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Group analysis of differential equations

SYMMETRIES and CONSERVATION LAWS of NONVARIATIONAL EQUATIONS SYSTEM

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Summary

The second order ordinary differential equations systems that generally cannot be obtain from the variational principle are considered. Each equation of that system can be presented as the sum of two expressions. The first expression follows from the variational principle. The second expression does not follow from the variational principle. Such system we shall call the quasi Euler - Lagrange system. Under a certain requirements to the Lie transformations group the quasi Euler - Lagrange system conservation laws can be obtained in explicit form (similarly to the Noether theorem).

1 Introduction

One of the most known techniques of construction of conservation laws of differential equations system uses the Noether theorem or the generalization of Noether's theorem [1]. For using of the Noether theorem is necessary to obtain the initial equations system as the Euler - Lagrange equations system for some functional. We shall call the system of such class as the variational system or as the EL system.

The conservation laws of the EL system can be write if the functional variational symmetries are known [2].

The condition of the functional of initial equations system existence is certain restriction on using of the Noether theorem. Using of divergent symmetries of the functional [2] loosens this restriction. However, the initial equations system also should be the EL system.

In the present paper the second order ordinary differential equations systems are considered. Thus the equations systems that generally cannot be obtain from the variational principle are considered. Each equation of that system can be presented as the sum of two expressions. The first expression follows from the variational principle. The second expression does not follow from the variational principle. Such system we shall call the system with EL part or QEL ("quasi Euler -Lagrange") system. Under a certain requirements to the Lie transformations group the QEL system conservation laws can be obtained in explicit form (as well as for a system obtained from the variational principle).

The conservation laws of some class of QEL systems were considered also in [3].

2 Second order QEL systems

Let us write an ordinary differential equations system of second order as

$$F^j(t, x, p_1, p_2) = 0. \quad (j = 1, \ldots, m) \tag{1}$$

Here t is independent variable, x^j are coordinates of m-dimensional vector of dependent variables, p_1^j and p_2^j are first and second order derivatives of coordinates x^j .

A conservation law of system (1) is the expression

$$\theta^j F^j(t, x, p_1, p_2) = \frac{d}{dt} P(t, x, p_1). \tag{3}$$

Here $P(t, x, p_1)$ is a smooth function (conservation law component). The functions $\theta^j(t, x, p_1)$ satisfying to expression (3) are called conservation law characteristics [2].

Is known, that if the system (1) is the EL system, then the functions $F^j(t, x, p_1, p_2)$ can be write in the form

$$F^j(t, x, p_1, p_2) = E_j(g), \tag{4}$$

where the function $g(t, x, p_1)$ is the Lagrangian of certain functional

$$I[\,x\,] = \int_{t_0}^{t_1} g(t, x, p_1)dt. \tag{5}$$

The operator of the EL E_j with number j by the formula

$$E_j = \sum_{J=0}^{1} (-D)_J \frac{\partial}{\partial p_J^j} \tag{6}$$

is determined [2]. Here $(-D)_0 = 1$; $(-D)_1 = -D_t$, and

$$D_t = \frac{\partial}{\partial t} + p_1^j \frac{\partial}{\partial x^j} + p_2^j \frac{\partial}{\partial p_1^j} + \ldots$$

Besides we suppose $p_0^j = x^j$. If (1) is EL system and the variational symmetries of the functional (5) are known, then the conservation laws of system (1) are determined by the Noether theorem [2].

Let us present now the functions $F^j(t, x, p_1, p_2)$ as

$$F^j(t, x, p_1, p_2) = E_j(g) + F_n^j(t, x, p_1, p_2). \tag{7}$$

Here $g(t, x, p_1)$ is certain function, which is considered as the Lagrangian of functional (5); functions $F_n^j(t, x, p_1, p_2)$ determine nonvariational part of the system (1).

The functions $F_n^j(t, x, p_1, p_2)$ determine "deviations" of the initial system from the EL system. In accordance with this "deviations" the divergent invariance condition, which in the Noether theorem is used, should be write in a changed form. Let us define the QEL system (1) invariance conditions, which are allowed to determine the system (1) conservation laws in explicit form (similarly to the Noether theorem).

3 Symmetries and conservation laws of QEL systems

Let a transformations group G is determined by the vector field

$$X = \xi_t \frac{\partial}{\partial t} + \xi^j \frac{\partial}{\partial x^j},$$

where $\xi_t(t,x), \xi^j(t,x)$ are certain smooth functions. Let us define also the Lie vector field $\overline{X}$ as the continued vector field X [2].

If (1) is the EL system, then the conditions (4) are hold and the function g is the Lagrangian of functional $I[\,x\,]$. The vector field$\overline{X}$ is called the infinitesimal variational symmetry of functional $I[\,x\,]$, if the following condition hold:

$$\overline{X}\, g + gD_t(\xi_t) = D_t(B) \tag{8}$$

where $B(t,x,p_1)$ is certain function. [2].

According to the Noether theorem, if the functional $I[\,x\,]$ has a variational symmetries group, then there is a function P such that

$$\frac{d}{dt}P(t,x,p_1) = Q^j E_j(g).$$

Here the functions Q^j are vector field X characteristics defined by expression

$$Q^j = \xi^j - \xi_t p_1^j.$$

The function $P(t,x,p_1)$ has the form [2]

$$P = B - A - \xi_t g, \tag{9}$$

where

$$A = Q^j \frac{\partial g}{\partial p_1^j}. \tag{10}$$

Now we give the following definition.

Definition. *The vector field* $\overline{X}$ *is called an infinitesimal variational symmetry of QEL systems* (1) *or the QEL symmetry if on integrgral manifold of (1) be fullfield*

$$\overline{X}\, g + Q^j F_n^j + gD_t(\xi_t) = D_t(B). \tag{11}$$

The condition (11) is generalization of the divergent invariance condition [2] used by formulation of the Noether theorem. If the initial system is the EL system, then $F_n^j = 0$ and the expression (11) is the divergent invariance condition. If the initial system does not include the variational part ($g = 0$), then from (11) follows the expression (3).

Theorem 1. *Let the system* (1) *is the QEL system and the functions* F^j *are determined by expressions* (7)*. If the vector field* $\overline{X}$ *is the QEL symmetry of system* (1)*, then the vector field* $\overline{X}$ *characteristics* Q^j *are the characteristics of* *the conservation law of system* (1). *In this case the conservation law component* P *defined by expression* (9).

Proof: The proof of this theorem follows from expression [2]

$$\overline{X}\, g = Q^j E_j(g) + D_t(A) + \xi_t D_t(g), \tag{12}$$

where $A(t, x, p_1)$ is determined by expression (10). Substituting (12) in (11), we have

$$Q^j E_j(g) + D_t(A) + \xi_t D_t(g) + Q^j F_n^j + g D_t(\xi_t) = D_t(B),$$

whence, taking account of (7), follows

$$Q^j F^j = D_t(B - A - \xi_t g)$$

and the theorem is proved.

This theorem is allowed to determine the system (1) conservation laws in explicit form (similarly to the Noether theorem).

4 Second order ordinary differential equation

Let

$$p_2 - H(t, u, p_1) = 0 \tag{13}$$

be a second order ordinary differential equation, where $H(t, u, p_1)$ is the arbitrary function. Is known that we can transform this equation into the EL form if we multiply the initial equation by some function, which is called the integrating factor.

The problem of the integrating factor determination is reduced to a partial differential equation integration problem [2]. Let us consider a second order equations having the integrating factor in the form of certain function $\phi(t, u)$. Using the Theorem 1, we can obtain the equation conservation laws without integrating factor determination.

Let us present the equation (13) in the form of QEL equation

$$E(g) + F_n = 0. \tag{14}$$

Here $g(t, u, p_1)$, $F_n(t, u, p_1, p_2)$ are functions satisfying to the expression $E(g) + F_n = p_2 - H(t, u, p_1)$.

Proposition 1. *Let* Q *is the characteristic of vector field* X*, determining the QEL symmetry of equation* (14). *Let also* $\phi(t, u)$ *be the integrating factor of* *the equation* (14)*. Then* $Q^* = Q/\phi$ *is characteristic of vector field* X^**, defining the infinitesimal variational symmetry of EL equation*

$$\phi(E(g) + F_n) = 0. \tag{15}$$

Proof: Let the proposal conditions are hold and let $Q = \xi_u(t,u) - \xi_t(t,u)p_1$ be the characteristic of the vector field X. Then we can write

$$Q(E(g) + F_n) = \frac{Q}{\phi}\phi(E(g) + F_n) = \frac{dP}{dt}. \tag{16}$$

Sinse ϕ is the integrating factor, then

$$\phi(E(g) + F_n) = E(g^*),$$

where $g^*(t,u,p_1)$ is certain function (Lagrangian). Let us consider the function $Q^* = \xi_u^*(t,u) - \xi_t^*(t,u)p_1$, where $\xi_u^*(t,u) = \xi_u/\phi$, $\xi_t^*(t,u) = \xi_t/\phi$. Taking into account (16), we have that Q^* is characteristic of the vector field defining the infinitesimal variational symmetry of the EL equation (15). The proposition is proved.

From the proof of this proposition follows, that all vector fields obtained as multiplication the vector field X^* by arbitrary functions $\phi(t,u)$ can be considered as equivalent vector fields. These vector fields determine the QEL symmetries of equivalent QEL equations having the same conservation law.

As example we consider the Emden - Fowler equation [2]

$$u_{tt} + 2u_t/t + u^5 = 0.$$

We present this equation in the form

$$p_2 + 2p_1/t + u^5 = 0. \tag{17}$$

Integrating factor for equation (17) is the function $\phi = t^2$. Hence we have the EL equation

$$t^2 p_2 + 2tp_1 + t^2u^5 = 0. \tag{18}$$

For Lagrangian $g = -t^2p_1^2/2 + t^2u^6/6$ of functional (5) we see, that the equation (18) is the EL equation. The vector field characteristic defining the infinitesimal variational symmetry of functional has the form $Q = -u/2 - tp_1$. Thus in (8) the function $B \equiv 0$ and the conservation law component is

$$P = -(t^3u^6/3 + t^2up_1 + t^3(p_1)^2)/2.$$

However conservation law of Emden - Fowler equation can be obtained without integrating factor using. We write the Emden - Fowler equation in the form

$$tp_2 + 2p_1 + tu^5 = 0. \tag{19}$$

The field vector characteristic determining QEL symmetry of the equation (19) has the form $Q = -ut/2 - t^2 p_1$. Let us present (19) as the QEL equation and apply the Theorem 1.

The function F_n in (14) and the function B in (16) are determined by the form of Lagrangian g. For example if $g \equiv 0$, then $F_n = tp_2 + 2p_1 + tu^5$ and $B = P$. For $g = -t(p_1)^2/2$ we obtain $F_n = p_1 + tu^5$ and $B = -t^3u^6/6$.

Let us note that the Theorem 1 can be generalized on a case of partial differential equations sets of an arbitrary order.

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