DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 4, 2002
Electronic Journal,
reg. N P23275 at 07.03.97

http://www.neva.ru/journal
e-mail: diff@osipenko.stu.neva.ru

Ordinary differential equations

Robustness of Governing Equations of Envelope Surface Created by Nearly Monochromatic Waves

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Abstract

In this paper the author deals with the Schrödinger equation of the two-dimensional envelope surfaces of the water waves and discusses the robustness about the propagation direction of this equation. In the field of the water and/or the plasma, the Schrödinger equation governs the envelope created by the nearly monochromatic waves, which energy is almost concentrated in a single frequency. The two-dimensional Schrödinger equation, which governs the envelope surface instead of the envelope of the one-dimensional system, is obtained through the

straightforward process from the one-dimensional system. The obtained two-dimensional Schrödinger equation contains the parameter of the propagation direction. The author shows that the two-dimensional Schrödinger equation is robust about the propagation direction, i.e., the small variation of the propagation direction makes no change of the original equation.

1. INTRODUCTION

The Schrödinger equation governs the envelope of the group waves [1][2], which propagate in the water and the plasma, etc. Moreover, the non-linear Schrödinger equation governs the non-linearity of the envelope. The fact that the solution for the non-linear Schrödinger equation can be a soliton[3][4] is known and of interest [5]. Many studies of the group waves have been carried out in the water wave area and some other area as well. For example, in the fiber-optic communication system, the GDV (Group Velocity Dispersion), in which problem the launched pulse may spread outside its timing window due to dispersion, limits the transmission data rate caused by the pulse overlapping between adjacent timing windows. Non-linear refraction of SPM (Self-Phase Modulation) can also limit the system performance by causing spectral broadening of the optical pulse. Those effects are also described by the linear or non-linear Schrödinger equation and analyzed to achieve the optimal system performance [6].

In this paper, the governing equation of the envelope surface created by nearly monochromatic waves, which forms the two-dimensional Schrödinger type equation, are derived [7][8][9] and its characteristics of robustness about the propagation direction of the representative style are discussed. The spectrum of nearly monochromatic waves is almost concentrated on a single frequency. Such waves create the envelope in a one-dimensional system. In a two-dimensional system, the envelope surface instead of the envelope in a one-dimensional system can be considered. Two-dimensional nearly monochromatic waves also create the envelope surface. The obtained two-dimensional Schrödinger equation contains the parameter of the propagation direction. The author shows that the two-dimensional Schrödinger equation is robust about the propagation direction, i.e., the small variation of the propagation direction makes no change of the original equation.

The following section presents the background of the envelope equation (the Schrödinger equation) of one-dimensional nearly monochromatic waves. The author also gives the general form of the envelope equation and the comprehension of the meaning of the spatial derivative of the envelope. The third section deals with the straightforward expansion of the Schrödinger equation to the two-dimensional space of nearly monochromatic waves. The forth section discusses the robustness of the derived two-dimensional equation about the propagation direction. Finally, in the fifth section concluding remarks are presented.

2. REVIEW: GOVERNING EQUATIONS OF THE ENVELOPE OF NEARLY MONOCHROMATIC WAVES[10]

Plane traveling waves with dispersive characteristics of the form:

$$u(x,t) = A(x,t)e^{i\{k_0 x - \omega(k_0)t\}} \tag{1}$$

are considered. Here, A, k_0, ω, x and t denote amplitude, wave number, angular frequency, travelling direction of waves and time, respectively.

This is based on the assumption that most of the energy is concentrated in one wave number k_0 (nearly monochromatic waves) and the amplitude $A(x,t)$is not constant but varies slowly is space and time. The amplitude $A(x,t)$ acts as an envelope of the traveling wave.

The governing Equation of the envelope is obtained as follows. The exact solution of $u(x,t)$ is represented as

$$u(x,t) = \int_k S(k)e^{i\{kx - \omega(k)t\}}dk. \tag{2}$$

Here, $S(k)$ is the spectrum of the waves. So, $A(x,t)$ is derived from Eq.(1) and Eq.(2).

$$A(x,t) = \int_k S(k)e^{iP(x,k,t)}dk \tag{3}$$

where

$$P(x,k,t) = (k-k_0)x - \{\omega(k) - \omega(k_0)\}t \tag{4}$$

The time derivative of $A(x,t)$ is

$$\frac{\partial A(x,t)}{\partial t} = \int_k (-i)\{\omega(k) - \omega(k_0)\}S(k)e^{iP(x,k,t)}dk. \tag{5}$$

Moreover, the spatial derivative of $A(x,t)$ is obtained as follows.

$$\frac{\partial^n A(x,t)}{\partial x^n} = \int_k (i)^n (k-k_0)^n S(k)e^{iP(x,k,t)}dk, n = 1, 2, 3 \cdots . \tag{6}$$

On the other hand, the linear dispersion relation of $\omega(k)$ can be written as the following Taylor expansion:

$$\omega(k) = \omega(k_0) + \omega^{'}(k_0)(k-k_0) + \frac{1}{2!}\omega^{''}(k_0)(k-k_0)^2$$
$$+\frac{1}{3!}\omega^{'''}(k_0)(k-k_0)^3 + \cdots , \tag{7}$$

where

$$\omega^{(n)}(k_0) = \frac{\partial^n \omega(k)}{\partial k^n}\Big|_{k=k_0}. \tag{8}$$

Substituting the relations of Eq.(6) and Eq.(7) into Eq.(5) leads to the following:

$$\frac{\partial A(x,t)}{\partial t} = \sum_{n=1}^{\infty}(-1)^n (i)^{n-1}\frac{\omega^{(n)}(k_0)}{n!}\frac{\partial^n A(x,t)}{\partial x^n}. \tag{9}$$

Eq.(9) represents the linear higher order governing equation that governs the amplitude of nearly monochromatic waves, that is, the equation that the envelope of nearly monochromatic waves satisfies. Neglecting the third and higher order of spatial derivatives in Eq. (9), we obtain the linear Schrödinger equation [10]:

$$i(\frac{\partial A(x,t)}{\partial t} + \omega^{'}(k_0)\frac{\partial A(x,t)}{\partial x}) + \frac{1}{2!}\omega^{''}(k_0)\frac{\partial^2 A(x,t)}{\partial x^2} = 0 \tag{10}$$

In the remainder of this section, we treat the non-linear case [11] of the dispersion relation. The non-linear dispersion relation is applied to Eq.(5) instead of the linear dispersion relation of Eq.(7). We consider the non-linear dispersion relation to be the following:

$$\omega = \omega(k, A(x,t)^2) \tag{11}$$

and then we expand ω in a Taylor series about $k = k_0$ and $|A(x,t)|^2 = 0$:

$$\omega \cong \omega(k_0) + \omega^{'}(k_0)(k-k_0) + \frac{1}{2!}\omega^{''}(k_0)(k-k_0)^2$$
$$+\frac{\partial \omega(k_0, A(x,t)^2)}{\partial |A(x,t)|^2}\Big|_{|A|^2=0}|A(x,t)|^2. \tag{12}$$

This equation can be applied to Eq.(2), then the following expansion for $A(x,t)$ is obtained:

$$A(x,t) = \int_k S(k)e^{iP_n(x,k,t)}dk, \tag{13}$$

where

$$P_n(x,k,t) = (k-k_0)x - \{\omega^{'}(k_0)(k-k_0) + \frac{1}{2!}\omega^{''}(k_0)(k-k_0)^2 - \gamma|A(x,t)|^2\}t, \tag{14}$$

and

$$\gamma \cong -\frac{\partial\omega(k_0, A(x,t)^2)}{\partial|A(x,t)|^2}\Bigg|_{|A|^2=0} \tag{15}$$

Using the same approach described above, we obtain the following, time and spatial derivatives of $A(x,t)$ in Eq.(13).

$$\begin{aligned}\frac{\partial A(x,t)}{\partial t} = &\int_k (-i)\{\omega^{'}(k_0)(k-k_0) \\ &+\frac{i}{2!}\omega^{''}(k_0)(k-k_0)^2 - \gamma|A(x,t)|^2\}S(k)e^{iP_n(x,k,t)}dk \\ = &(-i)\omega^{'}(k_0)\int_k (k-k_0)S(k)e^{iP_n(x,k,t)}dk \\ &-\frac{i}{2!}\omega^{''}(k_0)\int_k (k-k_0)^2 S(k)e^{iP_n(x,k,t)}dk \\ &+i\gamma|A(x,t)|^2\int_k S(k)e^{iP_n(x,k,t)}dk \\ = &-\omega^{'}(k_0)\frac{\partial A(x,t)}{\partial x} + \frac{i}{2!}\omega^{''}(k_0)\frac{\partial^2 A(x,t)}{\partial x^2} + i\gamma|A(x,t)|^2 A(x,t)\end{aligned} \tag{16}$$

That is

$$i\left(\frac{\partial A(x,t)}{\partial t} + \omega^{'}(k_0)\frac{\partial A(x,t)}{\partial x}\right) + \frac{1}{2!}\omega^{''}(k_0)\frac{\partial^2 A(x,t)}{\partial x^2} + \gamma|A(x,t)|^2 A(x,t) = 0. \tag{17}$$

This is the non-linear Schrödinger equation with cubic non-linearity. The cubic non-linear term is added to the linear Schröodinger equation of Eq.(10).

3. EXPANSION TO EQUATIONS GOVERNING THE ENVELOPE SURFACE OF NEARLY MONOCHROMATIC WAVES

Here the treatment of one-dimensional, nearly monochromatic waves is expanded to two-dimensional space. The following equation is obtained by expanding Eq.(1).

$$u(x,y,t) = A^f(x,y,t)e^{i\{k_0 x\cos\theta_0 + k_0 y\sin\theta_0 - \omega(k_0)t\}} \tag{18}$$

Here, $A^f(x,y,t)$,(x,y) and $\theta_0$indicate the amplitude of a two-dimensional wave surface, the coordinate system and the propagation angle of the progressive wave $u(x,y,t)$ respectively. Suppose $\theta_0 \neq 0$. The amplitude $A^f(x,y,t)$ becomes the envelope surface of the progressive wave.

The exact solution of $u(x, y, t)$ should be written as follows.

$$u(x, y, t) = \int_k S(k) e^{i\{kx\cos\theta_0 + ky\sin\theta_0 - \omega(k)t\}} dk \tag{19}$$

From Eq.(18) and Eq.(19), it follows that $A^f(x, y, t)$ is given by

$$A^f(x, y, t) = \int_k S(k) e^{iP^f(x,y,k,t)} dk, \tag{20}$$

where

$$P^f(x, y, k, t) = (k - k_0)x\cos\theta_0 + (k - k_0)y\sin\theta_0 - \{\omega(k) - \omega(k_0)\}t. \tag{21}$$

The partial derivatives of $A^f(x, y, t)$ with respect to x and y are as follows:

$$\frac{\partial A^f(x, y, t)}{\partial x} = \cos\theta_0 \int_k i(k - k_0) S(k) e^{iP^f(x,y,k,t)} dk \tag{22}$$

$$\frac{\partial A^f(x, y, t)}{\partial y} = \sin\theta_0 \int_k i(k - k_0) S(k) e^{iP^f(x,y,k,t)} dk \tag{23}$$

Then Eq.(22) and Eq.(23) follow

$$\cos\theta_0 \frac{\partial A^f(x, y, t)}{\partial x} + \sin\theta_0 \frac{\partial A^f(x, y, t)}{\partial y} = \int_k i(k - k_0) S(k) e^{iP^f(x,y,k,t)} dk. \tag{24}$$

In a similar manner, the higher order integral terms about $(k - k_0)$ are obtained as follows

$$\frac{\partial^2 A^f(x, y, t)}{\partial x^2} + \frac{\partial^2 A^f(x, y, t)}{\partial y^2} = \int_k i^2 (k - k_0)^2 S(k) e^{iP^f(x,y,k,t)} dk \tag{25}$$

$$\frac{1}{\cos\theta_0} \frac{\partial^3 A^f(x, y, t)}{\partial x^3} + \frac{1}{\sin\theta_0} \frac{\partial^3 A^f(x, y, t)}{\partial y^3} = \int_k i^3 (k - k_0)^3 S(k) e^{iP^f(x,y,k,t)} dk, \cdots \tag{26}$$

and so on. Moreover the general term is written as

$$\begin{aligned}(\frac{1}{\cos\theta_0})^{n-2} \frac{\partial^n A^f(x, y, t)}{\partial x^n} + (\frac{1}{\sin\theta_0})^{n-2} \frac{\partial^n A^f(x, y, t)}{\partial y^n} \\ = \int_k i^n (k - k_0)^n S(k) e^{iP^f(x,y,k,t)} dk.\end{aligned} \tag{27}$$

On the other hand, the time derivative of $A^f(x,y,t)$ is

$$\frac{\partial A^f(x,y,t)}{\partial t} = \int_k (-i)\{\omega(k)-\omega(k_0)\}S(k)e^{iP^f(x,y,k,t)}dk. \tag{28}$$

Therefore the linear higher order governing equation of the envelope surface of nearly monochromatic waves is obtained by substituting Eq.(7), Eq.(24), Eq.(25) and Eq.(26) into Eq.(28).

$$\begin{aligned} i\frac{\partial A^f(x,y,t)}{\partial t} &= -i\omega^{'}(k_0)(\cos\theta_0\frac{\partial A^f(x,y,t)}{\partial x}+\sin\theta_0\frac{\partial A^f(x,y,t)}{\partial y}) \\ &-\frac{1}{2!}\omega^{''}(k_0)(\frac{\partial^2 A^f(x,y,t)}{\partial x^2}+\frac{\partial^2 A^f(x,y,t)}{\partial y^2}) \\ &+\frac{i}{3!}\omega^{'''}(k_0)(\frac{1}{\cos\theta_0}\frac{\partial^3 A^f(x,y,t)}{\partial x^3}+\frac{1}{\sin\theta_0}\frac{\partial^3 A^f(x,y,t)}{\partial y^3})+\cdots \end{aligned} \tag{29}$$

Using the general term of Eq.(27), we can rewrite Eq.(29) as follows:

$$\frac{\partial A^f(x,y,t)}{\partial t} = \sum_{n=1}^{\infty}(-1)^n(i)^{n-1}\frac{\omega^{(n)}(k_0)}{n!}\{(\frac{1}{\cos\theta_0})^{n-2}\frac{\partial^n A^f(x,y,t)}{\partial x^n}+(\frac{1}{\sin\theta_0})^{n-2}\frac{\partial^n A^f(x,y,t)}{\partial y^n}\} \tag{30}$$

This equation is expanded to two-dimensional spaces and time:(x,y,t) from Eq.(9).

The equation obtained by $n=2$ in Eq.(30)

$$\begin{aligned} i\{\frac{\partial A^f(x,y,t)}{\partial t}+\omega^{'}(k_0)(\cos\theta_0\frac{\partial A^f(x,y,t)}{\partial x}+\sin\theta_0\frac{\partial A^f(x,y,t)}{\partial y})\} \\ +\frac{1}{2!}\omega^{''}(k_0)(\frac{\partial^2 A^f(x,y,t)}{\partial x^2}+\frac{\partial^2 A^f(x,y,t)}{\partial y^2})=0 \end{aligned} \tag{31}$$

is the expansion of the linear Schrödinger equation [7][8].

In Eq.(30), the following relation can be obtained.

$$\frac{\partial^n A^f(x,y,t)}{\partial y^n}/\frac{\partial^n A^f(x,y,t)}{\partial x^n} = \tan^n\theta_0,\ n=1,2,3,\cdots \tag{32}$$

This shows the relation of the ratio of the spatial slope of the envelope surface and the tangent of the wave propagation direction. Therefore Eq.(30) describes how the envelope surface of nearly monochromatic waves evolves, satisfying Eq.(32) and varying with time.

Next, we consider the non-linear dispersion relation of Eq.(11), use Eq.(12) for the Taylor series of ω about $k = k_0$ and $|A^f(x,y,t)|^2 = 0$, and obtain the following:

$$A^f(x,y,t) = \int_k S(k)e^{iP_n^f(x,y,k,t)}dk, \tag{33}$$

where

$$\begin{aligned} P_n^f(x,y,k,t) &= (k-k_0)x\cos\theta_0 + (k-k_0)y\sin\theta_0 \\ &-\{\omega_0^{'}(k)(k-k_0) + \frac{1}{2!}\omega_0^{''}(k)(k-k_0)^2 - \gamma^f|A^f(x,y,t)|^2\}t. \end{aligned} \tag{34}$$

The time derivative of $A^f(x,y,t)$ in Eq.(33) can be obtained as follows:

$$\begin{aligned} \frac{\partial A^f(x,y,t)}{\partial t} &= \int_k (-i)\{\omega^{'}(k_0)(k-k_0) \\ &+\frac{1}{2!}\omega^{''}(k_0)(k-k_0)^2 - \gamma^f|A^f(x,y,t)|^2\}S(k)e^{iP_n^f(x,y,k,t)}dk \\ &= (-i)\omega^{'}(k_0)\int_k (k-k_0)S(k)e^{iP_n^f(x,y,k,t)}dk \\ &\quad -\frac{i}{2!}\omega^{''}(k_0)\int_k (k-k_0)^2S(k)e^{iP_n^f(x,y,k,t)}dk \\ &\quad +i\gamma^f|A^f(x,y,t)|^2\int_k S(k)e^{iP_n^f(x,y,k,t)}dk \end{aligned} \tag{35}$$

The forms of the partial derivatives of $A^f(x,y,t)$ in the non-linear case are identical to the forms given by Eq.(22) and Eq.(23) except $P_n^f(x,y,k,t)$ appears instead of $P^f(x,y,k,t)$. So, applying these equations to Eq.(35), then we have

$$\begin{aligned} &i\{\frac{\partial A^f(x,y,t)}{\partial t} + \omega^{'}(k_0)(\cos\theta_0\frac{\partial A^f(x,y,t)}{\partial x} + \sin\theta_0\frac{\partial A^f(x,y,t)}{\partial y})\} \\ &\qquad +\frac{1}{2!}\omega^{''}(k_0)(\frac{\partial^2 A^f(x,y,t)}{\partial x^2} + \frac{\partial^2 A^f(x,y,t)}{\partial y^2}) + \gamma^f|A^f(x,y,t)|^2A^f(x,y,t) = 0 \end{aligned} \tag{36}$$

This is the two-dimensional, non-linear Schrödinger equation with cubic non-linearity. Compared with the linear case of Eq.(31), the cubic non-linear term is just added in the non-linear case of Eq.(36) and Eq.(32) is also satisfied for $n = 1$ and 2.

4. ROBUSTNESS OF THE TWO-DIMENSIONAL GOVERNING EQUATION OF THE ENVELOPE SURFACE OF NEARLY MONOCHROMATIC WAVES

In the previous section, we treat the governing equation of the envelope surface of nearly monochromatic waves, in which waves its propagation direction is single. That is, Eq. (30) presents the linear dynamics of the envelope surface of nearly monochromatic waves with a single propagation direction. We deal with the governing equation of the envelope surface of nearly monochromatic waves for the robustness consideration of multi-directionality of the propagation direction in this section.

Here the traveling wave whose propagation direction exists in two closely separated angles is considered. Let $A^{\theta}(x, y, t)$ be the amplitude of an envelope of the traveling wave with θ_0 and θ_1 of the propagation direction, then we have

$$u(x, y, t) = A^{\theta}(x, y, t)\{e^{i(k_0 x\cos\theta_0 + k_0 y\sin\theta_0 - \omega(k_0)t)} + e^{i(k_0 x\cos\theta_1 + k_0 y\sin\theta_1 - \omega(k_0)t)}\}. \quad (37)$$

This equation is expanded to two propagation directions of Eq.(18). Here we suppose

$$\theta_1 = \theta_0 + \Delta.(\Delta : \text{small}) \quad (38)$$

The exact solution of $u(x, y, t)$ should be written as follows.

$$u(x, y, t) = \int_k S(k)\{e^{i(kx\cos\theta_0 + ky\sin\theta_0 - \omega(k)t)} + e^{i(kx\cos\theta_1 + ky\sin\theta_1 - \omega(k)t)}\}dk \quad (39)$$

The following approximations are stood up based on Eq.(38).

$$\cos\theta_1 \cong \cos\theta_0 - \Delta\sin\theta_0 \quad (40)$$

$$\sin\theta_1 \cong \sin\theta_0 + \Delta\cos\theta_0 \quad (41)$$

Using Eq.(40) and Eq.(41), we have

$$\begin{aligned} e^{i(k_0 x\cos\theta_1 + k_0 y\sin\theta_1 - \omega(k_0)t)} &= e^{i\{k_0 x(\cos\theta_0 - \Delta\sin\theta_0) + k_0 y(\sin\theta_0 + \Delta\cos\theta_0) - \omega(k_0)t\}} \\ &= e^{i(k_0 x\cos\theta_0 + k_0 y\sin\theta_0 - \omega(k_0)t) - i\Delta(k_0 x\sin\theta_0 - k_0 y\cos\theta_0)} \\ &= e^{i(k_0 x\cos\theta_0 + k_0 y\sin\theta_0 - \omega(k_0)t)}e^{-i\Delta(k_0 x\sin\theta_0 - k_0 y\cos\theta_0)}. \end{aligned} \quad (42)$$

So, the relation of Eq.(37) and Eq.(39) becomes

$$\begin{aligned} &A^{\theta}(x, y, t)\{1 + e^{-i\Delta(k_0 x\sin\theta_0 - k_0 y\cos\theta_0)}\}e^{i(k_0 x\cos\theta_0 + k_0 y\sin\theta_0 - \omega(k_0)t)} \\ &\quad = \int_k S(k)\{1 + e^{-i\Delta(kx\sin\theta_0 - ky\cos\theta_0)}\}e^{i(kx\cos\theta_0 + ky\sin\theta_0 - \omega(k)t)}dk. \end{aligned} \quad (43)$$

This equation follows that$A^{\theta}(x,y,t)$ is given by

$$A^{\theta}(x,y,t)=\frac{1}{1+e^{-i\Delta f_0(x,y)}}\int_k S(k)\{1+e^{-i\Delta f(x,y,k)}\}e^{iP(x,y,k,t)}dk \tag{44}$$

where

$$P(x,y,k,t)=(k-k_0)x\cos\theta_0+(k-k_0)y\sin\theta_0-\{\omega(k)-\omega(k_0)\}t \tag{45}$$

$$f(x,y,k)=kx\sin\theta_0-ky\cos\theta_0 \tag{46}$$

$$f_0(x,y)=k_0x\sin\theta_0-k_0y\cos\theta_0 \tag{47}$$

The partial derivatives of $A^{\theta}(x,y,t)$ with respect to x and y are as follows:

$$\begin{aligned}\frac{\partial A^{\theta}(x,y,t)}{\partial x}&=\frac{i\Delta\sin\theta_0}{(1+e^{-i\Delta f_0(x,y)})^2}\int_k S(k)[k_0e^{-i\Delta f_0(x,y)}-ke^{-i\Delta f(x,y,k)}\\&-(k-k_0)e^{-i\Delta\{f(x,y,k)+f_0(x,y)\}}]e^{iP(x,y,k,t)}dk\\&+\frac{\cos\theta_0}{(1+e^{-i\Delta f_0(x,y)})}\int_k i(k-k_0)S(k)\left\{1+e^{-i\Delta f(x,y,k)}\right\}e^{iP(x,y,k,t)}dk\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{\partial A^{\theta}(x,y,t)}{\partial y}&=\frac{-i\Delta\cos\theta_0}{1+e^{-i\Delta f_0(x,y)}}\int_k S(k)[k_0e^{-i\Delta f_0(x,y)}-ke^{-i\Delta f(x,y,k)}\\&-(k-k_0)e^{-i\Delta\{f(x,y,k)+f_0(x,y)\}}]e^{iP(x,y,k,t)}dk\\&+\frac{\sin\theta_0}{1+e^{-i\Delta f_0(x,y)}}\int_k i(k-k_0)S(k)\left\{1+e^{-i\Delta f(x,y,k)}\right\}e^{iP(x,y,k,t)}dk\end{aligned} \tag{49}$$

Then Eq.(48) and Eq.(49) follow

$$\begin{aligned}&\cos\theta_0\frac{\partial A^{\theta}(x,y,t)}{\partial x}+\sin\theta_0\frac{\partial A^{\theta}(x,y,t)}{\partial y}=\\&\frac{1}{1+e^{-i\Delta f_0(x,y)}}\int_k i(k-k_0)S(k)\left\{1+e^{-i\Delta f(x,y,k)}\right\}e^{iP(x,y,k,t)}dk.\end{aligned} \tag{50}$$

In a similar manner, the second partial derivatives of $A^{\theta}(x,y,t)$with respect to x and y are as follows:

$$\begin{aligned}\frac{\partial^2 A^{\theta}(x,y,t)}{\partial x^2}&=\frac{\Delta^2\sin^2\theta_0}{(1+e^{-i\Delta f_0(x,y)})^3}\int_k S(k)[\{-k_0^2e^{-i\Delta f_0(x,y)}-(k^2-2kk_0)e^{-i\Delta f(x,y,k)}\\&-(k-k_0)^2e^{-i\Delta\{f(x,y,k)+f_0(x,y)\}}\}e^{-i\Delta f_0(x,y)}+k_0^2e^{-i\Delta f_0(x,y)}\\&-k^2e^{-i\Delta f(x,y,k)}-(k^2-k_0^2)e^{-i\Delta\{f(x,y,k)+f_0(x,y)\}}]e^{iP(x,y,k,t)}dk\end{aligned}$$

$$-\frac{2\Delta\sin\theta_0\cos\theta_0}{(1+e^{-i\Delta f_0(x,y)})^2}\int_k (k-k_0)S(k)[k_0e^{-i\Delta f_0(x,y)}-ke^{-i\Delta f(x,y,k)} \quad (51)$$

$$-(k-k_0)e^{-i\Delta\{f(x,y,k)+f_0(x,y)\}}]e^{iP(x,y,k,t)}dk$$

$$+\frac{\cos^2\theta_0}{1+e^{-i\Delta f_0(x,y)}}\int_k i^2(k-k_0)^2S(k)\left\{1+e^{-i\Delta f(x,y,k)}\right\}e^{iP(x,y,k,t)}dk$$

$$\frac{\partial^2 A^\theta(x,y,t)}{\partial y^2}=\frac{\Delta^2\cos^2\theta_0}{(1+e^{-i\Delta f_0(x,y)})^3}\int_k S(k)[\{-k_0^2e^{-i\Delta f_0(x,y)}-(k^2-2kk_0)e^{-i\Delta f(x,y,k)}$$

$$-(k-k_0)^2e^{-i\Delta\{f(x,y,k)+f_0(x,y)\}}\}e^{-i\Delta f_0(x,y)}+k_0^2e^{-i\Delta f_0(x,y)}$$

$$-k^2e^{-i\Delta f(x,y,k)}-(k^2-k_0^2)e^{-i\Delta\{f(x,y,k)+f_0(x,y)\}}]e^{iP(x,y,k,t)}dk$$

$$+\frac{2\Delta\sin\theta_0\cos\theta_0}{(1+e^{-i\Delta f_0(x,y)})^2}\int_k (k-k_0)S(k)[k_0e^{-i\Delta f_0(x,y)}-ke^{-i\Delta f(x,y,k)} \quad (52)$$

$$-(k-k_0)e^{-i\Delta\{f(x,y,k)+f_0(x,y)\}}]e^{iP(x,y,k,t)}dk$$

$$+\frac{\sin^2\theta_0}{1+e^{-i\Delta f_0(x,y)}}\int_k i^2(k-k_0)^2S(k)\left\{1+e^{-i\Delta f(x,y,k)}\right\}e^{iP(x,y,k,t)}dk$$

We can obtain the following relations by neglecting the term of Δ^2 in Eq.(51) and Eq.(52).

$$\frac{\partial^2 A^\theta(x,y,t)}{\partial x^2}\cong-\frac{2\Delta\sin\theta_0\cos\theta_0}{(1+e^{-i\Delta f_0(x,y)})^2}\int_k (k-k_0)S(k)[k_0e^{-i\Delta f_0(x,y)}-ke^{-i\Delta f(x,y,k)}$$

$$-(k-k_0)e^{-i\Delta\{f(x,y,k)+f_0(x,y)\}}]e^{iP(x,y,k,t)}dk \quad (53)$$

$$+\frac{\cos^2\theta_0}{1+e^{-i\Delta f_0(x,y)}}\int_k i^2(k-k_0)^2S(k)\left\{1+e^{-i\Delta f(x,y,k)}\right\}e^{iP(x,y,k,t)}dk$$

$$\frac{\partial^2 A^\theta(x,y,t)}{\partial y^2}\cong\frac{2\Delta\sin\theta_0\cos\theta_0}{(1+e^{-i\Delta f_0(x,y)})^2}\int_k (k-k_0)S(k)[k_0e^{-i\Delta f_0(x,y)}-ke^{-i\Delta f(x,y,k)}$$

$$-(k-k_0)e^{-i\Delta\{f(x,y,k)+f_0(x,y)\}}]e^{iP(x,y,k,t)}dk \quad (54)$$

$$+\frac{\sin^2\theta_0}{1+e^{-i\Delta f_0(x,y)}}\int_k i^2(k-k_0)^2S(k)\left\{1+e^{-i\Delta f(x,y,k)}\right\}e^{iP(x,y,k,t)}dk$$

Then Eq.(53) and Eq.(54) follow

$$\frac{\partial^2 A^\theta(x,y,t)}{\partial x^2}+\frac{\partial^2 A^\theta(x,y,t)}{\partial y^2}=$$

$$\frac{1}{1+e^{-i\Delta f_0(x,y)}}\int_k i^2(k-k_0)^2S(k)\left\{1+e^{-i\Delta f(x,y,k)}\right\}e^{iP(x,y,k,t)}dk \quad (55)$$

Moreover the third spatial derivatives are obtained and let the second order term or higher of Δ neglect and then we have

$$\frac{1}{\cos\theta_0}\frac{\partial^3 A^\theta(x,y,t)}{\partial x^3}+\frac{1}{\sin\theta_0}\frac{\partial^3 A^\theta(x,y,t)}{\partial y^3}=$$

$$\frac{1}{1+e^{-i\Delta f_0(x,y)}}\int_k i^3(k-k_0)^3 S(k)\left\{1+e^{-i\Delta f(x,y,k)}\right\}e^{iP(x,y,k,t)}dk \qquad (56)$$

In general, the general term is written as

$$\left(\frac{1}{\cos\theta_0}\right)^{n-2}\frac{\partial^n A^\theta(x,y,t)}{\partial x^n}+\left(\frac{1}{\sin\theta_0}\right)^{n-2}\frac{\partial^n A^\theta(x,y,t)}{\partial y^n}=$$

$$\frac{1}{1+e^{-i\Delta f_0(x,y)}}\int_k i^n(k-k_0)^n S(k)\left\{1+e^{-i\Delta f(x,y,k)}\right\}e^{iP(x,y,k,t)}dk. \qquad (57)$$

On the other hand, the time derivative of $A^\theta(x,y,t)$ is

$$\frac{\partial A^\theta(x,y,t)}{\partial t}=$$

$$\frac{1}{1+e^{-i\Delta f_0(x,y)}}\int_k(-i)\left\{\omega(k)-\omega(k_0)\right\}S(k)\left\{1+e^{-i\Delta f(x,y,k)}\right\}e^{iP(x,y,k,t)}dk \qquad (58)$$

Therefore the two-dimensional governing equation of the envelope surface of nearly monochromatic waves with two closely separated propagation directions is obtained by substituting Eq.(7), Eq.(50), Eq.(55) and Eq.(56) into Eq.(58). Using the general term of Eq.(57), we have

$$\frac{\partial A^\theta(x,y,t)}{\partial t}=$$

$$\sum_{n=1}^{\infty}(-1)^n(i)^{n-1}\frac{\omega^{(n)}(k_0)}{n!}\left\{\left(\frac{1}{\cos\theta_0}\right)^{n-2}\frac{\partial^n A^\theta(x,y,t)}{\partial x^n}+\left(\frac{1}{\sin\theta_0}\right)^{n-2}\frac{\partial^n A^\theta(x,y,t)}{\partial y^n}\right\} \qquad (59)$$

Also the linear Schrödinger equation for $A^\theta(x,y,t)$ is obtained as follows.

$$i\left\{\frac{\partial A^\theta(x,y,t)}{\partial t}+\omega'(k_0)\left(\cos\theta_0\frac{\partial A^\theta(x,y,t)}{\partial x}+\sin\theta_0\frac{\partial A^\theta(x,y,t)}{\partial y}\right)\right\}$$

$$+\frac{1}{2!}w''(k_0)\left(\frac{\partial^2 A^\theta(x,y,t)}{\partial x^2}+\frac{\partial^2 A^\theta(x,y,t)}{\partial y^2}\right)=0 \qquad (60)$$

Equation (59) and Eq.(60) are expanded to two propagation directions from Eq.(30) and Eq.(31) with its one propagation direction, respectively. We can find that Eq.(59) and Eq.(60) hold the identical forms of Eq.(30) and Eq.(31).

This fact shows that two-dimensional governing equation of the envelope surface retains the robustness of directionality, that is, the governing equation of the envelope surface of the nearly monochromatic waves with two closely propagation directions is the same of the equation with a single propagation direction. However, this is based on the approximation of Eq.(38) and neglecting the second order term or higher of Δ. Moreover, Eq.(32) is not satisfied with in the case of two closely propagation directions, but stands up in the meaning of approximation of small Δ , that is

$$\frac{\partial^n A^\theta(x,y,t)}{\partial y^n} / \frac{\partial^n A^\theta(x,y,t)}{\partial x^n} \cong \tan^n \theta_0, \text{for small } \Delta, n = 1, 2, 3, \cdots \tag{61}$$

The non-linear version considered in this section is obtained in a similar way of applying Eq.(12) to Eq.(58). The result is also the identical form of Eq.(36) using $A^\theta(x,y,t)$ instead of $A(x,y,t)$.

5.CONCLUDING REMARK

This paper presents that the governing equation of the envelope surface (two-dimensional Schrödinger equation), which is created by the two-dimensional, nearly monochromatic waves, holds the robustness of the propagation direction in its representative form.

In the early stage of this paper, the author reviews the Schrödinger equation of one-dimensional nearly monochromatic waves. Then two-dimensional Schrödinger equation, which is formed by the two-dimensional, nearly monochromatic waves with a single propagation direction, is derived by the straightforward expansion from the one-dimensional case. In order to present the robustness in terms of the propagation direction of the two-dimensional Schrödinger equation, the author derives the governing equation of the envelope surface, which is created by the two-dimensional, nearly monochromatic waves with two closely separated propagation directions. The fact that two derived governing equations, which correspond to a single propagation direction and two propagation directions, have the same expression is shown.

References

[1] Rice, S. O., 'Mathematical analysis of random noise,' *Bell Syst. Tech. J.*, **24**, 1944, pp.46-156.

[2] Longuet-Higgins, M. S., 'The statistical analysis of a random moving surface,' *Philos.Trans. Roy. Soc. London*, Ser. A, **249**, 1957, pp.321-387.

[3] Zabusky, N. J. and Kruskal, M.D., 'Interaction of solitons in a collisionless plasma and the recurrence of initial states,' *Phys. Rev. Lett.*, **15**, 1965, pp.240-243.

[4] Zabusky, N. J., 'A synergetic approach to problems of nonlinear dispersive wave propagation and interaction,' *Proc. Symp. On Nonlinear Partial Difference Equations*, Academic Press, Boston, 1967, pp.223-258.

[5] Zakharov, V.E. and Shabat, A. B., 'Exact theory of two-dimensional self focusing and one-dimensional self modulation of waves in nonlinear media,' *Soviet Phys. J.E.T.P.*, **34**, 1972, pp.62-69.

[6] for example, Agrawal, G. P., *Fiber-Optic Communication System*, 2nd ed., John Wiley & Son, Inc., New York, 1997.

[7] Nohara, B.T., "Consideration of Higher Order Governing Equations of Envelope Surface and Governing Equations of Envelope Surface created by Directionally, Nearly Monochromatic Waves," *Thirteen International Colloquium on Differential Equations*, pp.43, Plovdiv, Bulgaria, August 18-23, 2002.

[8] Nohara, B.T., "Consideration of Higher Order Governing Equations of Envelope Surface and Governing Equations of Envelope Surface created by Directional, Nearly Monochromatic Waves," *International Journal of Differential Equations and Applications*, 2002. (in printing)

[9] Nohara, B. T., "Derivation and Consideration of Governing Equations of Envelope Surface Created by Directional, Nearly MonochromaticWaves," *International Journal of Nonlinear Dynamics and Chaos in Engineering Systems*, submitted.

[10] Haberman, R., *Elementary Applied Partial Differential Equations*, Prentice Hall, 1983.

[11] Whitham, G.B., *Linear and Nonlinear Waves*, John Wiley & Sons, Inc., New York, 1974.