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Fractional differential equations

Existence of solutions of IVPs of singular multi-term fractional differential equations with impulse effects

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Abstract. This paper is devoted to studying the existence of solutions of two classes of initial value problems for nonlinear fractional differential equations with impulse effects. Firstly we transform initial value problems into integral equations. Then by constructing a special Banach space and employing fixed-point theorem, we obtain some sufficient conditions that guarantee the existence of solutions of these problems of fractional differential equations involving Caputo fractional derivatives. One equation is involved in multiple base points and the other one is involved in a single base point.

Keywords. Initial value problem, impulsive fractional differential equation, single base point FDE, multiple base points FDE, Caputo fractional derivative, fixed-point theorem

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1 Introduction

In recent years many investigations on fractional differential equations have been made since it was shown that many physical systems can be represented more accurately through fractional derivative formulation [17]. Fractional differential equations, therefore find numerous applications in the field of visco-elasticity, feed back amplifiers, electrical circuits, electro analytical chemistry, fractional multi-poles, neuron modelling encompassing different branches of physics, chemistry and biological sciences [22]. There have been many excellent books and monographs available on this field such as [6, 7, 8, 10, 20, 21, 19, 23].

In the literature, ${}^cD^{\alpha}_{0^+}u(t)+f(t,u(t))=0$ is known as a **single term equation**. In certain cases, we find equations containing more than one differential terms. These equations are called **multi-term equations**. A classical example is the so-called **Basset equation**

$$AD^1_{0^+}y(x)+bD^n_{0^+}y(x)+cy(x)=f(x),\ y(0)=y_0,$$

where $0<n<1$. This equation is most frequently, but not exclusively, used with $n=\frac{1}{2}$. It describes the forces that occur when a spherical object sinks in a (relatively dense) incompressible viscous fluid, see [1, 19].

In the left and right fractional derivatives $D^{\alpha}_{a^+}x$ and $D^{\alpha}_{b^-}x$, a is called a left base point and b right starting point. Both a and b are called starting points of fractional derivatives. An FDE containing more than one base point is called a **multiple stsrting points FDE**. An FDE containing only one starting point is called a **single starting point FDE**.

In [10], Liu discussed existence of positive solutions to the initial value problems of the nonlinear multi-order fractional differential equation on half line

$$D^{\alpha}_{0^+}D^{\beta}_{0^+}D^{\gamma}_{0^+}x(t)+f(t,x(t),D^p_{0^+}x(t))=0,\quad t\in(0,\infty),$$

$$\lim_{t\to 0}t^{1-\gamma}x(t)=\int_0^{+\infty}g_0(t,x(t),D^p_{0^+}x(t))dt,$$

$$\lim_{t\to 0}t^{1-\beta}D^{\gamma}_{0^+}x(t)=\int_0^{+\infty}g_1(t,x(t),D^p_{0^+}x(t))dt,$$

$$\lim_{t\to 0}t^{1-\alpha}D^{\beta}_{0^+}D^{\gamma}_{0^+}x(t)=\int_0^{+\infty}g_2(t,x(t),D^p_{0^+}x(t))dt,$$

where $x_0,x_1,x_2\in\mathbb{R}$, $\alpha,\beta,\gamma,p\in(0,1)$, D_{0^+} is the standard Riemann-Liouville fractional derivative, and $f:(0,\infty)\times\mathbb{R}^3\to\mathbb{R}$ is a Caratheodory function,

$g_0, g_1, g_2 : (0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ are strong Caratheodory functions and f, g_0, g_1, g_2 may be singular at $t = 0$.

In [11], authors studied the solvability of the following initial value problems for singular fractional differential equation with multiple starting points

$$^cD^{\alpha}_{*+}x(t) = m(t)f(t, x(t), ^cD^{p}_{*+}x(t)), a.e., t \in (0, \infty),$$

$$x(0) = x_0.$$

In [15], authors studied existence and uniqueness of the following two initial value problems (IVPs for short) of nonlinear multi-term FDEs with impulses on half lines:

$$^cD^{\alpha}_{0+}x(t) = q(t)f(t, x(t), ^cD^{p}_{0+}x(t)), a.e., t \in (t_s, t_{s+1}], s \in \mathbf{Z}_0,$$

$$x(0) = x_0,$$

$$\Delta x(t_s) = \lim_{t \to t_s^+} x(t) - x(t_s) = I(t_s, x(t_s)), s \in \mathbf{Z},$$

and

$$^cD^{\alpha}_{*+}x(t) = q(t)f(t, x(t), ^cD^{p}_{*+}x(t)), a.e., t \in (t_s, t_{s+1}], s \in \mathbf{Z}_0,$$

$$x(0) = x_0,$$

$$\Delta x(t_s) = \lim_{t \to t_s^+} x(t) - x(t_s) = I(t_s, x(t_s)), s \in \mathbf{Z},$$

where $x_0 \in \mathbb{R}$, $\alpha \in (0, 1]$, $0 < p < \alpha$, $\mathbf{Z}_0 = \{0, 1, 2, \cdots\}$ and $\mathbf{Z} = \{1, 2, \cdots\}$, $0 = t_0 < t_1 < t_2 < t_3 < \cdots$ with $\lim_{s \to +\infty} t_s = +\infty$, $^cD_{*+}$ is the standard Caputo fractional derivative at the base points $t = *$, $q : (0, +\infty) \to \mathbb{R}$ satisfies that there exists $l > -\alpha$ such that $|q(t)| \le t^l$ for all $t \in (0, +\infty)$, $f : (0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function and $I : \bigcup_{s=0}^{+\infty} (t_s, t_{s+1}) \times \mathbb{R} \to \mathbb{R}$ a discrete Carathéodory function. The recent studies on solvability of boundary value problems for impulsive fractional differential equations may be found in [12, 13, 14] and the references therein.

In applications, our equation can be interpreted as the standard Malthus population model $y' = \lambda y$ subject to a perturbation $f(t, y)$. That is $y' =$

$\lambda y + f(t,y)$. This situation makes us to study fractional differential equation $D^\alpha y - \lambda y = f(t,y)$ with $\alpha \in (0,1]$ see [2, 24].

Motivated by mentioned papers, in this paper, we study the following initial value problems (IVPs for short) of the nonlinear multi-term fractional differential equation on half line

$$\begin{cases} {}^cD^\alpha_{0^+}x(t) - \lambda x(t) = m(t)f(t, x(t), {}^cD^p_{0^+}x(t)), a.e., t \in (0,+\infty), \\ \\ x(0) = x_0, \\ \\ \Delta x(t_s) = \lim\limits_{t\to t_s^+} x(t) - x(t_s) = I(t_s, x(t_s), {}^cD^p_{0^+}x(t_s)), s \in \mathbf{Z}, \end{cases} \tag{1}$$

and

$$\begin{cases} {}^cD^\alpha_{t_s^+}x(t) - \lambda x(t) = n(t)g(t, x(t), {}^cD^p_{t_s^+}x(t)), a.e., t \in (t_s, t_{s+1}), s \in \mathbf{Z}_0, \\ \\ x(0) = x_0, \\ \\ \Delta x(t_s) = \lim\limits_{t\to t_s^+} x(t) - x(t_s) = J(t_s, x(t_s), {}^cD^p_{t_{s-1}^+}x(t_s)), s \in \mathbf{Z}, \end{cases} \tag{2}$$

where

(i) $x_0 \in \mathbb{R}$, $\alpha \in (0,1]$, $0 < p < \alpha$, $\lambda > 0$, $\mathbf{Z}_0 = \{0,1,2,\cdots\}$ and $\mathbf{Z} = \{1,2,\cdots\}$, $0 = t_0 < t_1 < t_2 < t_3 < \cdots$ with $\lim\limits_{s\to+\infty} t_s = +\infty$,

(ii) ${}^cD_{*^+}$ is the standard Caputo fractional derivative at the starting points $t = *$,

(iii) $m : (0,+\infty) \to \mathbb{R}$ satisfies that there exists $k_1 > -\alpha$ such that $|m(t)| \le t^{k_1}$ for all $t \in (0,+\infty)$,

(iv) $n : \bigcup\limits_{s=0}^{+\infty}(t_s, t_{s+1}) \to \mathbb{R}$ satisfies that there exists $0 \ge l_2 > -\alpha, k_2 > p - \alpha - l_2$ such that $|n(t)| \le (t-t_s)^{k_2}(t_{s+1}-t)^{l_2}$ for all $t \in (t_s, t_{s+1})(s \in \mathbf{Z}_0)$,

(v) $f : (0,+\infty)\times\mathbb{R}^2 \to \mathbb{R}$ are I-Carathéodory function and $I : \{t_s\}\times\mathbb{R}^2 \to \mathbb{R}$ discrete I-Carathéodory function,

(vi) $g : \bigcup\limits_{s=0}^{+\infty}(t_s, t_{s+1}) \times \mathbb{R}^2 \to \mathbb{R}$ are II-Carathéodory function and $J : \{t_s\} \times \mathbb{R}^2 \to \mathbb{R}$ discrete II-Carathéodory functions. .

A function $x : (0, +\infty) \to \mathbb{R}$ is called a solution of IVP(1) if $x|_{(t_s,t_{s+1}]}$, ${}^cD^p_{0^+}x|_{t_s,t_{s+1}]} \in C^0(t_s, t_{s+1}]$ $(s \in \mathbf{Z}_0)$, the limits $\lim\limits_{t\to t_s^+} x(t)$, $\lim\limits_{t\to t_s^+} {}^cD^p_{0^+}x(t)(s \in Z_0)$ exist, ${}^cI^\alpha_{0^+}{}^cD^\alpha_{0^+}x(t)$ exists almost every point on $(0, +\infty)$ and all equations in (1) are satisfied. A function $x : (0, +\infty) \to \mathbb{R}$ is called a solution of IVP(2) if $x|_{(t_s,t_{s+1}]}$, ${}^cD^p_{t_s^+}x \in C^0(t_s, t_{s+1}]$ $(s \in \mathbf{Z}_0)$, the limits $\lim\limits_{t\to t_s^+} x(t)$, $\lim\limits_{t\to t_s^+} {}^cD^p_{t_s^+}x(t)$ $(s \in \mathbf{Z}_0)$ exist, ${}^cI^\alpha_{t_s^+}{}^cD^\alpha_{t_s^+}x(t)$ exists almost every point on $(t_s, t_{s+1})(s \in \mathbf{Z}_0)$ and all equations in (2) are satisfied.

Our purpose of this paper is to establish sufficient conditions for the existence and uniqueness of solutions (positive solutions) of IVP(1) (under assumptions (i), (ii), (iii) and (v)) and IVP(2) (under assumptions (i), (ii), (iv) and (vi)), respectively. Existence results for IVP(2) generalizes those ones (Theorem 3.1) obtained in [11]. Existence results for IVP(1) generalizes Theorem 11 in [15].

The remainder of this paper is organized as follows: the preliminary results are given in Section 2, the main results are presented in Section 3. In Section 4, examples are presented.

2 Preliminary results

For the convenience of the reader, we present here the necessary definitions from fixed point theory and fractional calculus theory. These definitions and properties can be found in the literatures [18, 20, 23]. Denote the Gamma function, Beta function and Mittag-Leffler functions respectively by

$$\Gamma(\alpha_1) = \int_0^{+\infty} s^{\alpha_1-1}e^{-s}ds,$$

$$\mathbf{B}(\alpha_2, \beta_2) = \int_0^1 (1-x)^{\alpha_2-1}x^{\beta_2-1}dx, \alpha_1 > 0, \alpha_2, \beta_2 > 0,$$

$$\mathbf{E}_{\alpha_1}(x) = \sum_{s=0}^{+\infty} \frac{x^s}{\Gamma(\alpha_1 s+1)}, \alpha_1 > 0, x \in \mathbb{R},$$

$$\mathbf{E}_{\alpha_1,\alpha_2}(x) = \sum_{s=0}^{+\infty} \frac{x^s}{\Gamma(\alpha_1 s+\alpha_2))}, \alpha_1 > 0, \alpha_2 > 0, x \in \mathbb{R}.$$

Definition 1 *[20]. Let $c \geq 0$. The Riemann-Liouville fractional integral of* *order $\alpha > 0$ of a function $f : (c, \infty) \to \mathbb{R}$ is given by*

$$I^{\alpha}_{c^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} f(s)ds,$$

provided that the right-hand side exists.

Definition 2 *[20]. Let $c \ge 0$. The Caputo's derivative of order α for a function $f : (c, \infty) \to \mathbb{R}$ is defined as*

$${}^cD^{\alpha}_{c^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

for $n-1 < \alpha < n$, $n \in \mathbb{N}$. If $0 < \alpha \le 1$, then

$$D^{\alpha}_{c^+} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_c^t (t-s)^{-\alpha} f^{(1)}(s)ds.$$

Obviously, the Caputo's derivative of a constant is equal to zero.

Definition 3 *Choose $\sigma > \max\{-k_1, \alpha + k_1, \alpha\}$. $h : (0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is called a I-Carathéodory function if it satisfies the following assumptions:*

(i) $t \to h\left(t,\ (1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})x, \frac{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})}{t^p} y\right)$ is continuous on $(t_s, t_{s+1}] (s \in \mathbf{Z}_0)$ and is bounded on $\mathbb{R}$,

(ii) $(x, y) \to h\left(t,\ (1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})x, \frac{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})}{t^p} y\right)$ is continuous on $\mathbb{R}$;

(iii) for each $r > 0$ there exists a constant $M_r \ge 0$ such that $|x|, |y| \le r$ imply

$$\left| h\left(t,\ (1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})x, \frac{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})}{t^p} y\right)\right| \le M_r, t \in (0, +\infty).$$

Definition 4 *Choose $\sigma > \max\{-k_1, \alpha + k_1, \alpha\}$. $H : \{t_s : s \in Z\} \times \mathbb{R}^2 \to \mathbb{R}$ is called a discrete I-Carathéodory function if it satisfies the following assumptions:*

(i) $(x, y) \to H\left(t_s,\ (1+t_s^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t_s^{\alpha})x, \frac{(1+t_s^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t_s^{\alpha})}{t_s^p} y\right)$ is continuous on $\mathbb{R}$ for all $s \in \mathbf{Z}$;

(ii) for each $r > 0$ there exists a constant $M_{rs} \ge 0$ such that $|x|, |y| \le r$ imply

$$\left| H\left(t_s,\ (1+t_s^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t_s^{\alpha})x, \frac{(1+t_s^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t_s^{\alpha})}{t_s^p} y\right)\right| \le M_{rs}, s \in Z, \sum_{s=1}^{+\infty} M_{rs} < +\infty.$$

Choose $\sigma > \max\{-k_1, \alpha + k_1, \alpha\}$. Let

$$X = \left\{ x : \begin{array}{c} x|_{(t_s,t_{s+1}]}, {}^cD_{0^+}^p x|_{(t_s,t_{s+1}]} \in C^0((t_s,t_{s+1}], \mathbb{R}), s \in \mathbf{Z}_0, \\ \lim_{t\to t_s^+} x(t), \lim_{t\to t_s^+} {}^cD_{0^+}^p x(t) \text{ exist}, s \in \mathbf{Z}_0, \\ \lim_{t\to+\infty} \frac{x(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)}, \lim_{t\to+\infty} \frac{t^p}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)} {}^cD_{0^+}^p x(t) \text{ exists} \end{array} \right\}.$$

For $x \in X$, define the norm by

$$||x|| = ||x||_X = \max\left\{ \sup_{t\in(0,+\infty)} \frac{|x(t)|}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)}, \sup_{t\in(0,+\infty)} \frac{t^p}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)} |{}^cD_{0^+}^p x(t)| \right\}.$$

Lemma 2.1. X is a Banach space with $||\cdot||_X$ defined.

Proof. It is easy to see that X is a normed linear space. Let $\{x_u\}$ be a Cauchy sequence in X. Then $||x_u - x_v|| \to 0,\ u, v \to +\infty$. We will prove that there exists $x_0 \in X$ such that $x_u \to x_0$ as $u \to +\infty$.

It follows from $x_u \in X$ and $\{x_u\}$ a Cauchy sequence that

$$\sup_{t\in(0,+\infty)} \frac{|x_u(t)-x_v(t)|}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)} \to 0, u, v \to +\infty, \lim_{t\to+\infty} \frac{x_u(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)} = A_{xu} \text{ exists},$$

and

$$\sup_{t\in(0,+\infty)} \frac{t^p|{}^cD_{0^+}^p x_u(t) - {}^cD_{0^+}^p x_v(t)|}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)} \to 0, u, v \to +\infty, \lim_{t\to+\infty} \frac{t^{pc}D_{0^+}^p x_u(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)} = B_{xu} \text{ exists}.$$

It follows that

$$\left| \lim_{t\to+\infty} \frac{x_u(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)} - \lim_{t\to+\infty} \frac{x_v(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)} \right| \to 0, u, v \to +\infty$$

and

$$\left| \lim_{t\to+\infty} \frac{t^{pc}D_{0^+}^p x_u(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)} - \lim_{t\to+\infty} \frac{t^{pc}D_{0^+}^p x_v(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)} \right| \to 0, u, v \to +\infty.$$

Thus both $\lim_{u\to+\infty} A_{xu}$ and $\lim_{u\to+\infty} B_{xu}$ exist.

Since $\lim_{t\to t_s^+} x(t)$ and $\lim_{t\to t_s^+} {}^cD_{0^+}^p x(t)$ exist, we know both $\frac{x(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)}$ and $\frac{t^{pc}D_{0^+}^p x(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)}$ are continuous on $[t_s, t_{s+1}]$. Thus there exist two functions

$x_{0s}, y_{0s}(s \in \mathbf{Z}_0)$ defined on $[t_s, t_{s+1}]$ such that

$$\lim_{u\to+\infty} \frac{x_u(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)} = x_{0s}(t),$$

$$\lim_{u\to+\infty} \frac{t^{pc}D^p_{0^+}x_u(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)} = y_{0s}(t), t \in [t_s, t+s+1], s \in \mathbf{Z}_0.$$

Let $\overline{x}_0(t) = x_{0s}(t)$ and $\overline{y}_0(t) = y_{0s}(t)$ for $t \in (t_s, t_{s+1}](s \in \mathbf{Z}_0)$. It follows that both x_0 and y_0 are defined on $[0, +\infty)$ and the limits $\lim\limits_{t\to t_s^+} x_0(t)$, $\lim\limits_{t\to t_s^+} y_0(t)(s \in \mathbf{Z}_0)$ exist.

Furthermore, we have

$$\sup_{t\in(0,+\infty)} \left| \frac{x_u(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)} - \overline{x}_0(t) \right| \to 0, u \to +\infty,$$

$$\sup_{t\in(0,+\infty)} \left| \frac{t^{pc}D^p_{0^+}x_u(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)} - \overline{y}_0(t) \right| \to 0, u \to +\infty.$$

We have that

$$\lim_{t\to+\infty} \overline{x}_0(t) = \lim_{t\to+\infty} \lim_{u\to+\infty} \frac{x_u(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)} = \lim_{u\to+\infty} \lim_{t\to+\infty} \frac{x_u(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)} = \lim_{u\to+\infty} A_{xu},$$

$$\lim_{t\to+\infty} \overline{y}_0(t) = \lim_{t\to+\infty} \lim_{u\to+\infty} \frac{t^{pc}D^p_{0^+}x_u(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)} = \lim_{u\to+\infty} \lim_{t\to+\infty} \frac{t^{pc}D^p_{0^+}x_u(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)} = \lim_{u\to+\infty} B_{xu}.$$

Denote

$$x_0(t) = ((1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha))\overline{x}_0(t), \quad y_0(t) = \frac{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)}{t^p}\overline{y}_0(t).$$

We prove that $y_0(t) = {}^cD^p_{0^+}x_0(t)$ for $t \in (t_s, t_{s+1}]$. In fact, there exists $c_u = \lim\limits_{t\to t_s^+} x_u(t)$ such that

$$|x_u(t) + c_u - {}^cI^p_{0^+}y_0(t)| = \left|{}^cI^p_{0^+}\,{}^cD^p_{0^+}x_u(t) - {}^cI^p_{0^+}y_0(t)\right|$$

$$= \left| \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)}[{}^cD^p_{0^+}x_u(s) - y_0(s)]ds \right|$$

$$\le \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} \left| {}^cD^p_{0^+}x_u(s) - \frac{(1+s^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda s^\alpha)}{s^p}\overline{y}_0(s) \right| ds$$

$$\le \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} \frac{s^p}{(1+s^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda s^\alpha)} \left| \frac{(1+s^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda s^\alpha)}{s^p}{}^cD^p_{0^+}x_u(s) - \overline{y}_0(t) \right| ds$$

$$\leq \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} \frac{(1+s^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda s^\alpha)}{s^p} ds \sup_{t\in(0,+\infty)} \left| \frac{t^p}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)} {}^cD_{0^+}^p x_u(s) - \overline{y}_0(t) \right|$$

$\to 0$ as $u \to +\infty$.

Hence $\lim\limits_{u\to+\infty} [x_u(t) + c_u - {}^cI_{0^+}^p y_0(t)] = 0$. Then $x_0(t) + c_0 = {}^cI_{0^+}^p y_0(t)$. Hence $y_0(t) = D_{0^+}^p x_0(t)$. Then $x_0 \in X$ and $x_u \to x_0$ as $u \to +\infty$ in X. It follows that X is a Banach space. Lemma 2.1 is proved. ∎

Lemma 2.2. Let M be a subset of X. Then M is relatively compact if and only if the following conditions are satisfied:

(i) both $\left\{t \to \frac{x(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)} : x \in M\right\}$ and $\left\{t \to \frac{t^{p\,c}D_{0^+}^p x(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)} : x \in M\right\}$ are uniformly bounded,

(ii) both $\left\{t \to \frac{x(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)} : x \in M\right\}$ and $\left\{t \to \frac{t^{p\,c}D_{0^+}^p x(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)} : x \in M\right\}$ are equicontinuous in any subinterval $(t_s, t_{s+1}]$($s \in \mathbf{Z}_0$),

(iii) both $\left\{t \to \frac{x(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)} : x \in M\right\}$ and $\left\{t \to \frac{t^{p\,c}D_{0^+}^p x(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)} : x \in M\right\}$ are equi-convergent as $t \to +\infty$.

Proof. " $\Leftarrow$ ". From Lemma 2.1, we know X is a Banach space. In order to prove that the subset M is relatively compact in X, we only need to show M is totally bounded in X, that is for all $\epsilon > 0$, M has a finite ϵ-net.

For any given $\epsilon > 0$, by **(i)**-**(iii)**, then there exists t_{s_0} and $\delta > 0$ such that

$$\left| \frac{x(\overline{t}_1)}{(1+\overline{t}_1^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\overline{t}_1^\alpha)} - \frac{x(\overline{t}_2)}{(1+\overline{t}_2^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\overline{t}_2^\alpha)} \right| \leq \frac{\epsilon}{3}, \overline{t}_1, \overline{t}_2 \geq t_{s_0}, x \in M,$$

$$\left| \frac{\overline{t}_1^{p\,c}D_{)^+}^p x(\overline{t}_1)}{(1+\overline{t}_1^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda\overline{t}_1^\alpha)} - \frac{\overline{t}_2^{p\,c}D_{0^+}^p x(\overline{t}_2)}{(1+\overline{t}_2^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda\overline{t}_2^\alpha)} \right| \leq \frac{\epsilon}{3}, \overline{t}_1, \overline{t}_2 \geq t_{s_0}, x \in M,$$

$$\left| \frac{x(\overline{t}_1)}{(1+\overline{t}_1^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\overline{t}_1^\alpha)} - \frac{x(\overline{t}_2)}{(1+\overline{t}_2^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\overline{t}_2^\alpha)} \right| \leq \frac{\epsilon}{3}, \overline{t}_1, \overline{t}_2 \leq (t_s, t_{s+1}], |\overline{t}_1 - \overline{t}_2| < \delta, x \in M,$$

$$\left| \frac{\overline{t}_1^{p\,c}D_{0^+}^p x(\overline{t}_1)}{(1+\overline{t}_1^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda\overline{t}_1^\alpha)} - \frac{\overline{t}_2^{p\,c}D_{0^+}^p x(\overline{t}_2)}{(1+\overline{t}_2^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda\overline{t}_2^\alpha)} \right| \leq \frac{\epsilon}{3}, \overline{t}_1, \overline{t}_2 \leq (t_s, t_{s+1}], |\overline{t}_1 - \overline{t}_2| < \delta, x \in M.$$

Define

$$X|_{(0,t_{s_0}]} = \left\{ x : \begin{array}{c} x|_{(t_s,t_{s+1}]}, {}^cD_{0^+}^p x|_{(t_s,t_{s+1}]} \in C^0((t_s, t_{s+1}], \mathbb{R}), s = 0, 1, \cdots, s_0, \\ \lim\limits_{t\to t_s^+} x(t), \lim\limits_{t\to t_s^+} {}^cD_{0^+}^p x(t) \text{ exist}, s = 0, 1, \cdots, s_0 \end{array} \right\}.$$

For $x \in X|_{(0,t_{s_0}]}$, define

$$||x||_{s_0} = \max\left\{\sup_{t\in(0,t_{s_0}]} \frac{|x(t)|}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)},\ \sup_{t\in(0,t_{s_0}]} \frac{t^p|^cD_{0+}^p x(t)|}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)}\right\}.$$

Similarly to Lemma 2.1, we can prove that $X_{(0,t_{s_0}]}$ is a Banach space.

Let $M|_{(0,t_{s_0}]} = \{t \to x(t), t \in (0,t_{s_0}] : x \in M\}$. Then $M|_{(0,t_{s_0}]}$ is a subset of $X|_{(0,t_{s_0}]}$. By **(i)** and **(ii)**, and Ascoli-Arzela theorem, we can know that $M|_{(0,t_{s_0}]}$ is relatively compact. Thus, there exist $x_1, x_2, \cdots, x_k \in M$ such that, for any $x \in M$, we have that there exists some $i = 1, 2, \cdots, k$ such that

$$||x - x_i||_{s_0} = \max\left\{\sup_{t\in(0,t_{s_0}]} \frac{|x(t)-x_i(t)|}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)},\ \sup_{t\in(0,t_{s_0}]} \frac{t^p|^cD_{0+}^p x(t) - {}^cD_{0+}^p x_i(t)|}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)}\right\} \le \frac{\epsilon}{3}.$$

Therefore, for $x \in M$, we have that

$$\sup_{t\in(0,+\infty)} \frac{|x(t)-x_i(t)|}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)} = \max\left\{\sup_{t\in(0,t_{s_0}]} \frac{|x(t)-x_i(t)|}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)}, \sup_{t\ge t_{s_0}} \frac{|x(t)-x_i(t)|}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)}\right\}$$

$$\le \max\left\{\frac{\epsilon}{3}, \sup_{t\ge t_{s_0}} \left|\frac{x(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)} - \frac{x(t_{s_0})}{(1+t_{s_0}^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t_{s_0}^\alpha)}\right|\right.$$

$$\left. + \left|\frac{x(t_{s_0})}{(1+t_{s_0}^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t_{s_0}^\alpha)} - \frac{x_i(t_{s_0})}{(1+t_{s_0}^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t_{s_0}^\alpha)}\right|, \sup_{t\ge t_{s_0}} \left|\frac{x_i(t_{s_0})}{(1+t_{s_0}^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t_{s_0}^\alpha)} - \frac{x_i(t)}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)}\right|\right\}$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Similarly we can get

$$\sup_{t\in(0,+\infty)} \frac{t^p|^cD_{0+}^p x(t) - {}^cD_{0+}^p x_i(t)|}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)} < \epsilon.$$

So, for any $\epsilon > 0$, M has a finite ϵ-net $\{U_{x_1}, U_{x_2}, \cdots, U_{x_k}\}$, that is, M is totally bounded in X. Hence M is relatively compact in X.

$\Rightarrow$. Assume that M is relatively compact, then for any $\epsilon > 0$, there exists a finite ϵ-net of M. Let the finite ϵ-net be $\{U_{x_1}, U_{x_2}, \cdots, U_{x_k}\}$ with $x_i \subset M$. Then for any $x \in M$, there exists U_{x_i} such that $x \in U_{x_i}$ and

$$||x|| \le ||x - x_i|| + ||x_i|| \le \epsilon + \max\{||x_i|| : i = 1, 2, \cdots, k\}.$$

It follows that M is uniformly bounded. Then **(i)** holds.

Furthermore, there exists $t_{s_0} > 0$ such that

$$\left|\frac{x_i(\bar{t}_1)}{(1+\bar{t}_1^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\bar{t}_1^\alpha)} - \frac{x_i(\bar{t}_2)}{(1+\bar{t}_2^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\bar{t}_2^\alpha)}\right| < \epsilon,$$

$$\left|\frac{\bar{t}_1^p\,{}^cD_{0+}^p x_i(\bar{t}_1)}{(1+\bar{t}_1^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda\bar{t}_1^\alpha)} - \frac{\bar{t}_2^p\,{}^cD_{0+}^p x_i(\bar{t}_2)}{(1+\bar{t}_2^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda\bar{t}_2^\alpha)}\right| < \epsilon$$

for all $\bar{t}_1, \bar{t}_2 \geq t_{s_0}$ or all $\bar{t}_1, \bar{t}_2 \in (t_s, t_{s+1}]$ with $|\bar{t}_1 - \bar{t}_2| < \delta$. Then we have for $\bar{t}_1, \bar{t}_2 \geq t_{s_0}$ that

$$\left|\frac{x(\bar{t}_1)}{(1+\bar{t}_1^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\bar{t}_1^\alpha)} - \frac{x(\bar{t}_2)}{(1+\bar{t}_2^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\bar{t}_2^\alpha)}\right|$$

$$\leq \left|\frac{x(\bar{t}_1)}{(1+\bar{t}_1^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\bar{t}_1^\alpha)} - \frac{x_i(\bar{t}_1)}{(1+\bar{t}_1^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\bar{t}_1^\alpha)}\right| + \left|\frac{x_i(\bar{t}_1)}{(1+\bar{t}_1^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\bar{t}_1^\alpha)} - \frac{x_i(\bar{t}_2)}{(1+\bar{t}_2^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\bar{t}_2^\alpha)}\right|$$

$$+ \left|\frac{x_i(\bar{t}_2)}{(1+\bar{t}_2^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\bar{t}_2^\alpha)} - \frac{x(\bar{t}_2)}{(1+\bar{t}_2^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\bar{t}_2^\alpha)}\right|$$

$$\leq 3\epsilon,\ x \in M, \bar{t}_1, \bar{t}_2 \geq t_{s_0} \text{ or all } \bar{t}_1, \bar{t}_2 \in (t_s, t_{s+1}] \text{ with } |\bar{t}_1 - \bar{t}_2| < \delta.$$

Similarly we have

$$\left|\frac{x(\bar{t}_1)}{(1+\bar{t}_1^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\bar{t}_1^\alpha)} - \frac{x(\bar{t}_2)}{(1+\bar{t}_2^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda\bar{t}_2^\alpha)}\right| < 3\epsilon,$$

$$\bar{t}_1, \bar{t}_2 \geq t_{s_0} \text{ or all } \bar{t}_1, \bar{t}_2 \in (t_s, t_{s+1}] \text{ with } |\bar{t}_1 - \bar{t}_2| < \delta.$$

Thus **(ii)** and **(iii)** hold. Consequently, Lemma 2.2 is proved. ∎

Lemma 2.3. Suppose that $x \in X$. Then $u \in X$ is a solution of

$$\begin{cases} {}^cD_{0+}^\alpha u(t) - \lambda u(t) = m(t) f(t, x(t), {}^cD_{0+}^p x(t)), a.e., t \in (t_s, t_{s+1}], s \in \mathbf{Z}_0, \\ u(0) = x_0, \\ \Delta u(t_s) = I(t_s, x(t_s), {}^cD_{0+}^p x(t_s)), s \in \mathbf{Z}, \end{cases} \tag{3}$$

if and only if

$$u(t) = \int_0^t (t-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-v)^{\alpha})m(v)f(v,x(v),{}^cD_{0^+}^p x(v))dv + x_0\mathbf{E}_{\alpha}(\lambda t^{\alpha})$$

$$+\sum_{j=1}^{s}\mathbf{E}_{\alpha}(\lambda(t-t_j)^{\alpha})I(t_j,x(t_j),{}^cD_{0^+}^p x(t_s)), t\in(t_s,t_{s+1}], s\in\mathbf{Z}_0. \tag{4}$$

Proof. Step 1. Let $x\in X$. We prove that u satisfies (4) if u is a solution of (3).

From $x \in X$, we have that there exists $r > 0$ such that $||x|| = r < +\infty$. Since f is Carathéodory function and I a discrete Carathéodory function, then there exist $M_r, M_{rs} \geq 0$ such that

$$|f(t,x(t),{}^cD_{0^+}^p x(t))| = \left|f\left(t, \frac{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})x(t)}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})}\right)\right| \leq M_r,\ \ t\in[0,\infty),$$

$$|I(t_s,x(t_s),{}^cD_{0^+}^p x(t_s))| \leq M_{rs}, s\in\mathbf{Z}, \sum_{s=1}^{+\infty}M_{rs} < +\infty.$$

Firstly, we have for $t\in(t_s,t_{s+1}]$ that

$$\left|\int_0^t (t-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-v)^{\alpha})m(v)f(v,x(v),{}^cD_{0^+}^p x(v))dv\right|$$

$$\leq M_r\int_0^t (t-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-v)^{\alpha})v^{k_1}dv$$

$$\leq M_r\sum_{j=0}^{+\infty}\frac{\lambda^j}{\mathbf{\Gamma}((j+1)\alpha)}\int_0^t (t-v)^{\alpha-1}(t-v)^{\alpha j}v^{k_1}dv$$

$$\leq M_r\sum_{j=0}^{+\infty}\frac{\lambda^j}{\mathbf{\Gamma}((j+1)\alpha)}\int_0^t (t-v)^{\alpha+\alpha j-1}v^{k_1}dv$$

$$= M_r\sum_{j=0}^{+\infty}\frac{\lambda^j}{\mathbf{\Gamma}((j+1)\alpha)}t^{\alpha+\alpha j+k_1}\int_0^1 (1-w)^{\alpha+\alpha j-1}w^{k_1}dw$$

$$\leq M_r t^{\alpha+k_1}\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})\mathbf{B}(\alpha,k_1+1).$$

So $\int_0^t (t-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-v)^{\alpha})m(v)f(v,x(v),{}^cD_{0^+}^p x(v))dv$ is convergent.

Assume u is a solution of (3). We will prove (4). For $t\in(t_0,t_1]=(0,t_1]$, from Example 4.9 in [8], then

$$u(t)=\int_0^t(t-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-v)^{\alpha})m(v)f(v,x(v),{}^cD_{0^+}^px(v))dv+x_0\mathbf{E}_{\alpha}(\lambda t^{\alpha})$$

for , $t\in(t_0,t_1]$. Then (4) holds for $s=0$. Now we suppose that (4) holds for $s\le k$. That is

$$u(t)=\int_0^t(t-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-v)^{\alpha})m(v)f(v,x(v),{}^cD_{0^+}^px(v))dv+x_0\mathbf{E}_{\alpha}(\lambda t^{\alpha})$$

$$+\sum_{j=1}^{s}\mathbf{E}_{\alpha}(\lambda(t-t_j)^{\alpha})I(t_j,x(t_j),{}^cD_{0^+}^px(t_s)),t\in(t_s,t_{s+1}],s\le k.$$

For $t\in(t_{k+1},t_{k+2}]$, we let

$$u(t)=\int_0^t(t-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-v)^{\alpha})m(v)f(v,x(v),{}^cD_{0^+}^px(v))dv+x_0\mathbf{E}_{\alpha}(\lambda t^{\alpha})$$

$$+\sum_{j=1}^{k}\mathbf{E}_{\alpha}(\lambda(t-t_j)^{\alpha})I(t_j,x(t_j),{}^cD_{0^+}^px(t_s))+\Phi(t),t\in(t_{k+1},t_{k+2}], \tag{5}$$

where $\Phi:(t_{k+1},t_{k+2}]\to R$ is a differentiable function. By $\Delta u(t_{k+1})=I(t_{k+1},x(t_{k+1}),{}^cD_{0^+}^px(t_{k+1}))$, we get

$$\lim_{t\to t_{k+1}^+}\Phi(t)=I(t_{k+1},x(t_{k+1}),{}^cD_{0^+}^px(t_{k+1})). \tag{6}$$

For ease expression, denote $G_x(t)=G(t,x(t),{}^cD_{0^+}^px(t))$ for a function $G:(0,+\infty)\times R^2\to R$ and $x\in X$. Then (3) implies for $t\in(t_{k+1},t_{k+2}]$ that

$$m(t)f(t,x(t),{}^cD_{0^+}^px(t))+\lambda u(t)={}^cD_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\left(u(s)\right)'ds$$

$$=\frac{\sum\limits_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t-v)^{-\alpha}\left(\int_0^v(v-w)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(v-w)^{\alpha})m(w)f_x(w)dw+x_0\mathbf{E}_{\alpha}(\lambda v^{\alpha})+\sum\limits_{i=1}^{j}\mathbf{E}_{\alpha}(\lambda(v-t_i)^{\alpha})I_x(t_i)\right)'dv}{\Gamma(1-\alpha)}$$

$$+\frac{\int_{t_{k+1}}^{t}(t-v)^{-\alpha}\left(\int_0^v(v-w)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(v-w)^{\alpha})m(w)f_x(w)dw+x_0\mathbf{E}_{\alpha}(\lambda v^{\alpha})+\sum\limits_{i=1}^{k+1}\mathbf{E}_{\alpha}(\lambda(v-t_i)^{\alpha})I_x(t_i)+\Phi(v)\right)'dv}{\Gamma(1-\alpha)}$$

$$=\frac{\int_0^t(t-v)^{-\alpha}\left(\int_0^v(v-w)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(v-w)^{\alpha})m(w)f_x(w)dw\right)'dv}{\Gamma(1-\alpha)}$$

$$+\frac{\sum\limits_{j=0}^{k}\sum\limits_{i=1}^{j}I_x(t_i)\int_{t_j}^{t_{j+1}}(t-v)^{-\alpha}(\mathbf{E}_{\alpha}(\lambda(v-t_i)^{\alpha}))'dv}{\Gamma(1-\alpha)}$$

$$+\frac{x_0\int_0^t(t-v)^{-\alpha}(\mathbf{E}_\alpha(\lambda v^\alpha))'dv}{\Gamma(1-\alpha)}+\frac{\sum\limits_{i=1}^{k+1}I_x(t_i)\int_{t_{k+1}}^t(t-v)^{-\alpha}(\mathbf{E}_\alpha(\lambda(v-t_i)^\alpha))'dv}{\Gamma(1-\alpha)}+\frac{\int_{t_{k+1}}^t(t-v)^{-\alpha}\Phi'(v)dv}{\Gamma(1-\alpha)}$$

$$=\frac{\int_0^t(t-s)^{-\alpha}\left(\int_0^s(s-w)^{\alpha-1}\sum\limits_{v=0}^{+\infty}\frac{\lambda^v(s-w)^{\alpha v}}{\Gamma(\alpha(v+1))}m(w)f_x(w)dw\right)'ds}{\Gamma(1-\alpha)}$$

$$+\frac{\sum\limits_{j=0}^{k}\sum\limits_{i=1}^{j}I_x(t_i)\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left(\sum\limits_{v=0}^{+\infty}\frac{\lambda^v(s-t_i)^{\alpha v}}{\Gamma(\alpha v+1)}\right)'ds}{\Gamma(1-\alpha)}$$

$$+\frac{x_0\int_0^t(t-s)^{-\alpha}\left(\sum\limits_{v=0}^{+\infty}\frac{\lambda^v s^{\alpha v}}{\Gamma(\alpha v+1)}\right)'ds}{\Gamma(1-\alpha)}+\frac{\sum\limits_{i=1}^{k+1}I_x(t_i)\int_{t_{k+1}}^t(t-s)^{-\alpha}\left(\sum\limits_{v=0}^{+\infty}\frac{\lambda^v(s-t_i)^{\alpha v}}{\Gamma(\alpha v+1)}\right)'ds}{\Gamma(1-\alpha)}+{}^cD^\alpha_{t_{k+1}^+}\Phi(t)$$

$$=\frac{1}{(1-\alpha)\Gamma(1-\alpha)}\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\ \left[\int_0^t(t-s)^{1-\alpha}\left(\int_0^s(s-w)^{\alpha(v+1)-1}m(w)f_x(w)dw\right)'ds\right]'$$

$$+\frac{\sum\limits_{j=0}^{k}\sum\limits_{i=1}^{j}I_x(t_i)\sum\limits_{v=1}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha v)}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}(s-t_i)^{\alpha v-1}ds}{\Gamma(1-\alpha)}$$

$$+\frac{x_0\sum\limits_{v=1}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha v)}\int_0^t(t-s)^{-\alpha}s^{\alpha v-1}ds}{\Gamma(1-\alpha)}+\frac{\sum\limits_{i=1}^{k+1}I_x(t_i)\sum\limits_{v=1}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha v)}\int_{t_{k+1}}^t(t-s)^{-\alpha}(s-t_i)^{\alpha v-1}ds}{\Gamma(1-\alpha)}+{}^cD^\alpha_{t_{k+1}^+}\Phi(t).$$

By variable changing $\frac{s-t_i}{t-t_i}=w$ in second and fourth term and $\frac{s}{t}=w$ in third term, interchanging the sum order of the second term, integration by parts for the first term, we get

$$m(t)f(t,x(t),{}^cD^p_{0^+}x(t))+\lambda u(t)$$

$$=\frac{1}{\Gamma(2-\alpha)}\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\ \left[(1-\alpha)\int_0^t(t-s)^{-\alpha}\int_0^s(s-w)^{\alpha(v+1)-1}m(w)f_x(w)dwds\right]'$$

$$+\frac{\sum\limits_{i=1}^{k}\sum\limits_{j=i}^{k}I_x(t_i)\sum\limits_{v=1}^{+\infty}\frac{\lambda^v(t-t_i)^{\alpha(v-1)}}{\Gamma(\alpha v)}\int_{\frac{t_j-t_i}{t-t_i}}^{\frac{t_{j+1}-t_i}{t-t_i}}(1-w)^{-\alpha}w^{\alpha v-1}dw}{\Gamma(1-\alpha)}+\frac{x_0\sum\limits_{v=1}^{+\infty}\frac{\lambda^v t^{\alpha(v-1)}}{\Gamma(\alpha v)}\int_0^1(1-w)^{-\alpha}w^{\alpha v-1}ds}{\Gamma(1-\alpha)}$$

$$+\frac{\sum\limits_{i=1}^{k+1}I_x(t_i)\sum\limits_{v=1}^{+\infty}\frac{\lambda^v(t-t_i)^{\alpha(v-1)}}{\Gamma(\alpha v)}\int_{\frac{t_{k+1}-t_i}{t-t_i}}^1(1-w)^{-\alpha}w^{\alpha v-1}dw}{\Gamma(1-\alpha)}+{}^cD^\alpha_{t_{k+1}^+}\Phi(t)$$

$$= \frac{1}{\Gamma(2-\alpha)} \sum_{v=0}^{+\infty} \frac{\lambda^v}{\Gamma(\alpha(v+1))} \left[(1-\alpha)\int_0^t (t-s)^{-\alpha} \int_0^s (s-w)^{\alpha(v+1)-1} m(w) f_x(w) dw ds\right]'$$

$$+ \frac{\sum_{i=1}^{k} I_x(t_i) \sum_{v=1}^{+\infty} \frac{\lambda^v (t-t_i)^{\alpha(v-1)}}{\Gamma(\alpha v)} \int_0^{\frac{t_{k+1}-t_i}{t-t_i}} (1-w)^{-\alpha} w^{\alpha v-1} dw}{\Gamma(1-\alpha)}$$

$$+ \frac{\sum_{i=1}^{k+1} I_x(t_i) \sum_{v=1}^{+\infty} \frac{\lambda^v (t-t_i)^{\alpha(v-1)}}{\Gamma(\alpha v)} \int_{\frac{t_{k+1}-t_i}{t-t_i}}^1 (1-w)^{-\alpha} w^{\alpha v-1} dw}{\Gamma(1-\alpha)} + x_0 \sum_{v=1}^{+\infty} \frac{\lambda^v t^{\alpha(v-1)}}{\Gamma(\alpha(v-1)+1)} + {}^cD^{\alpha}_{t_{k+1}^+} \Phi(t)$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{v=0}^{+\infty} \frac{\lambda^v}{\Gamma(\alpha(v+1))} \left[\int_0^t \int_s^t (t-s)^{-\alpha} (s-w)^{\alpha(v+1)-1} ds m(w) f_x(w) dw\right]'$$

$$+ \frac{\sum_{i=1}^{k} I_x(t_i) \sum_{v=1}^{+\infty} \frac{\lambda^v (t-t_i)^{\alpha(v-1)}}{\Gamma(\alpha v)} \int_0^1 (1-w)^{-\alpha} w^{\alpha v-1} dw}{\Gamma(1-\alpha)} + \frac{I_x(t_{k+1}) \sum_{v=1}^{+\infty} \frac{\lambda^v (t-t_j)^{\alpha(v-1)}}{\Gamma(\alpha v)} \int_0^1 (1-w)^{-\alpha} w^{\alpha v-1} dw}{\Gamma(1-\alpha)}$$

$$+ x_0 \sum_{v=1}^{+\infty} \frac{\lambda^v t^{\alpha(v-1)}}{\Gamma(\alpha(v-1)+1)} + {}^cD^{\alpha}_{t_{k+1}^+} \Phi(t)$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{v=0}^{+\infty} \frac{\lambda^v}{\Gamma(\alpha(v+1))} \left[\int_0^t (t-s)^{\alpha v} \int_0^1 (1-u)^{-\alpha} u^{\alpha(v+1)-1} du m(w) f_x(w) dw\right]'$$

$$+ \sum_{i=1}^{k+1} I_x(t_i) \sum_{v=1}^{+\infty} \frac{\lambda^v (t-t_i)^{\alpha(v-1)}}{\Gamma(\alpha(v-1)+1)} + \lambda x_0 \mathbf{E}_{\alpha}(\lambda t^{\alpha}) + {}^cD^{\alpha}_{t_{k+1}^+} \Phi(t)$$

$$= \sum_{v=0}^{+\infty} \frac{\lambda^v}{\Gamma(\alpha v+1)} \left[\int_0^t (t-s)^{\alpha v} m(w) f_x(w) dw\right]' + \lambda \sum_{i=1}^{k+1} I_x(t_i) \mathbf{E}_{\alpha}(\lambda(t-t_i)^{\alpha})$$

$$+ \lambda x_0 \mathbf{E}_{\alpha}(\lambda t^{\alpha}) + {}^cD^{\alpha}_{t_{k+1}^+} \Phi(t)$$

$$= m(t) f_x(t) + \lambda \int_0^t (t-s)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha}) m(s) f_x(s) ds + \lambda x_0 \mathbf{E}_{\alpha}(\lambda t^{\alpha})$$

$$+ \lambda \sum_{j=1}^{k+1} \mathbf{E}_{\alpha}(\lambda(t-t_j)^{\alpha}) I_x(t_j) + {}^cD^{\alpha}_{t_{k+1}^+} \Phi(t).$$

It follows from (5) that ${}^cD^{\alpha}_{t_{k+1}^+} \Phi(t) - \lambda \Phi(t) = 0$ for $t \in (t_{k+1}, t_{k+2}]$. Then (6) and Example 4.9 in [8] imply that $\Phi(t) = I(t_{k+1}, x(t_{k+1}), {}^cD^p_{0^+} x(t_{k+1}))\mathbf{E}_{\alpha}(\lambda(t -$

$t_{k+1})^{\alpha})$. Substituting Φ into (5), we know that (4) holds for $s = k+1$. By mathematical induction method, we know that (4) holds for all $s \in \mathbf{Z}_0$.

Step 2. Now assume u satisfies (4). We will prove that u is a solution of (3) and $u \in X$.

We firstly prove that $u \in X$. In fact, we have

$$\left|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})m(s)f_x(s)ds\right|$$

$$\leq M_r t^{\alpha+k_1}\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})\mathbf{B}(\alpha, k_1+1) \to 0, t \to 0^+,$$

$$\left|\int_0^t (t-s)^{\alpha-p-1}\mathbf{E}_{\alpha,\alpha-p}(\lambda(t-s)^{\alpha})m(s)f_x(s)ds\right|$$

$$\leq M_r t^{\alpha-p+k_1}\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})\mathbf{B}(\alpha-p, k_1+1) \to 0, t \to 0^+.$$

We know that $\lim\limits_{t\to 0^+} u(t)$ exists and is finite. By a direct computation, it follows for $t \in (t_s, t_{s+1}]$ that

$$^cD_{0^+}^p u(t) = \frac{1}{\Gamma(1-p)}\int_0^t (t-s)^{-p}u'(s)ds$$

$$= \frac{\sum\limits_{j=0}^{s-1}\int_{t_j}^{t_{j+1}}(t-s)^{-p}\left[\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})m(u)f_x(u)du + x_0\mathbf{E}_{\alpha}(\lambda s^{\alpha}) + \sum\limits_{i=1}^{j} I_x(t_i)\mathbf{E}_{\alpha}(\lambda(s-t_i)^{\alpha})\right]' ds}{\Gamma(1-p)}$$

$$+ \frac{\int_{t_s}^t (t-s)^{-p}\left[\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})m(u)f_x(u)du + x_0\mathbf{E}_{\alpha}(\lambda s^{\alpha}) + \sum\limits_{i=1}^{s} I_x(t_i)\mathbf{E}_{\alpha}(\lambda(s-t_i)^{\alpha})\right]' ds}{\Gamma(1-p)}$$

$$= \frac{1}{\Gamma(1-p)}\int_0^t (t-s)^{-p}\left[\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})m(u)f_x(u)du\right]' ds$$

$$+ \frac{1}{\Gamma(1-p)}x_0\int_0^t (t-s)^{-p}\left[\mathbf{E}_{\alpha}(\lambda s^{\alpha})\right]' ds$$

$$+ \frac{1}{\Gamma(1-p)}\sum_{j=0}^{s-1}\int_{t_j}^{t_{j+1}}(t-s)^{-p}\left[\sum_{i=1}^{j}\mathbf{E}_{\alpha}(\lambda(s-t_i)^{\alpha})I_x(t_i)\right]' ds$$

$$+ \frac{1}{\Gamma(1-p)}\int_{t_s}^t (t-s)^{-p}\left[\sum_{i=1}^{s}\mathbf{E}_{\alpha}(\lambda(s-t_i)^{\alpha})I_x(t_i)\right]' ds$$

$$= \frac{1}{(1-p)\Gamma(1-p)} \left[\int_0^t (t-s)^{1-p} \left(\int_0^s (s-u)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha) m(u) f_x(u) du \right)' ds \right]'$$

$$+ \frac{1}{\Gamma(1-p)} x_0 \int_0^t (t-s)^{-p} \left[\sum_{v=0}^{+\infty} \frac{\lambda^v s^{\alpha v}}{\Gamma(\alpha v+1)} \right]' ds$$

$$+ \frac{1}{\Gamma(1-p)} \sum_{j=0}^{s-1} \int_{t_j}^{t_{j+1}} (t-s)^{-p} \left[\sum_{i=1}^{j} \sum_{v=0}^{+\infty} \frac{\lambda^v (s-t_i)^{\alpha} v}{\Gamma(\alpha v+1)} I_x(t_i) \right]' ds$$

$$+ \frac{1}{\Gamma(1-p)} \int_{t_s}^t (t-s)^{-p} \left[\sum_{i=1}^{s} \sum_{v=0}^{+\infty} \frac{\lambda^v (s-t_i)^{\alpha v}}{\Gamma(\alpha v+1)} I_x(t_i) \right]' ds$$

(use integration by parts for first term)

$$= \frac{1}{\Gamma(1-p)} \left[\int_0^t (t-s)^{-p} \int_0^s (s-u)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha) m(u) f_x(u) du ds \right]'$$

$$+ \frac{1}{\Gamma(1-p)} x_0 \sum_{v=1}^{+\infty} \int_0^t (t-s)^{-p} \frac{\lambda^v s^{\alpha v-1}}{\Gamma(\alpha v)} ds$$

$$+ \frac{1}{\Gamma(1-p)} \sum_{j=0}^{s-1} \sum_{i=1}^{j} \int_{t_j}^{t_{j+1}} (t-s)^{-p} \sum_{v=1}^{+\infty} \frac{\lambda^v (s-t_i)^{\alpha v-1}}{\Gamma(\alpha v)} I_x(t_i) ds$$

$$+ \frac{1}{\Gamma(1-p)} \int_{t_s}^t (t-s)^{-p} \sum_{i=1}^{s} \sum_{v=1}^{+\infty} \frac{\lambda^v (s-t_i)^{\alpha v-1}}{\Gamma(\alpha v)} I_x(t_i) ds$$

$$= \frac{1}{\Gamma(1-p)} \left[\int_0^t \int_u^t (t-s)^{-p} (s-u)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha) ds\, m(u) f_x(u) du \right]'$$

$$+ \frac{1}{\Gamma(1-p)} x_0 \sum_{v=1}^{+\infty} t^{\alpha v-p} \int_0^1 (1-w)^{-p} \frac{\lambda^v w^{\alpha v-1}}{\Gamma(\alpha v)} dw$$

$$+ \frac{1}{\Gamma(1-p)} \sum_{i=1}^{s-1} \sum_{j=i}^{s-1} (t-t_i)^{\alpha v-p} \int_{\frac{t_j-t_i}{t-t_i}}^{\frac{t_{j+1}-t_i}{t-t_i}} (1-w)^{-p} \sum_{v=1}^{+\infty} \frac{\lambda^v w^{\alpha v-1}}{\Gamma(\alpha v)} I_x(t_i) dw$$

$$+ \frac{1}{\Gamma(1-p)} \sum_{i=1}^{s} (t-t_i)^{\alpha v-p} \int_{\frac{t_s-t_i}{t-t_i}}^{1} (1-w)^{-p} \sum_{v=1}^{+\infty} \frac{\lambda^v w^{\alpha v-1}}{\Gamma(\alpha v)} I_x(t_i) dw$$

$$= \frac{1}{\Gamma(1-p)} \left[\sum_{v=0}^{+\infty} \frac{\lambda^v}{\Gamma(\alpha(v+1))} \int_0^t (t-u)^{\alpha(v+1)-p} \int_0^1 (1-w)^{-p} w^{\alpha(v+1)-1} dw m(u) f_x(u) du \right]'$$

$$+\lambda x_0 t^{\alpha-p} \mathbf{E}_{\alpha,\alpha+1-p}(\lambda t^\alpha)$$

$$+\frac{1}{\Gamma(1-p)} \sum_{i=1}^{s-1} (t-t_i)^{\alpha v-p} \int_0^{\frac{t_s-t_i}{t-t_i}} (1-w)^{-p} \sum_{v=1}^{+\infty} \frac{\lambda^v w^{\alpha v-1}}{\Gamma(\alpha v)} I_x(t_i) dw$$

$$+\frac{1}{\Gamma(1-p)} \sum_{i=1}^{s} (t-t_i)^{\alpha v-p} \int_{\frac{t_s-t_i}{t-t_i}}^1 (1-w)^{-p} \sum_{v=1}^{+\infty} \frac{\lambda^v w^{\alpha v-1}}{\Gamma(\alpha v)} I_x(t_i) dw$$

$$= \frac{1}{\Gamma(1-p)} \left[\sum_{v=0}^{+\infty} \frac{\lambda^v}{\Gamma(\alpha(v+1))} \int_0^t (t-u)^{\alpha(v+1)-p} \int_0^1 (1-w)^{-p} w^{\alpha(v+1)-1} dw m(u) f_x(u) du \right]'$$

$$+\lambda x_0 t^{\alpha-p} \mathbf{E}_{\alpha,\alpha+1-p}(\lambda t^\alpha) + \lambda \sum_{i=1}^{s} I_x(t_i)(t-t_i)^{\alpha-p} \mathbf{E}_{\alpha,\alpha+1-p}(\lambda (t-t_i)^\alpha)$$

$$= \int_0^t (t-u)^{\alpha-p-1} \mathbf{E}_{\alpha,\alpha-p}(\lambda(t-u)^\alpha) m(u) f_x(u) du$$

$$+\lambda x_0 t^{\alpha-p} \mathbf{E}_{\alpha,\alpha+1-p}(\lambda t^\alpha) + \lambda \sum_{i=1}^{s} I_x(t_i)(t-t_i)^{\alpha-p} \mathbf{E}_{\alpha,\alpha+1-p}(\lambda (t-t_i)^\alpha),$$

$t \in (t_s, t_{s+1}], s \in \mathbf{Z}_0.$

So $\lim\limits_{t\to 0^+} {}^cD_{0^+}^p u(t)$ exists and is finite.

It is easy to see that $u|_{(t_s,t_{s+1}]}, {}^cD_{0^+}^p u|_{(t_s,t_{s+1}]} \in C^0((t_s,t_{s+1}])(s \in \mathbf{Z}_0)$ and limits $\lim\limits_{t\to t_s^+} u(t)$, $\lim\limits_{t\to t_s^+} {}^cD_{0^+}^p u(t)(s \in \mathbf{Z}_0)$ exist. Note

$$\mathbf{E}_\alpha(\lambda t^\alpha) = \sum_{v=0}^{+\infty} \frac{\lambda^v t^{\alpha v}}{\Gamma(\alpha v+1)} = 1 + \sum_{v=1}^{+\infty} \frac{\lambda^v t^{\alpha v}}{\alpha v \Gamma(\alpha v)} \le 1 + \frac{\lambda t^\alpha}{\alpha} \mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha),$$

$$\mathbf{E}_{\alpha,\alpha+1-p}(\lambda t^\alpha) = \sum_{v=0}^{+\infty} \frac{\lambda^v t^{\alpha v}}{\Gamma(\alpha v+\alpha+1-p)} \le \sum_{v=0}^{+\infty} \frac{\lambda^v t^{\alpha v}}{(\alpha v+\alpha-p)\Gamma(\alpha v+\alpha-p)}$$

$$\le \sum_{v=0}^{+\infty} \frac{\lambda^v t^{\alpha v}}{(\alpha-p)\Gamma(\alpha v+\alpha-p)} = \frac{1}{\alpha-p} \mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha).$$

It follows that

$$\frac{\mathbf{E}_{\alpha}(\lambda t^{\alpha})}{\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})} \leq \Gamma(\alpha) + \frac{\lambda t^{\alpha}}{\alpha}, \quad \frac{\mathbf{E}_{\alpha,\alpha+1-p}(\lambda t^{\alpha})}{\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})} \leq \frac{1}{\alpha-p}. \tag{7}$$

Hence, we have, for $t \in (t_s, t_{s+1}]$, use (7), that

$$\frac{|u(t)|}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})} \leq \frac{1}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})} \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha}) s^{k_1} M_r ds$$

$$+\frac{|x_0|\mathbf{E}_{\alpha}(\lambda t^{\alpha})}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})} + \frac{1}{(1+t^{\sigma})\mathbf{E}_{\alpha}(\lambda t^{\alpha})} \sum_{j=1}^{s} \mathbf{E}_{\alpha}(\lambda(t-t_j)^{\alpha}) M_{rj}$$

$$\leq \frac{\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})} \int_0^t (t-s)^{\alpha-1} s^{k_1} M_r ds + \frac{|x_0|\mathbf{E}_{\alpha}(\lambda t^{\alpha})}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})}$$

$$+\frac{1}{(1+t^{\sigma})\mathbf{E}_{\alpha}(\lambda t^{\alpha})} \sum_{j=1}^{s} \mathbf{E}_{\alpha}(\lambda t^{\alpha}) M_{rj}$$

$$\leq \frac{M_r}{1+t^{\sigma}} t^{\alpha+k_1}\mathbf{B}(\alpha, k_1+1) + \frac{|x_0|}{1+t^{\sigma}}\left(\Gamma(\alpha) + \frac{\lambda t^{\alpha}}{\alpha}\right) + \frac{\Gamma(\alpha)+\frac{\lambda t^{\alpha}}{\alpha}}{1+t^{\sigma}} \sum_{j=1}^{+\infty} M_{rj}.$$

So $\lim\limits_{t\to+\infty} \frac{|u(t)|}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})} = 0$. Similarly we get, for $t \in (t_s, t_{s+1}]$ and using (7), that

$$\frac{t^p|{}^cD^p_{0^+}u(t)|}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})} \leq \frac{t^p}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})} \int_0^t (t-u)^{\alpha-p-1}\mathbf{E}_{\alpha,\alpha-p}(\lambda(t-u)^{\alpha}) u^{k_1} M_r du$$

$$+\frac{\lambda|x_0|t^{\alpha}}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})}\mathbf{E}_{\alpha,\alpha+1-p}(\lambda t^{\alpha})$$

$$+\lambda\frac{t^p}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})} \sum_{j=1}^{s} M_{rj}(t-t_j)^{\alpha-p}\mathbf{E}_{\alpha,\alpha+1-p}(\lambda(t-t_i)^{\alpha})$$

$$\leq \frac{t^p\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})} \int_0^t (t-u)^{\alpha-p-1} u^{k_1} M_r du$$

$$+\frac{\lambda|x_0|t^{\alpha}}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})}\mathbf{E}_{\alpha,\alpha+1-p}(\lambda t^{\alpha}) + \lambda\frac{t^{\alpha}}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})} \sum_{j=1}^{+\infty} M_{rj}\mathbf{E}_{\alpha,\alpha+1-p}(\lambda t^{\alpha})$$

$$\leq \frac{t^{\alpha+k_1}}{(\alpha-p)(1+t^{\sigma})}\mathbf{B}(\alpha-p, k_1+1)M_r + \frac{\lambda|x_0|t^{\alpha}}{(\alpha-p)(1+t^{\sigma})} + \frac{\lambda t^{\alpha}}{(\alpha-p)(1+t^{\sigma})} \sum_{j=1}^{+\infty} M_{rj}.$$

So $\lim\limits_{t\to+\infty} \frac{t^p}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})} {}^cD^p_{0^+}u(t) = 0$. Then $u \in X$.

We secondly prove that u satisfies the system (3). We know easily that $u(0) = x_0$. From (4), it is obviously that $\Delta u(t_s) = I(t_s, x(t_s), {}^cD^p_{0+}x(t_s))$ for all $s \in Z$. By direct computation, we have for $t \in (t_j, t_{j+1}](j \in Z)$ similarly to above discussion that

$$
\begin{aligned}
&{}^cD^{\alpha}_{0+}u(t) = \tfrac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u'(s)ds \\
&= \tfrac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{j}\int_{t_{i-1}}^{t_i}(t-s)^{-\alpha}u'(s)ds + \tfrac{1}{\Gamma(1-\alpha)}\int_{t_j}^{t}(t-s)^{-\alpha}u'(s)ds \\
&= \tfrac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{j}\int_{t_{i-1}}^{t_i}(t-s)^{-\alpha}\Big(\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})m(u)f_x(u)du \\
&+x_0\mathbf{E}_{\alpha}(\lambda s^{\alpha}) + \sum_{j=1}^{i-1}\mathbf{E}_{\alpha}(\lambda(t-t_j)^{\alpha})I_x(t_j)\Big)' ds \\
&+\tfrac{1}{\Gamma(1-\alpha)}\int_{t_j}^{t}(t-s)^{-\alpha}\Big(\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})m(u)f_x(u)du \\
&+x_0\mathbf{E}_{\alpha}(\lambda s^{\alpha}) + \sum_{i=1}^{j}\mathbf{E}_{\alpha}(\lambda(s-t_i)^{\alpha})I_x(t_i)\Big)' ds \\
&= \frac{\int_0^t (t-s)^{-\alpha}\left(\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})m(u)f_x(u)du\right)' ds}{\Gamma(1-\alpha)} \\
&+\frac{\sum_{i=1}^{j}\sum_{j=1}^{i-1}I_x(t_j)\int_{t_{i-1}}^{t_i}(t-s)^{-\alpha}(\mathbf{E}_{\alpha}(\lambda(t-t_j)^{\alpha}))' ds}{\Gamma(1-\alpha)} \\
&+\frac{x_0\int_0^t (t-s)^{-\alpha}(\mathbf{E}_{\alpha}(\lambda s^{\alpha}))' ds}{\Gamma(1-\alpha)} + \frac{\sum_{i=1}^{j}I_x(t_i)\int_{t_j}^{t}(t-s)^{-\alpha}(\mathbf{E}_{\alpha}(\lambda(s-t_i)^{\alpha}))' ds}{\Gamma(1-\alpha)} \\
&= \lambda u(t) + m(t)f(t, x(t), {}^cD^p_{0+}x(t)).
\end{aligned}
$$

Thus $u \in X$ and u is a solution of (3). This completes the proof. ∎

Let us define an operator T on X by

$$(Tx)(t) = \int_0^t (t-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-u)^\alpha)m(u)f(u,x(u),{}^cD_{0^+}^p x(u))du$$
$$+x_0\mathbf{E}_\alpha(\lambda t^\alpha) + \sum_{j=1}^{s}\mathbf{E}_\alpha(\lambda(t-t_j)^\alpha)I(t_j,x(t_j),{}^cD_{0^+}^p x(t_j) \qquad (8)$$

for $t \in (t_s, t_{s+1}], s \in \mathbf{Z}_0$.

Lemma 2.4. Suppose that f is a I-Carathéodory function and I a discrete I-Carathéodory function. Then

(i) $T : X \to X$ is well defined;

(ii) the fixed point of the operator T coincides with the solution of IVP(1);

(iii) $T : X \to X$ is completely continuous.

Proof. From Lemma 2.1, X is a Banach space. (i) the proof comes from Step 2 in the proof of Lemma 2.3. (ii) from Lemma 2.3, the proof is obvious. (iii) the proof is divided into following three steps:

Step 1. Prove that T is continuous. Use the facts that f is a Carathéodory function and I a discrete Carathéodory function.

Step 2. Let M be a bounded subset of X. Prove that TM is bounded set. Use the facts that f is a Carathéodory function and I a discrete Carathéodory function.

Step 3. Let M be a bounded subset of X. Use Lemma 2.2, prove that TM is relatively compact based upon Lemma 2.2. Use the facts that f is a Carathéodory function and I a discrete Carathéodory function. The details are omitted. ∎

Let

$$\delta_0(t) = \sum_{j=0}^{s} t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^\alpha)\prod_{v=1}^{s+1}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha),$$

$t \in (t_s, t_{s+1}], s \in \mathbf{Z}_0$,

and

$$\delta_1(t) = [\lambda(t_{s+1}-t_s)^{\alpha-p}+1]\sum_{j=0}^{s} t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}\times$$

$$[\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^\alpha)+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^\alpha)]\times$$

$$\prod_{v=1}^{s} \mathbf{E}_{\alpha}(\lambda(t_v - t_{v-1})^{\alpha})\mathbf{E}_{\alpha,\alpha+1-p}(\lambda(t-t_s)^{\alpha}),\ t \in (t_s, t_{s+1}], s \in \mathbf{Z}_0. \qquad (9)$$

$$Y = \left\{ x : \begin{array}{c} y|_{(t_s,t_{s+1}]}, {}^cD_{*+}^p y|_{(t_s,t_{s+1}]} = {}^cD_{t_s^+}^p y \in C^0((t_s,t_{s+1}],\mathbb{R}), s \in \mathbf{Z}_0, \\ \lim_{t\to t_s^+} y(t), \lim_{t\to t_s^+} {}^cD_{t_s^+}^p y(t) \text{ exist, } s \in \mathbf{Z}_0, \\ \lim_{t\to+\infty} \frac{y(t)}{\delta_0(t)}, \lim_{t\to+\infty} \frac{{}^cD_{*+}^p y(t)}{\delta_1(t)} \text{ exists} \end{array} \right\}.$$

For $y \in Y$, let z_0, z_1 be defined by defined by (11), the norm by

$$||y|| = ||y||_Y = \max\left\{ \sup_{t\in(0,+\infty)} \frac{|y(t)|}{\delta_0(t)}, \sup_{t\in(0,+\infty)} \frac{|{}^cD_{*+}^p y(t)|}{\delta_1(t)} \right\}.$$

Definition 2.5. $h : (0,+\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is called a II-Carathéodory function if it satisfies the following assumptions:

(i) $t \to h(t,\ \delta_0(t)x, \delta_1(t)y)$ is continuous on $(t_s, t_{s+1}](s \in \mathbf{Z}_0)$ and bounded on $\mathbb{R}$,

(ii) $(x,y) \to h(t,\ \delta_0(t)x, \delta_1(t)y)$ is continuous on $\mathbb{R}$;

(iii) for each $r > 0$ there exists a constant $M_r \geq 0$ such that $|x|, |y| \leq r$ imply

$$|h(t,\ \delta_0(t)x, \delta_1(t)y)| \leq M_r, t \in (0,+\infty).$$

Definition 2.6. $H : \{t_s : s \in \mathbf{Z}\} \times \mathbb{R}^2 \to \mathbb{R}$ is called a discrete II-Carathéodory function if it satisfies the following assumptions:

(i) $(x,y) \to H(t_s,\ \delta_0(t_s)x, \delta_1(t_s)y)$ is continuous on $\mathbb{R}$ for all $s \in \mathbf{Z}$;

(ii) for each $r > 0$ there exists a constant $M_{rs} \geq 0$ with $\sum_{s=1}^{+\infty} M_{rs} < +\infty$ such that $|x|, |y| \leq r$ imply

$$|H(t_s,\ \delta_0(t_s)x, \delta_1(t_s)y)| \leq M_{rs}, s \in \mathbf{Z}.$$

Lemma 2.6. Y is a Banach space with $||\cdot||_Y$ defined.

Proof. The proof is similar to that used in the proof of Lemma 2.1. The details are omitted. ∎

Lemma 2.7. Let M be a subset of Y. Then M is relatively compact if and only if the following conditions are satisfied:

(i) both $\left\{t \to \frac{y(t)}{\delta_0(t)} : y \in M\right\}$ and $\left\{t \to \frac{{}^cD_{*+}^p y(t)}{\delta_1(t)} : y \in M\right\}$ are uniformly bounded,

(ii) both $\left\{t \to \frac{y(t)}{\delta_0(t)} : y \in M\right\}$ and $\left\{t \to \frac{{}^cD^p_{*+}y(t)}{\delta_1(t)} : y \in M\right\}$ are equicontinuous in any subinterval $(t_s, t_{s+1}](s \in \mathbf{Z}_0)$,

(iii) both $\left\{t \to \frac{y(t)}{\delta_0(t)} : y \in M\right\}$ and $\left\{t \to \frac{{}^cD^p_{*+}y(t)}{\delta_1(t)} : y \in M\right\}$ are equi-convergent as $t \to +\infty$.

Proof. The proof is similar to that used in the proof of Lemma 2.2. The details are omitted. ∎

Lemma 2.8. Suppose that $x \in Y$. Then $u \in Y$ is a solution of

$$\begin{cases} {}^cD^\alpha_{t_s^+}u(t) - \lambda u(t) = n(t)g(t, x(t), {}^cD^p_{t_s^+}x(t)), a.e., t \in (t_s, t_{s+1}), s \in \mathbf{Z}_0, \\ u(0) = x_0, \;\; \Delta u(t_s) = J(t_s, x(t_s), {}^cD^p_{t_{s-1}^+}x(t_s)), s \in \mathbf{Z}. \end{cases} \tag{10}$$

if and only if

$$u(t) = \begin{cases} \int_0^t (t-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-v)^\alpha)n(v)g(v, x(v), {}^cD^p_{0^+}x(v))dv + x_0\mathbf{E}_\alpha(\lambda t^\alpha), \\ t \in (t_0, t_1], \\ \int_{t_s}^t (t-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-v)^\alpha)n(v)g(v, x(v), {}^cD^p_{t_s^+}x(v))dv \\ + \left[x_0 \prod_{v=1}^s \mathbf{E}_\alpha(\lambda(t_v - t_{v-1})^\alpha) \right. \\ + \sum_{j=1}^s \left(\prod_{v=j+1}^s \mathbf{E}_\alpha(\lambda(t_v - t_{v-1})^\alpha)\right) J(t_j, x(t_j), {}^cD^p_{t_{j-1}^+}x(t_j)) \\ + \sum_{j=1}^s \left(\prod_{v=j+1}^s \mathbf{E}_\alpha(\lambda(t_v - t_{v-1})^\alpha)\right) \times \\ \left. \int_{t_{j-1}}^{t_j} (t_j - v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j - v)^\alpha)n(v)g(v, x(v), {}^cD^p_{t_{j-1}^+}x(v))dv \right] \times \\ \mathbf{E}_\alpha(\lambda(t - t_s)^\alpha), \\ t \in (t_s, t_{s+1}], s \in \mathbf{Z}, x \in X. \end{cases} \tag{11}$$

Proof. In fact, for $x \in Y$, we have $||y|| = r < +\infty$. Since g is a II-Carathéodory function and J a discrete II-Carathéodory function, we know that there exists $M_r \geq 0$, $M_{rs} \geq 0$ such that

$$|f(t, x(t), {}^cD^p_{t_s^+}x(t))| \leq M_r, t \in (t_s, t_{s+1}], s \in \mathbf{Z}_0,$$

$$|J(t_s, x(t_s), {}^cD^p_{t_{s-1}^+}x(t_s))| \leq M_{rs}, s \in \mathbf{Z}_0.$$

Then

$$\left|\int_{t_s}^t (t-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-v)^\alpha)n(v)g(v, x(v), {}^cD^p_{t_s^+}x(v))dv\right|$$

$$\leq M_r \int_{t_s}^t (t-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-v)^\alpha)(v-t_s)^{k_2}(t_{s+1}-v)^{l_2}dv$$

$$\leq M_r \int_{t_s}^t (t-v)^{\alpha+l_2-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{s+1}-t_s)^\alpha)(v-t_s)^{k_2}dv$$

$$\leq M_r(t_{s+1}-t_s)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{s+1}-t_s)^\alpha)\mathbf{B}(\alpha+l_2, k_2+1).$$

Furthermore, we have

$$\left|\int_{t_s}^t (t-v)^{\alpha-p-1}\mathbf{E}_{\alpha,\alpha-p}(\lambda(t-v)^\alpha)n(v)g(v, x(v), {}^cD^p_{t_s^+}x(v))dv\right|$$

$$\leq M_r(t_{s+1}-t_s)^{\alpha+k_2+l_2-p}\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{s+1}-t_s)^\alpha)\mathbf{B}(\alpha+l_2-p, k_2+1).$$

If u is a solution of (10), then from Example 4.9 in [8] we have

$$u(t) = \int_0^t (t-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-v)^\alpha)n(v)g(v, x(v), {}^cD^p_{t_s^+}x(v))dv + x_0\mathbf{E}_\alpha(\lambda t^\alpha),$$
$$t \in (t_0, t_1].$$

When $t \in (t_s, t_{s+1}](s \geq 1)$, we have similarly that there exist numbers $c_s \in \mathbb{R}$ such that

$$u(t) = \int_{t_s}^t (t-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-v)^\alpha)n(v)g(v, x(v), {}^cD^p_{t_s^+}x(v))dv$$

$$+c_s\mathbf{E}_\alpha(\lambda(t-t_s)^\alpha),\ t \in (t_s, t_{s+1}].$$

By $\Delta u(t_s) = J(t_s, x(t_s), {}^cD^p_{t_{s-1}^+}x(t_s))$, we see

$$c_s = c_{s-1}\mathbf{E}_\alpha(\lambda(t_s-t_{s-1})^\alpha) + J(t_s, x(t_s), {}^cD^p_{t_{s-1}^+}x(t_s))$$

$$+\int_{t_{s-1}}^{t_s}(t_s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_s-v)^\alpha)n(v)g(v, x(v), {}^cD^p_{t_{s-1}^+}x(v))dv.$$

It follows that

$$c_s = x_0 \prod_{v=1}^{s} \mathbf{E}_\alpha(\lambda(t_v - t_{v-1})^\alpha)$$

$$+ \sum_{j=1}^{s} \left(\prod_{v=j+1}^{s} \mathbf{E}_\alpha(\lambda(t_v - t_{v-1})^\alpha) \right) J(t_j, x(t_j), {}^cD^p_{t^+_{j-1}} x(t_j))$$

$$+ \sum_{j=1}^{s} \left(\prod_{v=j+1}^{s} \mathbf{E}_\alpha(\lambda(t_v - t_{v-1})^\alpha) \right) \times$$

$$\int_{t_{j-1}}^{t_j} (t_j - w)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(t_j - w)^\alpha) n(w) g(w, x(w), {}^cD^p_{t^+_{j-1}} x(w)) dw.$$

Then u satisfies (11). We have furthermore that

$${}^cD^p_{t^+_s} u(t) =$$

$$\begin{cases} \int_0^t (t-w)^{\alpha-p-1} \mathbf{E}_{\alpha,\alpha-p}(\lambda(t-w)^\alpha) n(w) g(w, x(w), {}^cD^p_{0^+} x(w)) dw \\ +\lambda x_0 t^{\alpha-p} \mathbf{E}_{\alpha,\alpha+1-p}(\lambda t^\alpha),\ t \in (t_0, t_1], \\ \int_{t_s}^t (t-w)^{\alpha-p-1} \mathbf{E}_{\alpha,\alpha-p}(\lambda(t-w)^\alpha) n(w) g(w, x(w), {}^cD^p_{t^+_s} x(w)) dw \\ +\lambda(t-t_s)^{\alpha-p} \left[x_0 \prod_{v=1}^{s} \mathbf{E}_\alpha(\lambda(t_v - t_{v-1})^\alpha) \right. \\ + \sum_{j=1}^{s} \left(\prod_{v=j+1}^{s} \mathbf{E}_\alpha(\lambda(t_v - t_{v-1})^\alpha) \right) J(t_j, x(t_j), {}^cD^p_{t^+_{j-1}} x(t_j)) \\ + \sum_{j=1}^{s} \left(\prod_{v=j+1}^{s} \mathbf{E}_\alpha(\lambda(t_v - t_{v-1})^\alpha) \right) \times \\ \left. \int_{t_{j-1}}^{t_j} (t_j - w)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(t_j - w)^\alpha) n(w) g(w, x(w), {}^cD^p_{t^+_{j-1}} x(w)) dw \right] \times \\ \mathbf{E}_{\alpha,\alpha+1-p}(\lambda(t-t_s)^\alpha), t \in (t_s, t_{s+1}], s \in \mathbf{Z}, x \in X. \end{cases}$$

If u satisfies (11), we have

$$y|_{(t_s,t_{s+1}]}, {}^cD^p_{*^+}y|_{(t_s,t_{s+1}]} = {}^cD^p_{t_s^+}y \in C^0((t_s,t_{s+1}],\mathbb{R}),$$

$$\lim_{t\to t_s^+} y(t), \lim_{t\to t_s^+} {}^cD^p_{t_s^+}y(t) \text{ exist}, \ s \in \mathbf{Z}_0.$$

Note $\prod\limits_{v=1}^{s+1}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha) \geq 1$, we have

$$\frac{|u(t)|}{\delta_0(t)} \leq \frac{M_r}{\delta_0(t)}\int_{t_s}^t (t-w)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-w)^\alpha)(w-s_s)^{k_2}(t_{s+1}-w)^{l_2}dw$$

$$+\frac{1}{\delta_0(t)}\left[|x_0|\prod_{v=1}^{s}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha)+\sum_{j=1}^{s}M_{rj}\left(\prod_{v=j+1}^{s}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha)\right)\right.$$

$$+M_r\sum_{j=1}^{s}\left(\prod_{v=j+1}^{s}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha)\right)\times$$

$$\left.\int_{t_{j-1}}^{t_j}(t_j-w)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-w)^\alpha)(w-s_s)^{k_2}(t_{s+1}-w)^{l_2}dw\right]\mathbf{E}_\alpha(\lambda(t_{s+1}-t_s)^\alpha)$$

$$\leq \frac{M_r\mathbf{E}_{\alpha,\alpha}(\lambda(t_{s+1}-t_s)^\alpha)\int_{t_s}^t(t-w)^{\alpha-1}(w-s_s)^{k_2}(t_{s+1}-w)^{l_2}dw}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^\alpha)}$$

$$+\frac{|x_0|+\sum\limits_{j=1}^{+\infty}M_{rj}}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^\alpha)}$$

$$+\frac{M_r\sum\limits_{j=1}^{s}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^\alpha)\int_{t_{j-1}}^{t_j}(t_j-w)^{\alpha-1}(w-s_s)^{k_2}(t_{s+1}-w)^{l_2}dw}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^\alpha)}$$

$$\leq \frac{|x_0|+\sum\limits_{j=1}^{+\infty}M_{rj}+M_r\sum\limits_{j=1}^{s+1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^\alpha)(t_j-t_{j-1})^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^\alpha)}.$$

Use Stolz theorem, we have

$$\lim_{s\to+\infty}\frac{|x_0|+\sum\limits_{j=1}^{+\infty}M_{rj}+M_r\sum\limits_{j=1}^{s+1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^{\alpha})(t_j-t_{j-1})^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})}$$

$$=\lim_{s\to+\infty}\frac{\mathbf{E}_{\alpha,\alpha}(\lambda(t_{s+1}-t_s)^{\alpha})(t_{s+1}-t_s)^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)}{t_{s+1}(t_{s+1}-t_s)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{s+1}-t_s)^{\alpha})}=\lim_{s\to+\infty}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{t_{s+1}}=0,$$

we know that $\lim\limits_{t\to+\infty}\frac{|u(t)|}{\delta_0(t)}=0$. Similarly we can prove that

$$\frac{|cD_{*+}^{p}u(t)|}{\delta_1(t)}$$

$$\leq\frac{|x_0|+\sum\limits_{j=1}^{+\infty}M_{rj}+M_r\sum\limits_{j=1}^{s+1}[\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_j-t_{j-1})^{\alpha})+\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^{\alpha})](t_j-t_{j-1})^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}[\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})]}.$$

Then Stolz theorem implies that $\lim\limits_{t\to+\infty}\frac{|^{c}D_{*+}^{p}u_{(t_s,t_{s+1}]}(t)|}{\delta_1(t)}=0$. It follows that $u\in Y$. The remainder of the proof is similar to that used in the proof of Lemma 2.3. The details are omitted. ∎

Let us define $T_1 : Y \to Y$ by

$$(T_1x)(t) = \begin{cases} \int_0^t (t-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-u)^\alpha)n(u)g(u,x(u),{}^cD^p_{t_0^+}x(u))du + x_0\mathbf{E}_\alpha(\lambda t^\alpha), \\ t \in (t_0,t_1], \\ \int_{t_s}^t (t-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-u)^\alpha)n(u)g(u,x(u),{}^cD^p_{t_s^+}x(u))du \\ + \Bigg[x_0 \prod_{v=1}^s \mathbf{E}_\alpha(\lambda(t_v - t_{v-1})^\alpha) \\ + \sum_{j=1}^s \left(\prod_{v=j+1}^s \mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha) \right) J(t_j, x(t_j), {}^cD^p_{t_{j-1}^+}x(t_j)) \\ + \sum_{j=1}^s \left(\prod_{v=j+1}^s \mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha) \right) \times \\ \int_{t_{j-1}}^{t_j} (t_j-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-u)^\alpha)n(u)g(u,x(u),{}^cD^p_{t_{j-1}^+}x(u))du \Bigg] \mathbf{E}_\alpha(\lambda(t-t_s)^\alpha), \\ t \in (t_s, t_{s+1}], s \in \mathbf{Z}, x \in X. \end{cases} \tag{12}$$

Lemma 2.9. Suppose that g is a II-Carathéodory function and J a discrete II-Carathéodory function. Then

(i) $T_1 : Y \to Y$ is well defined;

(ii) the fixed point of the operator T_1 coincides with the solution of IVP(2);

(iii) $T_1 : Y \to Y$ is completely continuous.

Proof. By Lemma 2.6, Y is a Banach space. Use Lemma 2.7 and Lemma 2.8, it is similar to those of the proofs of (i), (ii) and (iii) in Lemma 2.4 and details are omitted. ∎

3 Main theorems

In this section, we are in the position to prove the existence and uniqueness results for solutions of IVP(1) and IVP(2) respectively. We firstly present the main assumptions which will be used in Theorem 3.1 and Theorem 3.2 for existence of solutions IVP(1).

(H1). Let $\sigma > \max\{-k_1, \alpha + k_1, \alpha\}$. Suppose that there exist non-decreasing functions $\Phi, \Psi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ such that

f is a I-Carathéodory function and there exists a piecewise continuous bounded function $r : (0, +\infty) \to \mathbb{R}$ such that

$$\left| f\left(t, (1 + t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})u, \tfrac{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})}{t^p} v\right) - r(t)\right| \le \Phi(|u|, |v|),$$
$$t \in (0, +\infty), u, v \in \mathbb{R}.$$

I is a discrete I-Carathéodory function and there exists a sequence $\{I_s\}$ and a sequence $\{\psi_s\}$ with $\sum\limits_{s=1}^{+\infty} \psi_s < +\infty$ such that

$$\left| I\left(t_s, (1 + t_s^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t_s^{\alpha})u, \tfrac{(1+t_s^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t_s^{\alpha})}{t_s^p} v\right) - I_s\right| \le \psi_s \Psi(|u|, |v|),$$
$$s \in \mathbf{Z}, u, v \in \mathbb{R}.$$

(H2). Let $\sigma > \max\{-k_1, \alpha + k_1, \alpha\}$. Suppose that $\delta_{1i}(i = 1, 2, \cdots, m)$ with $\delta_i = \delta_{1i} + \delta_{2i} > 0 (i = 1, 2, \ldots, m)$ and $\delta_1 \le \delta_2 \le \cdots \le \delta_m$. The following assumptions are satisfied:

f is a I-Carathéodory function and there exist nonnegative numbers $A_i(i = 1, 2, \ldots, m)$, piecewise continuous bounded function $r : (0, +\infty) \to R$ such that

$$\left| f\left(t, (1 + t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})u, \tfrac{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})}{t^p} v\right) - r(t)\right| \le \sum_{i=1}^{m} A_i |u|^{\delta_{1i}} |v|^{\delta_{2i}}$$

hold for all $t \in (0, +\infty), u, v \in \mathbb{R}$.

I is a discrete I-Carathéodory function and there exist nonnegative numbers $B_i(i = 1, 2, \ldots, m)$, a sequence $\{I_s\}$ and a sequence $\{\psi_s\}$ with $\sum\limits_{j=1}^{+\infty} \psi_j < +\infty$ such that

$$\left| I\left(t_s, (1 + t_s^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t_s^{\alpha})u, \tfrac{(1+t_s^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t_s^{\alpha})}{t_s^p} v\right) - I_s\right| \le \psi_s \sum_{i=1}^{m} B_i |u|^{\delta_{1i}} |v|^{\delta_{2i}}$$

hold for all $s \in \mathbf{Z}, u, v \in \mathbb{R}$.

Let

$$\begin{aligned} \phi(t) &= \int_0^t (t - s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t - s)^{\alpha})m(s)r(s)ds + x_0\mathbf{E}_{\alpha}(\lambda t^{\alpha}) \\ &+ \sum_{j=1}^{s} \mathbf{E}_{\alpha}(\lambda(t - t_j)^{\alpha})I_j, t \in (t_s, t_{s+1}], s \in \mathbf{Z}_0. \end{aligned} \tag{13}$$

Then

$$
\begin{aligned}
&{}^cD^p_{0^+}\phi(t) = \int_0^t (t-u)^{\alpha-p-1}\mathbf{E}_{\alpha,\alpha-p}(\lambda(t-u)^\alpha)m(u)r(u)du \\
&+\lambda x_0 t^{\alpha-p}\mathbf{E}_{\alpha,\alpha+1-p}(\lambda t^\alpha) + \lambda\sum_{i=1}^{s} I_i(t-t_i)^{\alpha-p}\mathbf{E}_{\alpha,\alpha+1-p}(\lambda(t-t_i)^\alpha), \\
&t\in(t_s,t_{s+1}], s\in\mathbf{Z}_0.
\end{aligned}
\tag{14}
$$

Denote

$$
M = \max\left\{\frac{(\sigma-\alpha-k_1)\mathbf{B}(\alpha,k_1+1)}{\sigma},\ \frac{(\sigma-\alpha-k_1)\mathbf{B}(\alpha-p,k_1+1)}{\sigma(\alpha-p)}\right\},
$$

$$
N = \max\left\{\left(\Gamma(\alpha)+\frac{\lambda}{\alpha}\frac{\sigma-\alpha}{\sigma}\left(\frac{\alpha}{\sigma-\alpha}\right)^{\alpha/\sigma}\right),\ \frac{\lambda(\sigma-p)}{\sigma(\alpha-p)}\left(\frac{p}{\sigma-p}\right)^{p/\sigma}\right\}.
$$

Theorem 3.1. Suppose that (H1) holds. Then IVP(1) has at least one solution if

$$
M\left(\frac{\alpha+k_1}{\sigma-\alpha-k_1}\right)^{(\alpha+k_1)/\sigma}\Phi\left(r+||\phi||,r+||\phi||\right)+N\sum_{j=1}^{+\infty}\psi_j\Psi\left(r+||\phi||,r+||\phi||\right)\le r
\tag{15}
$$

has a positive solution $r_0>0$.

Proof. Let X and T be defined in Section 2. From Lemma 2.3 and 2.4, we need to prove that T has at least one fixed point in X. By (7) and direct computation, we have

$$
\begin{aligned}
&{}^cD^p_{0^+}(Tx)(t) = \int_0^t (t-u)^{\alpha-p-1}\mathbf{E}_{\alpha,\alpha-p}(\lambda(t-u)^\alpha)m(u)f_x(u)du \\
&+\lambda x_0 t^{\alpha-p}\mathbf{E}_{\alpha,\alpha+1-p}(\lambda t^\alpha) + \lambda\sum_{i=1}^{s} I_x(t_i)(t-t_i)^{\alpha-p}\mathbf{E}_{\alpha,\alpha+1-p}(\lambda(t-t_i)^\alpha), \\
&t\in(t_s,t_{s+1}], s\in\mathbf{Z}_0.
\end{aligned}
\tag{16}
$$

It is easy to show that

$$
\sup_{t\in(0,+\infty)}\frac{t^a}{1+t^b} = \frac{b-a}{b}\left(\frac{a}{b-a}\right)^{a/b},\ b>a>0,\ \ \frac{\mathbf{E}_\alpha(\lambda(t-t_j)^\alpha)}{\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)}\le\Gamma(\alpha)+\frac{\lambda t^\alpha}{\alpha}.
\tag{17}
$$

One has for $t\in(t_s,t_{s+1}]$ from (8) and (12), (16) and (7), that

$$
\frac{|(Tx)(t)-\phi(t)|}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)}\le\frac{1}{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)}\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)|m(s)|f_x(s)-r(s)|ds
$$

$$+\frac{1}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})}\sum_{j=1}^{s}\mathbf{E}_{\alpha}(\lambda(t-t_j)^{\alpha})|I_x(t_j)-I_j|$$

$$\leq \frac{1}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})}\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})s^{k_1}\Phi\left(\frac{|x(s)|}{(1+s^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda s^{\alpha})},\ \frac{s^p|{}^cD_{0+}^p x(s)|}{(1+s^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda s^{\alpha})}\right)ds$$

$$+\frac{1}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})}\sum_{j=1}^{s}\mathbf{E}_{\alpha}(\lambda(t-t_j)^{\alpha})\Psi\left(\frac{|x(t_j)|}{(1+t_j^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t_j^{\alpha})},\ \frac{t_j^p|{}^cD_{0+}^p x(t_j)|}{(1+t_j^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t_j^{\alpha})}\right)$$

$$\leq \frac{1}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})}\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})s^{k_1}ds\Phi\left(||x||,||x||\right)$$

$$+\frac{1}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})}\sum_{j=1}^{s}\mathbf{E}_{\alpha}(\lambda(t-t_j)^{\alpha})\psi_j\Psi\left(||x||,||x||\right)$$

$$\leq \frac{1}{1+t^{\sigma}}\int_0^t (t-s)^{\alpha-1}s^{k_1}ds\Phi\left(||x||,||x||\right)+\frac{\Gamma(\alpha)+\frac{\lambda t^{\alpha}}{\alpha}}{1+t^{\sigma}}\Psi\left(||x||,||x||\right)\sum_{j=1}^{+\infty}\psi_j$$

$$\leq \frac{t^{\alpha+k_1}}{1+t^{\sigma}}\mathbf{B}(\alpha,k_1+1)\Phi\left(||x||,||x||\right)+\left(\Gamma(\alpha)+\frac{\lambda t^{\alpha}}{\alpha(1+t^{\sigma})}\right)\Psi\left(||x||,||x||\right)\sum_{j=1}^{+\infty}\psi_j$$

$$\leq \frac{\sigma-\alpha-k_1}{\sigma}\left(\frac{\alpha+k_1}{\sigma-\alpha-k_1}\right)^{(\alpha+k_1)/\sigma}\mathbf{B}(\alpha,k_1+1)\Phi\left(||x||,||x||\right)$$

$$+\left(\Gamma(\alpha)+\frac{\lambda}{\alpha}\frac{\sigma-\alpha}{\sigma}\left(\frac{\alpha}{\sigma-\alpha}\right)^{\alpha/\sigma}\right)\Psi\left(||x||,||x||\right)\sum_{j=1}^{+\infty}\psi_j.$$

Furthermore, we have from (13) and (15), (16) and (7), that

$$\frac{t^p|{}^cD_{0+}^p(Tx)(t)-{}^cD_{0+}^p\phi(t)|}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})}$$

$$\leq \frac{t^p}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})}\int_0^t (t-s)^{\alpha-p-1}\mathbf{E}_{\alpha,\alpha-p}(\lambda(t-s)^{\alpha})|m(s)||f_x(s)-r(s)|ds$$

$$+\frac{t^p}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})}\lambda\sum_{j=1}^{s}\mathbf{E}_{\alpha,\alpha+1-p}(\lambda(t-t_j)^{\alpha})|I_x(t_j)-I_j|$$

$$\leq \frac{t^p}{(\alpha-p)(1+t^{\sigma})}\int_0^t (t-s)^{\alpha-p-1}s^{k_1}ds\Phi\left(||x||,||x||\right)+\frac{t^p}{(\alpha-p)(1+t^{\sigma})}\lambda\Psi\left(||x||,||x||\right)\sum_{j=1}^{s}\psi_j$$

$$\leq \frac{t^{\alpha+k_1}}{(\alpha-p)(1+t^\sigma)}\mathbf{B}(\alpha-p,k_1+1)\Phi\left(||x||,||x||\right)+\frac{t^p}{(\alpha-p)(1+t^\sigma)}\lambda\Psi\left(||x||,||x||\right)\sum_{j=1}^{+\infty}\psi_j$$

$$\leq \frac{\sigma-\alpha-k_1}{\sigma(\alpha-p)}\left(\frac{\alpha+k_1}{\sigma-\alpha-k_1}\right)^{(\alpha+k_1)/\sigma}\mathbf{B}(\alpha-p,k_1+1)\Phi\left(||x||,||x||\right)$$

$$+\frac{\lambda(\sigma-p)}{\sigma(\alpha-p)}\left(\frac{p}{\sigma-p}\right)^{p/\sigma}\Psi\left(||x||,||x||\right)\sum_{j=1}^{+\infty}\psi_j.$$

Hence

$$||Tx-\phi||\leq M\left(\frac{\alpha+k_1}{\sigma-\alpha-k_1}\right)^{(\alpha+k_1)/\sigma}\Phi\left(||x||,||x||\right)+N\sum_{j=1}^{+\infty}\psi_j\Psi\left(||x||,||x||\right). \quad (18)$$

It is easy to show that $\phi\in X$. Let $r>0$ and define $\overline{\Omega}_r=\{x\in X:\|x-\phi\|\leq r\}$. For $x\in\overline{\Omega}_r$, we have $\|x-\phi\|\leq r$. Then $||x||\leq ||x-\phi||+||\phi||\leq r+||\phi||$. Let $r_0>0$ be a solution of (14). From above discussion, we have

$$||Tx-\phi||\leq M\left(\frac{\alpha+k_1}{\sigma-\alpha-k_1}\right)^{(\alpha+k_1)/\sigma}\Phi\left(r_0+||\phi||,r_0+||\phi||\right)$$

$$+N\sum_{j=1}^{+\infty}\psi_j\Psi\left(r_0+||\phi||,r_0+||\phi||\right)\leq r_0.$$

It is easy to see that $T\overline{\Omega}_{r_0}\subset\overline{\Omega}_{r_0}$. Then, Schauder fixed point theorem implies that T has a fixed point $x\in\overline{\Omega}_{r_0}$, which is a solution of IVP(1). The proof is completed. ∎

Theorem 3.2. Suppose that (H2) holds. Then IVP(1) has at least one solution $x\in X$ if

$$\delta_m<1 \text{ or } \delta_m=1 \text{ with } N_0<1 \text{ or } \delta_m>1 \text{ with } \frac{\|\phi\|^{1-\delta_m}(\delta_m-1)^{\delta_m-1}}{\delta_m^{\delta_m}}\geq N_0, \quad (19)$$

where

$$N_0=\sum_{i=1}^{m}\left[\max\left\{\frac{(\sigma-\alpha-k_1)\mathbf{B}(\alpha,k_1+1)}{\sigma},\ \frac{(\sigma-\alpha-k_1)\mathbf{B}(\alpha-p,k_1+1)}{\sigma(\alpha-p)}\right\}\left(\frac{\alpha+k_1}{\sigma-\alpha-k_1}\right)^{(\alpha+k_1)/\sigma}A_i\right.$$

$$\left.+\max\left\{\left(\Gamma(\alpha)+\frac{\lambda}{\alpha}\frac{\sigma-\alpha}{\sigma}\left(\frac{\alpha}{\sigma-\alpha}\right)^{\alpha/\sigma}\right),\ \frac{\lambda(\sigma-p)}{\sigma(\alpha-p)}\left(\frac{p}{\sigma-p}\right)^{p/\sigma}\right\}B_i\sum_{j=1}^{+\infty}\psi_j\right]||\phi||^{\delta_i-\delta_m}.$$

Proof. Let X and T be defined in Section 2. By Lemma 2.3 and Lemma 2.4 we seek solutions of IVP(1) by getting the fixed point of T in X. Let ϕ be defined by (11). We can get (12). Let $r > 0$ and define $\overline{\Omega}_r = \{x \in X : \|x - \phi\| \le r\}$.

For $x \in \overline{\Omega}_r$, we have $\|x - \phi\| \le r$. Then $||x|| \le ||x - \phi|| + ||\phi|| \le r + ||\phi||$. Using (H2), we find similarly to the proof of Theorem 3.1 that

$$\frac{|(Tx)(t)-\phi(t)|}{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(\lambda t^{\alpha})} \le \sum_{i=1}^{m} \Bigg[\frac{(\sigma-\alpha-k_1)\mathbf{B}(\alpha,k_1+1)}{\sigma} \left(\frac{\alpha+k_1}{\sigma-\alpha-k_1}\right)^{(\alpha+k_1)/\sigma} A_i$$

$$+\left(\Gamma(\alpha) + \frac{\lambda}{\alpha}\frac{\sigma-\alpha}{\sigma}\left(\frac{\alpha}{\sigma-\alpha}\right)^{\alpha/\sigma}\right) B_i \sum_{j=1}^{+\infty}\psi_j \Bigg] ||x||^{\delta_{1i}+\delta_{2i}}.$$

Furthermore, we have

$$\frac{t^p}{(1+t^{\sigma}\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})}|{}^cD_{0^+}^p(Tx)(t) - {}^cD_{0^+}^p\phi(t)|$$

$$\le \sum_{i=1}^{m} \Bigg[\frac{(\sigma-\alpha-k_1)\mathbf{B}(\alpha-p,k_1+1)}{\sigma(\alpha-p)} \left(\frac{\alpha+k_1}{\sigma-\alpha-k_1}\right)^{(\alpha+k_1)/\sigma} A_i$$

$$+\frac{\lambda(\sigma-p)}{\sigma(\alpha-p)}\left(\frac{p}{\sigma-p}\right)^{p/\sigma} B_i \sum_{j=1}^{+\infty}\psi_j . \Bigg] ||x||^{\delta_{1i}+\delta_{2i}}.$$

It follows that

$$||Tx-\phi|| \le$$
$$\sum_{i=1}^{m}\Bigg[\max\left\{\frac{(\sigma-\alpha-k_1)\mathbf{B}(\alpha,k_1+1)}{\sigma}, \ \frac{(\sigma-\alpha-k_1)\mathbf{B}(\alpha-p,k_1+1)}{\sigma(\alpha-p)}\right\}\left(\frac{\alpha+k_1}{\sigma-\alpha-k_1}\right)^{(\alpha+k_1)/\sigma} A_i$$

$$+\max\left\{\left(\Gamma(\alpha)+\frac{\lambda}{\alpha}\frac{\sigma-\alpha}{\sigma}\left(\frac{\alpha}{\sigma-\alpha}\right)^{\alpha/\sigma}\right), \ \frac{\lambda(\sigma-p)}{\sigma(\alpha-p)}\left(\frac{p}{\sigma-p}\right)^{p/\sigma}\right\} B_i \sum_{j=1}^{+\infty}\psi_j\Bigg] \times$$

$$[r+||\phi||]^{\delta_i}.$$

Then

$$||Tx-\phi|| \le [r+||\phi||]^{\delta_m}\times$$
$$\sum_{i=1}^{m}\Bigg[\max\left\{\frac{(\sigma-\alpha-k_1)\mathbf{B}(\alpha,k_1+1)}{\sigma},\ \frac{(\sigma-\alpha-k_1)\mathbf{B}(\alpha-p,k_1+1)}{\sigma(\alpha-p)}\right\}\left(\frac{\alpha+k_1}{\sigma-\alpha-k_1}\right)^{(\alpha+k_1)/\sigma}A_i$$

$$+\max\left\{\left(\Gamma(\alpha)+\frac{\lambda}{\alpha}\frac{\sigma-\alpha}{\sigma}\left(\frac{\alpha}{\sigma-\alpha}\right)^{\alpha/\sigma}\right),\ \frac{\lambda(\sigma-p)}{\sigma(\alpha-p)}\left(\frac{p}{\sigma-p}\right)^{p/\sigma}\right\}B_i\sum_{j=1}^{+\infty}\psi_j\Bigg]\,||\phi||^{\delta_i-\delta_m}$$

$$=N_0[r+||\phi||]^{\delta_m}.$$

(i) If $\delta_m < 1$, we can choose $r_0 > 0$ sufficiently large such that $[r_0 + ||\phi||]^{\delta_m}N_0 < r_0$. Let $\Omega_{r_0} = \{x \in Y : ||x-\phi|| < r_0\}$. It is easy to see that $T\overline{\Omega}_{r_0} \subset \overline{\Omega}_{r_0}$. Then, Schauder fixed point theorem implies that T has a fixed point $x \in \overline{\Omega}_{r_0}$, which is a bounded solution of IVP (1).

(ii) If $\delta_m = 1$, we choose

$$r_0 \ge \frac{||\phi||N_0}{1-N_0}.$$

Let $\Omega_{r_0} = \{x \in Y : ||x-\phi|| < r_0\}$. It is easy to see that $T\overline{\Omega}_{r_0} \subset \overline{\Omega}_{r_0}$. Then, Schauder fixed point theorem implies that T has a fixed point $x \in \overline{\Omega}_{r_0}$, which is a bounded solution of IVP (1).

(iii) If $\delta_m > 1$, we choose $r = r_0 = \frac{\|\phi\|}{\delta_m-1}$. By assumption,

$$\frac{r_0}{(r_0+\|\phi\|)^{\delta_m}} = \frac{\|\phi\|^{1-\delta_m}(\delta_m-1)^{\delta_m-1}}{\delta_m^{\delta_m}} \ge N_0.$$

Let $\Omega_{r_0} = \{x \in Y : ||x-\phi|| < r_0\}$. It is easy to see that $T\overline{\Omega}_{r_0} \subset \overline{\Omega}_{r_0}$. Then, Schauder fixed point theorem implies that F has a fixed point $x \in \overline{\Omega}_{r_0}$, which is a solution of IVP(1). The proof is completed. ∎

Now, we establish existence results for IVP(2) under some suitable assumptions. Let $\delta_0(t)$ and $\delta_1(t)$ be defined by (9).

(H3). Suppose that there exist non-decreasing functions $\Phi, \Psi : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ such that

g is a II-Carathéodory function and there exists a piecewise continuous bounded function $r : (0,+\infty) \to R$ such that $|g\,(t, \delta_0(t)u, \delta_1(t)v) - r(t)| \le \Phi(|u|, |v|), t \in (0,+\infty), u, v \in \mathbb{R}$.

J is a discrete II-Carathéodory function and there exists sequence $\{J_s\}$ and $\{\psi_s\}$ with $\sum\limits_{s=1}^{+\infty} J_s < +\infty$ and $\sum\limits_{s=1}^{+\infty} \psi_s < +\infty$ such that $|I\left(t_s, \delta_0(t)u, \delta_1(t)v\right) - J_s| \leq \psi_s \Psi(|u|, |v|), s \in \mathbf{Z}, u, v \in \mathbb{R}$.

(H4). Suppose that $\delta_{1i}(i = 1, 2, \cdots, m)$ with $\delta_i = \delta_{1i} + \delta_{2i} > 0(i = 1, 2, \ldots, m)$ and $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_m$. The following assumptions are satisfied:

g is a II-Carathéodory function and there exist nonnegative numbers $A_i(i = 1, 2, \ldots, m)$, a piecewise continuous bounded function $r : (0, +\infty) \to R$ such that $|g\left(t, (\delta_0(t)u, \delta_1(t)v\right) - r(t)| \leq \sum\limits_{i=1}^{m} A_i |u|^{\delta_{1i}} |v|^{\delta_{2i}}$ hold for all $t \in (0, +\infty), u, v \in \mathbb{R}$.

J is a discrete II-Carathéodory function and there exist nonnegative numbers $B_i(i = 1, 2, \ldots, m)$, sequences $\{J_s\}$ and $\{\psi_s\}$ with $\sum\limits_{s=1}^{+\infty} J_s, \sum\limits_{s=1}^{+\infty} \psi_s < +\infty$ such that $|J\left(t_s, \delta_0(t_s)u, \delta_1(t_s)v\right) - J_s| \leq \psi_s \sum\limits_{i=1}^{m} B_i |u|^{\delta_{1i}} |v|^{\delta_{1i}}$ hold for all $s \in \mathbf{Z}, u, v \in \mathbb{R}$.

Denote

$$\phi(t) = \begin{cases} \int_0^t (t-u)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(t-u)^\alpha) n(u) r(u) du + x_0 \mathbf{E}_\alpha(\lambda t^\alpha), \ t \in (t_0, t_1], \\ \int_{t_s}^t (t-u)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(t-u)^\alpha) n(u) r(u) du + \left[x_0 \prod\limits_{v=1}^{s} \mathbf{E}_\alpha(\lambda(t_v - t_{v-1})^\alpha) \right. \\ + \sum\limits_{j=1}^{s} \left(\prod\limits_{v=j+1}^{s} \mathbf{E}_\alpha(\lambda(t_v - t_{v-1})^\alpha) \right) J_j + \sum\limits_{j=1}^{s} \left(\prod\limits_{v=j+1}^{s} \mathbf{E}_\alpha(\lambda(t_v - t_{v-1})^\alpha) \right) \times \\ \left. \int_{t_{j-1}}^{t_j} (t_j - u)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(t_j - u)^\alpha) n(u) r(u) du \right] \mathbf{E}_\alpha(\lambda(t - t_s)^\alpha), \\ t \in (t_s, t_{s+1}], s \in \mathbf{Z}, x \in X. \end{cases} \tag{20}$$

Then

$$
{}^{c}D^{p}_{t_s+}\phi(t) = \begin{cases} \int_0^t (t-u)^{\alpha-p-1}\mathbf{E}_{\alpha,\alpha-p}(\lambda(t-u)^{\alpha})n(u)r(u)du \\ +\lambda x_0 t^{\alpha-p}\mathbf{E}_{\alpha,\alpha+1-p}(\lambda t^{\alpha}),\ t\in(t_0,t_1], \\ \int_{t_s}^t (t-u)^{\alpha-p-1}\mathbf{E}_{\alpha,\alpha-p}(\lambda(t-u)^{\alpha})n(u)r(u)du \\ +\lambda(t-t_s)^{\alpha-p}\left[x_0\prod_{v=1}^{s}\mathbf{E}_{\alpha}(\lambda(t_v-t_{v-1})^{\alpha})\right. \\ +\sum_{j=1}^{s}\left(\prod_{v=j+1}^{s}\mathbf{E}_{\alpha}(\lambda(t_v-t_{v-1})^{\alpha})\right)J_j \\ +\sum_{j=1}^{s}\left(\prod_{v=j+1}^{s}\mathbf{E}_{\alpha}(\lambda(t_v-t_{v-1})^{\alpha})\right)\times \\ \left.\int_{t_{j-1}}^{t_j}(t_j-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-u)^{\alpha})n(u)r(u)du\right]\mathbf{E}_{\alpha,\alpha+1-p}(\lambda(t-t_s)^{\alpha}), \\ t\in(t_s,t_{s+1}], s\in\mathbf{Z}, x\in X. \end{cases} \tag{21}
$$

Theorem 3.3 Suppose that (H3) holds. Then IVP(2) has at least one solution if

$$
A\Phi\left(r+||\phi||, r+||\phi||\right)+B\Psi\left(r+||\phi||, r+||\phi||\right)\le r \tag{22}
$$

has a positive solution $r_0>0$, where

$$
A=\max\left\{\sup_{s\in\mathbf{Z}_0}\frac{\sum_{j=1}^{s+1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^{\alpha})(t_j-t_{j-1})^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)}{\sum_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})},\right.
$$

$$
\left.\sup_{s\in\mathbf{Z}_0}\frac{\sum_{j=1}^{s+1}[\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_j-t_{j-1})^{\alpha})+\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^{\alpha})](t_j-t_{j-1})^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)}{\sum_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}[\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})]}\right\},
$$

$$B = \max\left\{ \sup_{s\in \mathbf{Z}_0} \frac{\sum\limits_{j=1}^{+\infty}\psi_j}{\sum\limits_{j=0}^{s} t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})}, \right.$$

$$\left. \sup_{s\in \mathbf{Z}_0} \frac{\sum\limits_{j=1}^{+\infty}\psi_j}{\sum\limits_{j=0}^{s} t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}[\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})]} \right\}.$$

Proof. Let Y and T_1 be defined in Section 2. From Lemma 2.8 and 2.9, we need to prove that T_1 has at least one fixed point in Y.One gets

$$\begin{array}{l} {}^cD^p_{t_s^+}(T_1x)(t) = \\ \begin{cases} \int_0^t (t-s)^{\alpha-p-1}\mathbf{E}_{\alpha,\alpha-p}(\lambda(t-s)^{\alpha})n(s)g(s,x(s),{}^cD^p_{t_s^+}x(s))ds \\ +\lambda x_0 t^{\alpha-p}\mathbf{E}_{\alpha,\alpha+1-p}(\lambda t^{\alpha}),\ t\in(t_0,t_1], \\ \int_{t_s}^t (t-s)^{\alpha-p-1}\mathbf{E}_{\alpha,\alpha-p}(\lambda(t-s)^{\alpha})n(s)g(s,x(s),{}^cD^p_{t_s^+}x(s))ds \\ +\lambda(t-t_s)^{\alpha-p}\left[x_0\prod\limits_{v=1}^{s}\mathbf{E}_{\alpha}(\lambda(t_v-t_{v-1})^{\alpha})\right. \\ +\sum\limits_{j=1}^{s}\left(\prod\limits_{v=j+1}^{s}\mathbf{E}_{\alpha}(\lambda(t_v-t_{v-1})^{\alpha})\right)J(t_j,x(t_j),{}^cD^p_{t_{j-1}^+}x(t_j)) \\ +\sum\limits_{j=1}^{s}\left(\prod\limits_{v=j+1}^{s}\mathbf{E}_{\alpha}(\lambda(t_v-t_{v-1})^{\alpha})\right)\times \\ \left.\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-s)^{\alpha})n(s)g(s,x(s),{}^cD^p_{t_{j-1}^+}x(s))ds\right]\times \\ \mathbf{E}_{\alpha,\alpha+1-p}(\lambda(t-t_s)^{\alpha}),t\in(t_s,t_{s+1}],s\in\mathbf{Z},x\in X. \end{cases} \end{array} \tag{23}$$

By (H3) and direct computation, from (9), (12) and (20), we have for $t\in(t_s,t_{s+1}]$ that

$$\frac{|(Tx)(t)-\phi(t)|}{\delta_0(t)} \le \frac{\int_{t_s}^t (t-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-u)^{\alpha})|n(u)||g(u,x(u),{}^cD^p_{t_{s-1}^+}x(u))-r(u)|du}{\sum\limits_{j=0}^{s} t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})\prod\limits_{v=1}^{s+1}\mathbf{E}_{\alpha}(\lambda(t_v-t_{v-1})^{\alpha})}$$

$$+\frac{\mathbf{E}_\alpha(\lambda(t-t_s)^\alpha)}{\sum\limits_{j=0}^{s} t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^\alpha)\prod\limits_{v=1}^{s+1}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha)}\times$$

$$\left[\sum_{j=1}^{s}\left(\prod_{v=j+1}^{s}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha)\right)|J(t_j,x(t_j),{}^cD^p_{t^+_{j-1}}x(t_j))-J_j|\right.$$

$$+\sum_{j=1}^{s}\left(\prod_{v=j+1}^{s}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha)\right)\times$$

$$\left.\int_{t_{j-1}}^{t_j}(t_j-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-u)^\alpha)|n(u)||g(u,x(u),{}^cD^p_{t^+_{j-1}}x(u))-r(u)|du\right]$$

$$\leq\frac{\mathbf{E}_{\alpha,\alpha}(\lambda(t_{s+1}-t_s)^\alpha)\int_{t_s}^{t}(t-u)^{\alpha-1}(u-t_s)^{k_2}(t_{s+1}-u)^{l_2}\Phi(||x||,||x||)du}{\sum\limits_{j=0}^{s} t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^\alpha)\prod\limits_{v=1}^{s+1}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha)}$$

$$+\frac{\mathbf{E}_\alpha(\lambda(t_{s+1}-t_s)^\alpha)}{\sum\limits_{j=0}^{s} t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^\alpha)\prod\limits_{v=1}^{s+1}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha)}\times$$

$$\left[\sum_{j=1}^{s}\left(\prod_{v=j+1}^{s}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha)\right)\psi_j\Psi(||x||,||x||)\right.$$

$$+\sum_{j=1}^{s}\left(\prod_{v=j+1}^{s}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha)\right)\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^\alpha)\times$$

$$\left.\int_{t_{j-1}}^{t_j}(t_j-u)^{\alpha-1}(u-t_{j-1})^{k_2}(t_j-u)^{l_2}\Phi(||x||,||x||)du\right]$$

$$\leq\frac{\mathbf{E}_{\alpha,\alpha}(\lambda(t_{s+1}-t_s)^\alpha)\int_{t_s}^{t}(t-u)^{\alpha+l_2-1}(u-t_s)^{k_2}du\Phi(||x||,||x||)}{\sum\limits_{j=0}^{s} t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^\alpha)}$$

$$+\frac{\sum\limits_{j=1}^{+\infty}\psi_j\Psi(||x||,||x||)+\sum\limits_{j=1}^{s}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^\alpha)\int_{t_{j-1}}^{t_j}(t_j-u)^{\alpha+l_2-1}(u-t_{j-1})^{k_2}du\Phi(||x||,||x||)}{\sum\limits_{j=0}^{s} t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^\alpha)}$$

$$\leq\frac{\mathbf{E}_{\alpha,\alpha}(\lambda(t_{s+1}-t_s)^\alpha)(t-t_s)^{\alpha+k_2+l_2}\int_0^1(1-w)^{\alpha+l_2-1}w^{k_2}dw\Phi(||x||,||x||)}{\sum\limits_{j=0}^{s} t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^\alpha)}$$

$$+\frac{\sum\limits_{j=1}^{+\infty}\psi_j\Psi(||x||,||x||)+\sum\limits_{j=1}^{s}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^{\alpha})(t_j-t_{j-1})^{\alpha+k_2+l_2}\int_0^1(1-w)^{\alpha+l_2-1}w^{k_2}dw\Phi(||x||,||x||)}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})}$$

$$\leq\frac{\sum\limits_{j=1}^{+\infty}\psi_j\Psi(||x||,||x||)+\sum\limits_{j=1}^{s+1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^{\alpha})(t_j-t_{j-1})^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)\Phi(||x||,||x||)}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})}$$

$$\leq\sup_{s\in\mathbf{Z}_0}\frac{\sum\limits_{j=1}^{+\infty}\psi_j}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})}\Psi(||x||,||x||)$$

$$+\sup_{s\in\mathbf{Z}_0}\frac{\sum\limits_{j=1}^{s+1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^{\alpha})(t_j-t_{j-1})^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})}\Phi(||x||,||x||).$$

Furthermore, from (9), (21) and (23), we have

$$\frac{|{}^cD^p_{*+}(Tx)(t)-{}^cD^p_{*+}\phi(t)|}{\delta_1(t)}\leq\frac{\int_{t_s}^t(t-u)^{\alpha-p-1}\mathbf{E}_{\alpha,\alpha-p}(\lambda(t-u)^{\alpha})|n(u)||g(u,x(u),{}^cD^p_{t_s^+}x(u))-r(u)|du}{\delta_1(t)}$$

$$+\frac{\lambda(t-t_s)^{\alpha-p}}{\delta_1(t)}\left[\sum_{j=1}^{s}\left(\prod_{v=j+1}^{s}\mathbf{E}_{\alpha}(\lambda(t_v-t_{v-1})^{\alpha})\right)|J(t_j,x(t_j),{}^cD^p_{t_{j-1}^+}x(t_j))-J_j|\right.$$

$$+\sum_{j=1}^{s}\left(\prod_{v=j+1}^{s}\mathbf{E}_{\alpha}(\lambda(t_v-t_{v-1})^{\alpha})\right)\times$$

$$\left.\int_{t_{j-1}}^{t_j}(t_j-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-u)^{\alpha})|n(u)||g(u,x(u),{}^cD^p_{t_{j-1}^+}x(u))-r(u)|du\right]\times$$

$$\mathbf{E}_{\alpha,\alpha+1-p}(\lambda(t-t_s)^{\alpha})$$

$$\leq\frac{\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{s+1}-t_s)^{\alpha})\int_{t_s}^t(t-u)^{\alpha-p-1}u-t_s)^{k_2}(t_{s+1}-u)^{l_2}du\Phi(||x||,||x||)}{[\lambda(t_{s+1}-t_s)^{\alpha-p}+1]\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}[\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})]}$$

$$+\frac{\lambda(t-t_s)^{\alpha-p}}{[\lambda(t_{s+1}-t_s)^{\alpha-p}+1]\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}[\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})]}\times$$

$$\left[\sum_{j=1}^{+\infty}\psi_j\Psi(||x||,||x||)\right.$$

$$\left.+\sum_{j=1}^{s}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^{\alpha})\int_{t_{j-1}}^{t_j}(t_j-u)^{\alpha-1}(u-t_{j-1})^{k_2}(t_j-u)^{l_2}du\Phi(||x||,||x||)\right]$$

$$\leq\frac{\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{s+1}-t_s)^{\alpha})(t_{s+1}-t_s)^{\alpha-p+k_2+l_2}\mathbf{B}(\alpha-p+l_2,k_2+1)}{[\lambda(t_{s+1}-t_s)^{\alpha-p}+1]\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}[\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})]}\Phi(||x||,||x||)$$

$$+\frac{\lambda(t_{s+1}-t_s)^{\alpha-p}\sum\limits_{j=1}^{s}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^{\alpha})(t_j-t_{j-1})^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)}{[\lambda(t_{s+1}-t_s)^{\alpha-p}+1]\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}[\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})]}\Phi(||x||,||x||)$$

$$+\frac{\lambda(t_{s+1}-t_s)^{\alpha-p}\sum\limits_{j=1}^{+\infty}\psi_j}{[\lambda(t_{s+1}-t_s)^{\alpha-p}+1]\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}[\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})]}\Psi(||x||,||x||)$$

$$\leq\frac{\sum\limits_{j=1}^{s+1}[\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_j-t_{j-1})^{\alpha})+\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^{\alpha})](t_j-t_{j-1})^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}[\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})]}\Phi(||x||,||x||)$$

$$+\frac{\sum\limits_{j=1}^{+\infty}\psi_j}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}[\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})]}\Psi(||x||,||x||)$$

$$\leq\sup_{s\in\mathbf{Z}_0}\frac{\sum\limits_{j=1}^{s+1}[\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_j-t_{j-1})^{\alpha})+\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^{\alpha})](t_j-t_{j-1})^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}[\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})]}\Phi(||x||,||x||)$$

$$+\sup_{s\in\mathbf{Z}_0}\frac{\sum\limits_{j=1}^{+\infty}\psi_j}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}[\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})]}\Psi(||x||,||x||).$$

Hence

$$||T_1x-\phi||\leq A\Phi(||x||,||x||)+B\Psi(||x||,||x||). \tag{24}$$

It is easy to show that $\phi\in X$. Let $r>0$ and define $\overline{\Omega}_r=\{x\in X:\|x-\phi\|\leq r\}$. For $x\in\overline{\Omega}_r$, we have $\|x-\phi\|\leq r$. Then $||x||\leq$

$||x-\phi||+||\phi|| \le r+||\phi||$. Let $r_0>0$ be a solution of (14). From above discussion, we have

$$||Tx-\phi|| \le A\Phi\left(r_0+||\phi||, r_0+||\phi||\right)+B\Psi\left(r_0+||\phi||, r_0+||\phi||\right). \le r_0.$$

It is easy to see that $T\overline{\Omega}_{r_0} \subset \overline{\Omega}_{r_0}$. Then, Schauder fixed point theorem implies that T has a fixed point $x \in \overline{\Omega}_{r_0}$, which is a solution of IVP(2). The proof is completed. ∎

Theorem 3.4 Suppose that (H4) holds. Let $N_0=\sum\limits_{i=1}^{m}\left[AA_i+BB_i\right]||\phi||^{\delta_i-\delta_m}$ and A, B are defined in Theorem 3.3. Then IVP(2) has at least one solution $x \in Y$ if

$$\delta_m<1 \text{ or } \delta_m=1 \text{ with } N_0<1 \text{ or } \delta_m>1 \text{ with } \frac{\|\phi\|^{1-\delta_m}(\delta_m-1)^{\delta_m-1}}{\delta_m^{\delta_m}} \ge N_0, \quad (25)$$

Proof. Let Y and T_1 be defined in Section 2. By Lemma 2.8 and Lemma 2.9 we seek solutions of IVP(2) by getting the fixed point of T_1 in Y. Let ϕ be defined by (20). We can get (21). Let $r>0$ and define $\overline{\Omega}_r=\{x \in X : \|x-\phi\| \le r\}$.

For $x \in \overline{\Omega}_r$, we have $\|x-\phi\| \le r$. Then $||x|| \le ||x-\phi||+||\phi|| \le r+||\phi||$. Using (H4), use (9), (20) and (12), we find similarly to the proof of Theorem 3.3 that

$$\frac{|(Tx)(t)-\phi(t)|}{\delta_0(t)} \le \sum_{i=1}^{m}\left[\sup_{s\in Z_0}\frac{\sum\limits_{j=1}^{s+1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^{\alpha})(t_j-t_{j-1})^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})}A_i\right.$$

$$\left.+\sup_{s\in\mathbf{Z}_0}\frac{\sum\limits_{j=1}^{+\infty}\psi_j}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^{\alpha})}B_i\right]||x||^{\delta_{1i}+\delta_{2i}}.$$

Furthermore, use (9), (21) and (23), similarly to the proof of Theorem 3.3, we have

$$\frac{|{}^cD^p_{t_s^+}(Tx)(t)-{}^cD^p_{t_s^+}\phi(t)|}{\delta_1(t)}$$

$$\le \sum_{i=1}^{m}\left[\sup_{s\in\mathbf{Z}_0}\frac{\sum\limits_{j=1}^{s+1}[\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_j-t_{j-1})^{\alpha})\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^{\alpha})](t_j-t_{j-1})^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}[\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})]}A_i\right.$$

$$\left.+\sup_{s\in\mathbf{Z}_0}\frac{\sum\limits_{j=1}^{+\infty}\psi_j}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}[\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^{\alpha})]}B_i\right]||x||^{\delta_{1i}+\delta_{2i}}.$$

It follows that

$$||Tx-\phi|| \leq \sum_{i=1}^{m} [AA_i + BB_i] [r + ||\phi||]^{\delta_i}.$$

$$\leq [r+||\phi||]^{\delta_m} \sum_{i=1}^{m} [AA_i + BB_i] ||\phi||^{\delta_i-\delta_m} = N_0[r+||\phi||]^{\delta_m}.$$

The remainder of the proof is similar to that of Theorem 3.2 and is omitted. The proof is complete. ∎

4 Examples

In this section, we present examples to illustrate Theorem 3.2 and Theorem 3.4.

Example 4.1. Consider the following IVP

$$\begin{cases} {}^cD_{0^+}^{\frac{2}{3}}x(t) - x(t) = \\ t^{-\frac{1}{6}} \left[B + A\left(\frac{|x(t)|}{(1+t^2)\mathbf{E}_{2/3,2/3}(t^{2/3})}\right)^{\sigma} \left(\frac{t^{1/6}|{}^cD_{0^+}^{\frac{1}{6}}x(t)|}{(1+t^2)\mathbf{E}_{2/3,5/6}(t^{2/3})}\right)^{\tau}\right], a.e., t \in (0,+\infty), \\ x(0) = x_0, \\ \Delta x(s) = 2^{-s}\left[b + a\left(\frac{|x(s)|}{(1+s^2)\mathbf{E}_{2/3,2/3}(s^{2/3})}\right)^{\sigma} \left(\frac{s^{1/6}|{}^cD_{0^+}^{\frac{1}{6}}x(s)|}{(1+s^2)\mathbf{E}_{2/3,5/6}(s^{2/3})}\right)^{\tau}\right], s \in \mathbf{Z}, \end{cases} \tag{26}$$

where $A, B, a, b \in \mathbb{R}$ are constants.

Corresponding to IVP(1), we have $\alpha = \frac{2}{3}$, $p = \frac{1}{6}$, $\lambda = 1$, $k_1 = -\frac{1}{6}$, $m(t) = t^{-\frac{1}{6}}$, $t_s = s, s \in Z$, and

$$f(t,u,v) = B + A\left(\frac{u}{(1+t^2)\mathbf{E}_{2/3,2/3}(t^{2/3})}\right)^{\sigma} \left(\frac{t^{1/6}|v|}{(1+t^2)\mathbf{E}_{2/3,5/6}(t^{2/3})}\right)^{\tau},$$

$$I(s,u,v) = 2^{-s}\left[b + a\left(\frac{u}{(1+s^2)\mathbf{E}_{2/3,2/3}(s^{2/3})}\right)^{\sigma} \left(\frac{s^{1/6}|v}{(1+s^2)\mathbf{E}_{2/3,5/6}(s^{2/3})}\right)^{\tau}\right].$$

Hence (i), (ii) and (iii) in Section 1 are satisfied.

Choose $\sigma = 2$. Then $\sigma > \max\{-k_1, \alpha + k_1, \alpha\}$ and

$$f\left(t, (1+t^2)\mathbf{E}_{2/3,2/3}(t^{2/3})u, \frac{(1+t^2)\mathbf{E}_{2/3,5/6}(t^{2/3})}{t^{1/6}}v\right) = B + A\,|u|^{\sigma}\,|v|^{\tau},$$

$$I\left(s, (1+s^2)\mathbf{E}_{2/3,2/3}(s^{2/3})u, \frac{(1+s^2)\mathbf{E}_{2/3,5/6}(s^{2/3})}{s^{1/6}}v\right) = 2^{-s}\left[b + a\,|u|^{\sigma}\,|v|^{\tau}\right].$$

Then (v) in Section 1 holds.

Choose $m = 1$, $\delta_{11} = \delta$, $\delta_{21} = \tau$, $\delta_1 = \sigma + \tau$, $r(t) = A$ and $I_s = 2^{-s}b$, $\psi_s = 2^{-s}$, $A_1 = |A|$ and $B_1 = |b|$. Then

$$\left|f\left(t, (1+t^{\sigma})\mathbf{E}_{\alpha,\alpha}(t^{\alpha})u, \frac{(1+t^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(\lambda t^{\alpha})}{t^{p}}v\right) - r(t)\right| \leq A_1|u|^{\delta_{11}}|v|^{\delta_{21}},$$

$$\left|I\left(s, (1+s^{\sigma})\mathbf{E}_{\alpha,\alpha}(s^{\alpha})u, \frac{(1+t_s^{\sigma})\mathbf{E}_{\alpha,\alpha-p}(s^{\alpha})}{s^{p}}v\right) - I_s\right| \leq \psi_s B_1|u|^{\delta_{11}}|v|^{\delta_{21}}.$$

Hence (H2) in Section 3 holds.

By direct computation we have

$$N_0 = \max\left\{\frac{(\sigma-\alpha-k_1)\mathbf{B}(\alpha,k_1+1)}{\sigma}, \ \frac{(\sigma-\alpha-k_1)\mathbf{B}(\alpha-p,k_1+1)}{\sigma(\alpha-p)}\right\}\left(\frac{\alpha+k_1}{\sigma-\alpha-k_1}\right)^{(\alpha+k_1)/\sigma}|A|$$

$$+\max\left\{\left(\Gamma(\alpha) + \frac{\lambda}{\alpha}\frac{\sigma-\alpha}{\sigma}\left(\frac{\alpha}{\sigma-\alpha}\right)^{\alpha/\sigma}\right), \ \frac{\lambda(\sigma-p)}{\sigma(\alpha-p)}\left(\frac{p}{\sigma-p}\right)^{p/\sigma}\right\}|a|\sum_{j=1}^{+\infty}2^{-s}$$

$$= \frac{\max\left\{\frac{3\mathbf{B}(2/3,5/6)}{4}, \frac{3\mathbf{B}(1/2,5/6)}{2}\right\}}{\sqrt[4]{3}}|A| + \max\left\{\Gamma(2/3) + \frac{1}{\sqrt[3]{2}}, \ \frac{11}{6}\frac{1}{\sqrt[12]{11}}\right\}|a|.$$

Using Mathlab7.0, we have $N_0 < 2.6|A| + 2.2|b|$.

Furthermore, we have for $t \in (s, s+1]$ that

$$\frac{|\phi(t)|}{(1+t^2)\mathbf{E}_{2/3,2/3}(t^{2/3})} \leq \frac{|A|}{(1+t^2)\mathbf{E}_{2/3,2/3}(t^{2/3})}\int_0^t (t-s)^{-\frac{1}{3}}\mathbf{E}_{2/3,2/3}((t-s)^{2/3})s^{-\frac{1}{6}}ds$$

$$+\frac{|x_0|}{(1+t^2)\mathbf{E}_{2/3,2/3}(t^{2/3})}\mathbf{E}_{2/3}(t^{2/3}) + \frac{|b|}{(1+t^2)\mathbf{E}_{2/3,2/3}(t^{2/3})}\sum_{j=1}^{s}\mathbf{E}_{2/3}((t-j)^{2/3})2^{-s}$$

$$\leq \frac{|A|}{1+t^2}\int_0^t (t-s)^{-\frac{1}{3}}s^{-\frac{1}{6}}ds + \frac{|x_0|}{(1+t^2)} + \frac{|b|}{(1+t^2)}\sum_{j=1}^{s}2^{-s}$$

$$\leq \frac{|A|}{1+t^2}t^{\frac{1}{2}}\mathbf{B}(2/3,5/6) + |x_0| + |b|$$

$$\leq 1.8|A| + |x_0| + |b|$$

and use (7), we have

$$\frac{t^p|^cD_{0^+}^{\frac{1}{6}}\phi(t)|}{(1+t^2)\mathbf{E}_{2/3,1/2}(t^{2/3})} \le |A|\frac{t^p}{1+t^2}\int_0^t (t-u)^{-\frac{1}{2}}u^{-\frac{1}{6}}du + |x_0|\frac{t^{1/6}}{1+t^2}t^{\frac{1}{2}}\frac{1}{2/3-1/6}$$

$$+|b|\frac{t^{1/6}}{1+t^2}\sum_{i=1}^{s}2^{-s}(t-i)^{\frac{1}{2}}\frac{1}{2/3-1/6}$$

$$\le 2.3|A| + 2|x_0| + 2|b|.$$

Then

$$\|\phi\| = \max\left\{\sup_{t\in(0,+\infty)}\frac{|\phi(t)|}{(1+t^2)\mathbf{E}_{2/3,2/3}(t^{2/3})},\ \sup_{t\in(0,+\infty)}\frac{t^{1/6}}{(1+t^2)\mathbf{E}_{2/3,11/6}(t^{2/3})}|^cD_{0^+}^{\frac{1}{6}}\phi(t)|\right\}$$

$$\le 2.3|A| + 2|x_0| + 2|b|.$$

Since

$$\sigma+\tau<1 \text{ or}$$

$$\sigma+\tau=1 \text{ with } 2.6|A|+2.2|a|<1 \text{ or} \tag{27}$$

$$\sigma+\tau>1 \text{ with } \frac{(2.3|A|+2|x_0|+2|b|)^{1-\sigma-\tau}(\sigma+\tau-1)^{\sigma+\tau-1}}{(\sigma+\tau)^{\sigma+\tau}} \ge 2.6|A|+2.2|a|$$

implies (19) holds, by Theorem 3.2, IVP(26) has at least one solution $x \in X$ if (27) holds. This example is ended.

Example 4.2. Consider the following IVP

$$\begin{cases} {}^cD_{s^+}^{\frac{2}{3}}x(t) - x(t) \\ = (t-s)^{-\frac{1}{2}}(s+1-t)^{-\frac{1}{12}}\left[B + A\left(\frac{|x(t)|}{\delta_0(t)}\right)^{\sigma}\left(\frac{|^cD_{s^+}^{\frac{1}{6}}x(t)|}{\delta_1(t)}\right)^{\tau}\right], \\ a.e., t\in(s,s+1), s\in \mathbf{Z}_0, \\ x(0)=x_0, \\ \Delta x(s) = 2^{-s}\left[b + a\left(\frac{|x(s)|}{\delta_0(s)}\right)^{\sigma}\left(\frac{|^cD_{(s-1)^+}^{\frac{1}{6}}x(s)|}{\delta_1(s)}\right)^{\tau}\right], s\in\mathbf{Z}, \end{cases} \tag{28}$$

where $a, b, A, B \in \mathbb{R}$, $\sigma, \tau \geq 0$ and

$$\delta_0(t) = \frac{\mathbf{E}_{2/3,2/3}(1)(s+1)(s+2)\left[\mathbf{E}_{2/3}(1)\right]^{s+1}}{2}, \ t \in (s, s+1], s \in \mathbf{Z}_0,$$

$$\delta_1(t) = (\mathbf{E}_{2/3,1/2}(1) + \mathbf{E}_{2/3,2/3}(1))(s+1)(s+2)\left[\mathbf{E}_{2/3}(1)\right]^s \mathbf{E}_{2/3,3/2}((t-t_s)^{2/3}),$$

$$t \in (s, s+1], s \in \mathbf{Z}_0.$$

Corresponding to IVP(2), we have $\alpha = \frac{2}{3}$, $p = \frac{1}{6}$, $k_2 = l_2 = -\frac{1}{12}$, $\lambda = 1$, $t_s = s$ for $s \in \mathbf{Z}_0$, $n(t) = (t-s)^{-\frac{1}{2}}(s+1-t)^{-\frac{1}{12}}$ for $t \in (s, s+1)$ and

$$g(t, u, v) = B + A\left(\frac{|u|}{\delta_0(t)}\right)^\sigma \left(\frac{|v|}{\delta_1(t)}\right)^\tau,$$

$$J(s, u, v) = 2^{-s}\left[b + a\left(\frac{|u|}{\delta_0(s)}\right)^\sigma \left(\frac{|v|}{\delta_1(s)}\right)^\tau\right].$$

Choose $m = 1$, $r(t) = B$ and $J_s = 2^{-s}b$. Then

$$|J\left(t_s, \delta_0(t_s)u, \delta_1(t_s)v\right) - J_s| \leq |a|\psi_s|u|^\sigma|v|^\tau,$$

$$|g\left(t, (\delta_0(t)u, \delta_1(t)v\right) - r(t)| \leq |A||u|^\sigma|v|^\tau.$$

One sees that (i), (ii) (iv), (vi) and (H4) hold. By Mathlab 7.0, we have

$$\overline{A} = \max\left\{\sup_{s\in\mathbf{Z}_0} \frac{2\mathbf{E}_{2/3,2/3}(1)\mathbf{B}(7/12,11/12)}{(s+2)\mathbf{E}_{2/3,2/3}(1)}, \sup_{s\in\mathbf{Z}_0} \frac{2[\mathbf{E}_{2/3,1/2}(1)+\mathbf{E}_{2/3,2/3}(1)]\mathbf{B}(7/12,11/12)}{(s+2)[\mathbf{E}_{2/3,2/3}(1)+\mathbf{E}_{2/3,1/2}(1)]}\right\} < 1.9,$$

$$\overline{B} = \max\left\{\sup_{s\in\mathbf{Z}_0} \frac{2\sum\limits_{j=1}^{+\infty} 2^{-j}}{(s+1)(s+2)\mathbf{E}_{2/3,2/3}(1)}, \sup_{s\in\mathbf{Z}_0} \frac{2\sum\limits_{j=1}^{+\infty} 2^{-j}}{(s+1)(s+2)[\mathbf{E}_{2/3,2/3}(1)+\mathbf{E}_{2/3,1/2}(1)]}\right\}$$

$$\leq \frac{1}{\mathbf{E}_{2/3,2/3}(1)} < 0.8.$$

We have for $t \in (s, s+1]$ that

$$\frac{|\phi(t)|}{\delta_0(t)} \leq \frac{1}{\delta_0(t)}\left|\int_{t_s}^t (t-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-u)^\alpha)n(u)Bdu + \left[x_0 \prod_{v=1}^s \mathbf{E}_\alpha(\lambda(t_v - t_{v-1})^\alpha)\right.\right.$$

$$+\sum_{j=1}^{s}\left(\prod_{v=j+1}^{s}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha)\right)2^{-j}b+\sum_{j=1}^{s}\left(\prod_{v=j+1}^{s}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha)\right)\times$$

$$\left.\left.\int_{t_{j-1}}^{t_j}(t_j-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-u)^\alpha)n(u)Bdu\right]\mathbf{E}_\alpha(\lambda(t-t_s)^\alpha)\right|$$

$$\leq\frac{|x_0|+\sum\limits_{j=1}^{+\infty}2^{-j}|b|+|B|\sum\limits_{j=1}^{s+1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^\alpha)(t_j-t_{j-1})^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha+k_2+l_2}\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^\alpha)}$$

$$=\frac{2|x_0|+2|b|+4(s+1)|B|\mathbf{E}_{2/3,2/3}(1)\mathbf{B}(7/12,11/12)}{(j+1)(s+2)\mathbf{E}_{2/3,2/3}(1)}\leq\frac{|x_0|+|b|+2|B|\mathbf{E}_{2/3,2/3}(1)\mathbf{B}(7/12,11/12)}{\mathbf{E}_{2/3,2/3}(1)}$$

and

$$\frac{|^cD^p_{t_s+}\phi(t)}{\delta_1(t)}\leq\frac{1}{\delta_1(t)}\left|\int_{t_s}^{t}(t-u)^{\alpha-p-1}\mathbf{E}_{\alpha,\alpha-p}(\lambda(t-u)^\alpha)n(u)Bdu\right.$$

$$+\lambda(t-t_s)^{\alpha-p}\left[x_0\prod_{v=1}^{s}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha)+\sum_{j=1}^{s}\left(\prod_{v=j+1}^{s}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha)\right)J_j\right.$$

$$+\sum_{j=1}^{s}\left(\prod_{v=j+1}^{s}\mathbf{E}_\alpha(\lambda(t_v-t_{v-1})^\alpha)\right)\times$$

$$\left.\left.\int_{t_{j-1}}^{t_j}(t_j-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-u)^\alpha)n(u)r(u)du\right]\mathbf{E}_{\alpha,\alpha+1-p}(\lambda(t-t_s)^\alpha)\right|$$

$$\leq\frac{|x_0|+\sum\limits_{j=1}^{+\infty}2^{-j}|b|+|B|\sum\limits_{j=1}^{s+1}[\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_j-t_{j-1})^\alpha)+\mathbf{E}_{\alpha,\alpha}(\lambda(t_j-t_{j-1})^\alpha)](t_j-t_{j-1})^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)}{\sum\limits_{j=0}^{s}t_{j+1}(t_{j+1}-t_j)^{\alpha-p+k_2+l_2}[\mathbf{E}_{\alpha,\alpha}(\lambda(t_{j+1}-t_j)^\alpha)+\mathbf{E}_{\alpha,\alpha-p}(\lambda(t_{j+1}-t_j)^\alpha)]}$$

$$\leq\frac{|x_0|+|b|+|B|\sum\limits_{j=1}^{s+1}[\mathbf{E}_{2/3,1/2}(1)+\mathbf{E}_{2/3,2/3}(1)]\mathbf{B}(7/12,11/12)}{\sum\limits_{j=0}^{s}(j+1)[\mathbf{E}_{2/3,2/3}(1)+\mathbf{E}_{2/3,1/2}(1)]}$$

$$\leq\frac{|x_0|+|b|+2|B|[\mathbf{E}_{2/3,1/2}(1)+\mathbf{E}_{2/3,2/3}(1)]\mathbf{B}(7/12,11/12)}{\mathbf{E}_{2/3,2/3}(1)+\mathbf{E}_{2/3,1/2}(1)}.$$

It follows that

$$||\phi||\leq\max\left\{\frac{|x_0|+|b|+2|B|\mathbf{E}_{2/3,2/3}(1)\mathbf{B}(7/12,11/12)}{\mathbf{E}_{2/3,2/3}(1)},\right.$$

$$\left.\frac{|x_0|+|b|+2|B|[\mathbf{E}_{2/3,1/2}(1)+\mathbf{E}_{2/3,2/3}(1)]\mathbf{B}(7/12,11/12)}{\mathbf{E}_{2/3,2/3}(1)+\mathbf{E}_{2/3,1/2}(1)}\right\}$$

$$< 0.8|x_0| + 0.8|b| + 3.7|B|.$$

Then $N_0 = \overline{A}|A| + \overline{B}|a|$. Since

$$\begin{aligned} &\sigma+\tau<1 \text{ or } \quad \sigma+\tau=1 \text{ with } 1.9|A|+0.8|a|<1 \text{ or} \\ &\sigma+\tau>1 \text{ with } \frac{(0.8|x_0|+0.8|b|+3.7|B|)^{1-(\sigma+\tau)}(\sigma+\tau)^{\sigma+\tau-1}}{(\sigma+\tau)^{\sigma+\tau)}} \geq 1.9|A|+0.8|a| \end{aligned} \tag{29}$$

imply (25) hold, we know from Theorem 3.4 that IVP(28) has at least one solution $x \in Y$ if (29) holds. This example is ended.

Remark 4.1. We can prove that IVP(1) has at least one solution if (i), (ii), (iii), (v) hold, $f\left(t,\ (1+t^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)x, \frac{(1+t^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t^\alpha)}{t^p}y\right)$ is bounded and there exists positive sequence $\{\psi_s\}$ with $\sum_{s=1}^{+\infty}\psi_s < +\infty$ such that $\frac{I\left(t_s,\ (1+t_s^\sigma)\mathbf{E}_{\alpha,\alpha}(\lambda t_s^\alpha)x, \frac{(1+t_s^\sigma)\mathbf{E}_{\alpha,\alpha-p}(\lambda t_s^\alpha)}{t_s^p}y\right)}{\psi_s}$ is bounded.

Remark 4.2. It is easy to prove that IVP(2) has at least one solution if (i), (ii), (iv), (vi) hold, $g(t,\delta_0(t)u,\delta_1(t)v)$ is bounded and there exists positive sequence $\{\psi_s\}$ with $\sum_{s=1}^{+\infty}\psi_s < +\infty$ such that $\frac{J(t_s,\delta_0(t_s)u,\delta_1(t_s)v)}{\psi_s}$ is bounded.

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