\mathbf{r}}) = 0$, the following equation is valid for $0 < t < t_f$

$$\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) = -\int_0^t dt'(\frac{c^2\nabla P(t', \overrightarrow{\mathbf{r}})}{\Phi_m(t', \overrightarrow{\mathbf{r}})} + \frac{G}{c^2}\int \frac{(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}')\Phi_m(t', \overrightarrow{\mathbf{r}}')}{\|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\|^3}dV') \tag{70}$$

with

$$0 = -\int_0^{t_f} dt'(\frac{c^2\nabla P(t', \overrightarrow{\mathbf{r}})}{\Phi_m(t', \overrightarrow{\mathbf{r}})} + \frac{G}{c^2}\int \frac{(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}')\Phi_m(t', \overrightarrow{\mathbf{r}}')}{\|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\|^3}dV') \tag{71}$$

and hence, for $t > t_f$, up to the next particle's perturbation (for example by interaction with some boson, we have that in any point $(t, \overrightarrow{\mathbf{r}})$ inside particle's body, this equation

$$0 = \frac{c^2 \nabla P(t, \overrightarrow{\mathbf{r}})}{\Phi_m(t, \overrightarrow{\mathbf{r}})} + \frac{G}{c^2} \int \frac{(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}')\Phi_m(t, \overrightarrow{\mathbf{r}}')}{\| \overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}' \|^3} dV' \qquad (72)$$

represents again the particle's hydrostatic equilibrium (now in (72) expressed not in proper frame of the particle with the center in particle's barycenter, but in Minkowski time-space of the observer 's laboratory) with spherically symmetric density $\Phi_m = \frac{K}{\sqrt{r}}$ where r is the distance from the particle's barycenter (the center of its spherical body shape) and with constant internal force in every point inside the particle's body.
In this self-gravitational stability assumption, the conservation law for rest-mass energy-density (4) becomes equal to the following integral-differential equation:

$$\begin{aligned} \frac{\partial \Phi_m(t, \overrightarrow{\mathbf{r}})}{\partial t} &= -\nabla \cdot (\Phi_m(t, \overrightarrow{\mathbf{r}})[\overrightarrow{\mathbf{v}}(t) + \overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})]) \\ &= -\nabla \cdot (\Phi_m(t, \overrightarrow{\mathbf{r}})[\frac{\frac{d}{dt} \int \overrightarrow{\mathbf{r}}' \Phi_m(t, \overrightarrow{\mathbf{r}}'))dV'}{\int \Phi_m(t, \overrightarrow{\mathbf{r}}'))dV'} \\ + \overrightarrow{\mathbf{u}}(0, \overrightarrow{\mathbf{r}}) - \int_0^t dt' (\frac{c^2 \nabla P(t', \overrightarrow{\mathbf{r}})}{\Phi_m(t', \overrightarrow{\mathbf{r}})} &+ \frac{G}{c^2} \int \frac{(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}')\Phi_m(t', \overrightarrow{\mathbf{r}}')}{\| \overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}' \|^3} dV')]) \quad (73) \end{aligned}$$

Note that the rest-mass energy-density Φ_m in equation (73) has the following self-referential aspect: its time evolution in the past (integral from time-instance 0 to current time t) determines the current variation-velocity $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})$, while this current variation-velocity determines the future state of the energy-density Φ_m as specified by the first row of equation above. This fact is analog to the General Relativity theory of gravitation and time-space curvature where the time-space is considered as a particular *field*. In effect, in GR time-space plays a dual role in this theory, because it constitutes both the dynamical object and the context within which the dynamics are defined. This self-referential aspect gives general relativity certain characteristics different from any other field theory. For example, in other theories we formulate a Cauchy initial value problem by specifying the condition of the field everywhere at a given instant $t = 0$, and then use the field equations to determine the future evolution of the field. In contrast, because of the inherent self-referential quality of the metrical field, we are not free to specify arbitrary initial conditions. Also in our case, we can not specify the field Φ_m and vector field $\overrightarrow{\mathbf{u}}$ everywhere at a given instant $t = 0$ for an unstable massive particle (we are able to specify them only for stationary cases of an inertial propagation in the vacuum), so that we are not able practically to compute Φ_m for any $t > 0$, also when, from (73) we have that for any infinitesimal interval of time δt,

$$\Phi_m(t+\delta t, \overrightarrow{\mathbf{r}}) = \Phi_m(t, \overrightarrow{\mathbf{r}}) - \delta t \nabla \cdot (\Phi_m(t, \overrightarrow{\mathbf{r}})[\frac{\frac{d}{dt}\int \overrightarrow{\mathbf{r}}'\Phi_m(t, \overrightarrow{\mathbf{r}}'))dV'}{\int \Phi_m(t, \overrightarrow{\mathbf{r}}'))dV'}$$
$$+ \overrightarrow{\mathbf{u}}(0, \overrightarrow{\mathbf{r}}) - \int_0^t dt'(\frac{c^2\nabla P(t', \overrightarrow{\mathbf{r}})}{\Phi_m(t', \overrightarrow{\mathbf{r}})} + \frac{G}{c^2}\int \frac{(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}')\Phi_m(t', \overrightarrow{\mathbf{r}}')}{\|\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\|^3}dV')]) \quad (74)$$

Note that we consider the velocity $\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}}) = \overrightarrow{\mathbf{v}}(t) + \overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})$ as a particular vector field, which is not observable in QM, thus can be considered as a 'hidden variable' in this theory.
The simply connected matter distribution [10] at any fixed instance of time t (the space where its matter/energy distribution in the 3-dimensional space-like hypersurface $\Sigma_t \subset \mathcal{M}$ is greater than zero) means that every closed 3-dimensional loop in it, such that for each point $\overrightarrow{\mathbf{r}} \in V_t \subset \Sigma_t$ of this loop, $\Phi(t, \overrightarrow{\mathbf{r}}) > 0$, can be deformed continuously to a small sphere. In fact, from the Poincare conjecture (Grigorij Perelman, [2, 4, 3]):

"*Every simply connected closed 3-manifold is homeomorphic to the 3-sphere*",

we have the fact that in stationary cases when a massive particle propagates with a constant speed (w.r.t. a given Minkowski frame), the natural topology of particle's body-volume V_t is a sphere with radius r_0 (with the spherically symmetric energy-density distribution Φ_m proportional to $\frac{1}{\sqrt{r}}$ for the distance r from the barycenter), like the topology of stars in universe. During acceleration this 'geometry' of V_t changes, but with autocohesive forces when the particle again returns in its inertial propagation, its 'geometry' again becomes perfectly a sphere (3-D space symmetry).
The physical explanation of this 'implosion' process can be, for example, provided by using the Ricci flow, defined by Richard Hamilton [5], expressed by the equation for this 3-D manifold V_t,

$$\partial_t h_{ij} = -2R^{(3)}_{ij} \quad (75)$$

where h_{ij} is the component of the Riemannian metric of V_t and $R^{(3)}_{ij}$ is the component of the 3-D Ricci curvature tensor (considered in unification of QM with Einstein's GR [11], Section 1.4, for the 4-D time-space pseudo-Riemannian metric). Ricci flow expands the negative curvature part of the manifold and contracts the positive curvature part, as can be seen in Fig.1

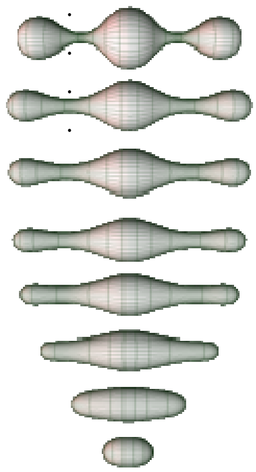

Figure 1: Ricci flow evolution

(from $https : //en.wikipedia.org/wiki/File : Ricci - flow.png)$[6], where we can see an example of the evolution of the stable state (sphere) into unstable particle's state, and vice versa.

However, in order to consider a kind of Richard Hamilton's Ricci flow (as that in equation (75)) we must first transform the "Newton's" equation (73) into its covariant form (similarly to the method applied in [11]). More work in this direction has to be done in the future.

In [10] has been considered the processes of spherical expansions and compressions for the massless bosons, and explained how unstationary bosons can obtain the properties of the massive particles during their interaction with another particles. Let us now consider the case of the self-compression process for the unstable (massive) bosons. In this case the restriction of its 3-D energy density Φ_m is guided by self-gravitational forces and by using the Ricci flow, expressed by the equation (75) for this 3-D manifold V_t, which reduces the energy-density topology into an infinitesimal but finite sphere with a radius $r_0 > 0$ with spher-

[6] The proof of Poincare conjecture built upon the program of Richard Hamilton to use the Ricci flow to attempt to solve the problem. In some cases Hamilton was able to show that this works; for example, if the manifold has positive Ricci curvature everywhere he showed that the manifold becomes extinct in finite time under Ricci flow without any other singularities. (In other words, the manifold collapses to a point in finite time; it is easy to describe the structure just before the manifold collapses.) This easily implies the Poincare conjecture in the case of positive Ricci curvature. However in general the Ricci flow equations lead to singularities of the metric after a finite time. Perelman showed how to continue past these singularities: very roughly, he cuts the manifold along the singularities, splitting the manifold into several pieces, and then continues with the Ricci flow on each of these pieces. This procedure is known as Ricci flow with surgery.

ically symmetric energy-density distribution $\Phi_m(t,r) = T(t)\frac{K}{\sqrt{r}}$, so that we obtain the new version of (16) in the Minkowski time-space reference system of the observer's laboratory

$$\overrightarrow{\mathbf{g}}(t, \overrightarrow{\mathbf{r}}) = -\frac{G}{c^2}\int \frac{(\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}')\Phi_m(t, \overrightarrow{\mathbf{r}}')}{\| \overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\|^3}dV' = -\frac{GKT(t)}{c^2}\int \frac{\overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'}{\| \overrightarrow{\mathbf{r}} - \overrightarrow{\mathbf{r}}'\|^{7/2}}dV' \tag{76}$$

The enough condition that the process of compression continues up to the restriction into a single point (boson's barycenter) in 3-D is to avoid the generation of a micro black-hole, in the way that the energy-density is progressively expelled from the 3-D into higher compactified dimensions [11]. It remains to provide mathematically the details of such a physical process in the future research.

Other important conclusion is that for the formal development of QM operators, in the conservative extension of current probabilistic/statistical theory based fundamentaly on the Schrödinger equation (and its extensions) valid for the ensemble of identically prepared particles for a statistical measurements, for an individual elementary particle, the only necessary new equation is the conservation of rest-mass energy (4) (the conservation of matter for which the rest-mass and rest-mass-energy are some of its fundamental properties), while the other two equations (conservation laws of momentum and *internal* energy) provided in Section 3, are useful only for the physical comprehension of the self-equilibrium process inside the massive particles.

In effect, the momentum conservation law is useful only to explain the internal dynamics of the density-flow speed $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})$ by using the two opposite internal forces generated by the self-gravity acceleration $\overrightarrow{\mathbf{g}}(t, \overrightarrow{\mathbf{r}})$ and the internal gradient-pressure force $\nabla P(t, \overrightarrow{\mathbf{r}})$ during particle's accelerations. In the hydrostatic equilibrium we have that, in the particle's proper frame (with coordinate center in particle's barycenter), $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}}) = 0$, and the particle's density is spherically symmetric and proportional to $\frac{K}{\sqrt{r}}$, so that the internal force is constant and equal in every point inside particle's body.

The internal energy law is useful also to consider the behavior of a massive particle with the rest mass m_0 and momentum p in a number of extreme perturbations and also in the cases of the so called virtual particles (when temporary is not valid the equation for particles total energy E, that is, when $E^2 \neq m_0^2c^4 + p^2c^2$), as provided in some examples in [10].

Moreover, in the general case of the particle's perturbation, the self-gravity acceleration $\overrightarrow{\mathbf{g}}(t, \overrightarrow{\mathbf{r}})$ and the internal gradient-pressure force $\nabla P(t, \overrightarrow{\mathbf{r}})$ are not known and can not be measured, so it is enough [10] to use only the density

speed $\overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})$ as a hidden (unobservable) variable in the formal theory of completion of QM for an individual elementary particle.
This fact explains why it was enough to use only this hidden variable and, derived from it, the absolute speed w.r.t. a given frame, $\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}}) = \overrightarrow{\mathbf{v}}(t) + \overrightarrow{\mathbf{u}}(t, \overrightarrow{\mathbf{r}})$, where $\overrightarrow{\mathbf{v}}(t)$ is observable particle's velocity (group velocity of particle's energy-density wave-packet), for the definition of the new TSPF quantum operators $\widehat{\mathbb{M}} = i\hbar(\overrightarrow{\mathbf{w}}(t, \overrightarrow{\mathbf{r}})\nabla - \frac{\nabla\cdot\overrightarrow{\mathbf{w}}(t,\overrightarrow{\mathbf{r}})}{2})$ [10] for a given individual elementary particle.

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