DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 2, 2002
Electronic Journal,
reg. N P23275 at 07.03.97

http://www.neva.ru/journal
e-mail: diff@osipenko.stu.neva.ru

Applications to physics, electrotechnics, and electronics

RELATIVE MOTION OF FLUID SPHERES WITH A FREE SURFACE

B. S. BHATT, A. SHIRLEY

Department Of Mathematics & Computer Science
The University Of The West Indies, St. Augustine
Trindad, W.I.
E-mail: bbhatt@centre.uwi.tt

Abstract

Relative motion of liquid spheres of different viscosities has been considered when the surface of the outer sphere is a free surface. The exact solution has been found using Stokes' approximation. Drag force experienced by the inner sphere has been determined and several cases of interest have been derived. The correction to the drag expression up to first order in Reynolds number has also been obtained. The effect of viscosity ratio and the Reynolds number on the drag has been shown graphically.

1 Introduction

Relative motion of solid and liquid spheres is of great importance to industrial and engineering applications, such as in the flow of fluids in fluidized beds or fixed beds, the sedimentation of fine particulate suspensions and the flow of oil in oil fields/reservoirs during oil recovery.

A great deal of work has been carried out on the flow of Newtonian fluids while considering the relative motion of solid/liquid spheres. One can refer to the problems given in the book of Happel and Brenner[1]. Some related problems for porous spheres have been done by Bhatt [2], Bhatt and Owen [3] and Neale et.al.[4].

In the present problem we have considered the relative motion of two liquid spheres of viscosity μ_i and μ_o (inner and outer fluid regions). The inner region is $0 < r < a$ whereas the outer region is $a < r < b$. The surface $r = b$ is a free surface. The drag force experienced by the inner sphere has been obtained and the result has been extended to the first order in Reynolds number following Verma and Bhatt [5].

2 Equations of motion

We consider the free surface cell (bounded) model of Happel [6] when a fluid sphere (of viscosity μ_i) is surrounded by a fluid (of viscosity μ_o) which is moving with velocity V. The two fluids are taken to be immiscible. Therefore there are two flow regions, namely, $0 < r < a$ and $a < r < b$. Using the Stokes' approximation for slow motion the flows are given as follows:

The region I: $0 < r < a$ is governed by the Navier-Stokes equations, given by

$$E^4\psi^i = 0, \tag{2.1}$$

the region II: $a < r < b$ is governed by the Navier-Stokes equations, given by

$$E^4\psi^{(o)} = 0, \tag{2.2}$$

where ψ^i, ψ^o are the stream functions in the regions I and II respectively, given by

$$
\begin{aligned}
(u_{r1}, u_{r2}) &= \frac{1}{r^2 \sin\theta}\left(\frac{\partial\psi^{(i)}}{\partial\theta}, \frac{\partial\psi^{(o)}}{\partial\theta}\right),\\
(u_{\theta 1}, u_{\theta 2}) &= -\frac{1}{r \sin\theta}\left(\frac{\partial\psi^{(i)}}{\partial r}, \frac{\partial\psi^{(o)}}{\partial r}\right),
\end{aligned} \tag{2.3}
$$

$(u_{ri}, u_{\theta i})$, $i = 1, 2$ are the velocity components. The boundary conditions used are:

$$
\text{as } r \to 0,\ u_{r1},\ u_{\theta 1} \text{ remain finite,} \tag{2.4}
$$

$$
\left.
\begin{aligned}
&u_{r1} = u_{r2} = 0,\\
&u_{\theta 1} = u_{\theta 2},\\
&\mu_i \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi^{(i)}}{\partial r}\right) = \mu_o \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi^{(o)}}{\partial r}\right)
\end{aligned}
\right\} \text{ at } r = a, \tag{2.5}
$$

$$
\left.
\begin{aligned}
&u_{r2} = V\cos\theta,\\
&r\frac{\partial}{\partial r}\left(\frac{u_{\theta 2}}{r}\right) + \frac{1}{r}\frac{\partial u_{r2}}{\partial\theta} = 0
\end{aligned}
\right\} \text{ at } r = b, \tag{2.6}
$$

3 Solutions

The solution of equations (2.1) and (2.2) can be obtained, using (see Happel and Brenner [1])

$$
(\psi^{(i)}, \psi^{(o)}) = (f^{(i)}(r), f^{(o)}(r))\sin^2\theta, \tag{3.1}
$$

which lead to

$$
\psi^{(i)} = \left[\frac{1}{10}A_1 r^4 - \frac{1}{2}B_1 r + C_1 r^2 + \frac{D_1}{r}\right] V \sin^2\theta \tag{3.2}
$$

$$
\psi^{(o)} = \left[\frac{1}{10}A_2 r^4 - \frac{1}{2}B_2 r + C_2 r^2 + \frac{D_2}{r}\right] V \sin^2\theta \tag{3.3}
$$

Using (2.3), (3.1) and (3.2), we obtain

$$
u_{ri} = V\cos\theta\left[\frac{1}{5}A_i r^2 - \frac{B_i}{r} + 2C_i + \frac{2D_i}{r^3}\right], \tag{3.4}
$$

$$
u_{\theta i} = -V\sin\theta\left[\frac{2}{5}A_i r^2 - \frac{B_i}{2r} + 2C_i - \frac{D_i}{r^3}\right], \tag{3.5}
$$

where $i = 1, 2$.

Using the boundary conditions (2.4)-(2.6) we get:

$$B_1 = D_1 = 0, \tag{3.6}$$

$$A_1 = -\frac{10C_1}{a^2}, \tag{3.7}$$

$$A_2 = -\frac{10D_2}{b^5}, \tag{3.8}$$

$$C_1 = \frac{D_2}{2a^3}\left(1 + 4\frac{a^5}{b^5}\right) + \frac{B_2}{4a}\left(1 - 2\frac{a}{b}\right) - \frac{1}{2}, \tag{3.9}$$

$$C_2 = \frac{1}{2}\left(1 + \frac{B_2}{b}\right), \tag{3.10}$$

$$\frac{B_2 a^2}{2}\left(\frac{a}{b} - 1\right) + D_2\left(1 - \frac{a^5}{b^5}\right) + \frac{a^3}{2} = 0, \tag{3.11}$$

$$B_2 = \frac{a\left[\frac{3}{2} + \eta^5 + \sigma(1 - \eta^5)\right]}{1 + \sigma - \left(\frac{3}{2} + \sigma\right)\eta + \left(\frac{3}{2} - \sigma\right)\eta^5 - (1 - \sigma)\eta^6}, \tag{3.12}$$

with $\sigma = \frac{\mu_o}{\mu_i}$ and $\eta = \frac{a}{b}$.

4 Drag force

The drag force experienced by the inner fluid sphere is given by (see Happel and Brenner [1]):

$$D_r = -4\mu_o \pi V B_2. \tag{4.1}$$

We define

$$\Omega_1 = \frac{D_r}{-6\pi\mu_o V a} = \frac{1 + \frac{2}{3}\eta^5 + \frac{2}{3}\sigma(1 - \eta^5)}{1 + \sigma - \left(\frac{3}{2} + \sigma\right)\eta + \left(\frac{3}{2} - \sigma\right)\eta^5 - (1 - \sigma)\eta^6}. \tag{4.2}$$

Some special cases:

For rigid sphere ($\sigma = 0$), (4.2) gives

$$\Omega_1 = \frac{1 + \frac{2}{3}\eta^5}{1 - \frac{3}{2}\eta + \frac{3}{2}\eta^5 - \eta^6}, \tag{4.3}$$

which is the well known Happel's [6] formula.

For sphere with viscosity equal to that of external medium ($\sigma = 1$), (4.2) gives

$$\Omega_1 = \frac{\frac{5}{3}}{2 - \frac{5}{2}\eta + \frac{1}{2}\eta^5 - \eta^6}. \tag{4.4}$$

For fluid sphere of vanishing viscosity ($\sigma = \infty$), (4.2)

$$\Omega_1 = \frac{\frac{2}{3}(1 - \eta^5)}{1 - \eta - \eta^5 + \eta^6}. \tag{4.5}$$

(4.5) is the drag force on a gaseous bubble rising through a liquid where $\mu_o >> \mu_i$.

Following the method of matched asymptotic technique of Proudman and Pearson [7] as used by Verma and Bhatt [5], we can extend the result in (4.2) to first order in the Reynolds number when the viscous terms are included in the equations of motion.

We use the formula as obtained by Verma and Bhatt [5] to get the drag force experienced by the inner fluid sphere as:

$$D_{r1} = -4\mu_o \pi V B_2 \left[1 + \frac{B_2}{4}R\right] = -6\mu_o \pi V a \Omega_1 \left(1 + \frac{3}{8}R\Omega_1\right), \tag{4.6}$$

where $R = \frac{Va}{\nu_o}$, ν_0 is kinematic viscosity of the outer liquid.

(4.6) leads to:

$$\Omega = \Omega_1 \left(1 + \frac{3}{8}R\Omega_1\right), \tag{4.7}$$

where $\Omega = \frac{D_{r1}}{-6\pi\mu_o Va}$. (4.6) gives the drag force on the inner fluid sphere up to first order in the Reynolds number.

For $\eta \to 0$, (4.6) reduces to

$$D_{r1} = -6\mu_o \pi V a \frac{1 + \frac{2}{3}\sigma}{1 + \sigma}\left[1 + \frac{3R(1 + \frac{2}{3}\sigma)}{8(1 + \sigma)}\right]. \tag{4.8}$$

(4.8) is the drag force on a fluid sphere up to first order in the Reynolds number which is surrounded by an infinite extent of different liquid. For $R = 0$, (4.8) agrees with the result obtained in [1].

Using (4.2) and (4.7) the behaviour of Ω verses η with σ and R has been given in figure 1. Ω remains constant up to $\eta = 0.4$ and then increases with R but decreases with σ (similar behaviour has been obtained in [1]). As σ increases $\mu_o > \mu_i$, therefore the inner sphere experiences less force, whereas when R increases the velocity of free surface increases and therefore the inner sphere experiences more force. In figure 2 and 3 we can see the effect of R on Ω when $\sigma = 0$ and $\sigma = \infty$ respectively [using the equations (4.3) and (4.5)].

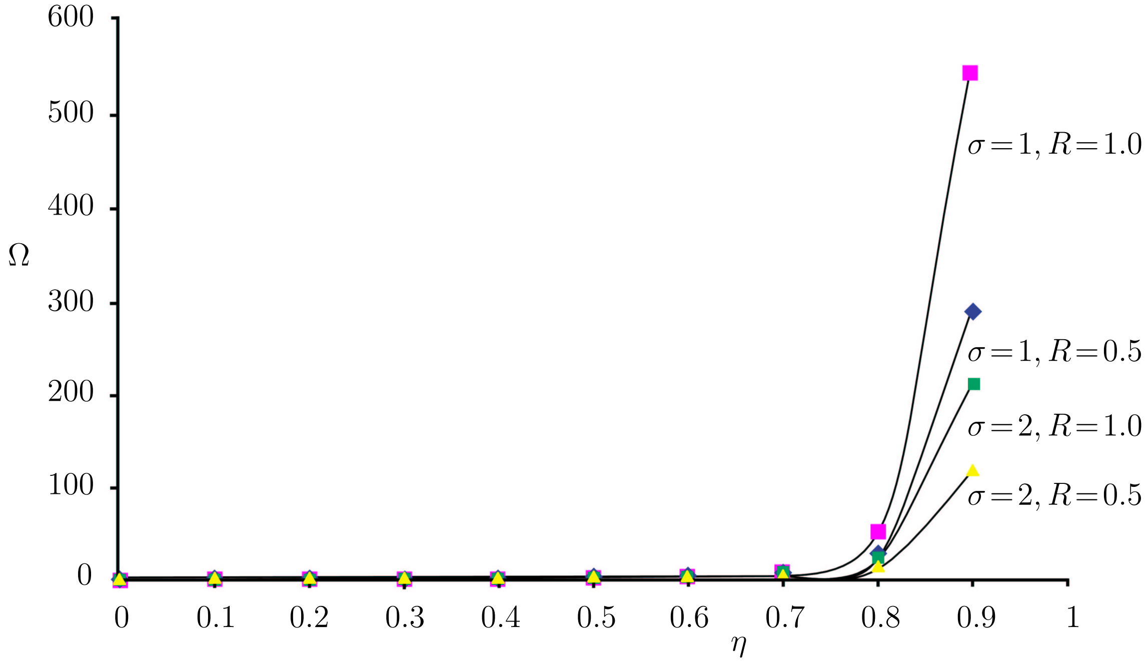

Fig.1

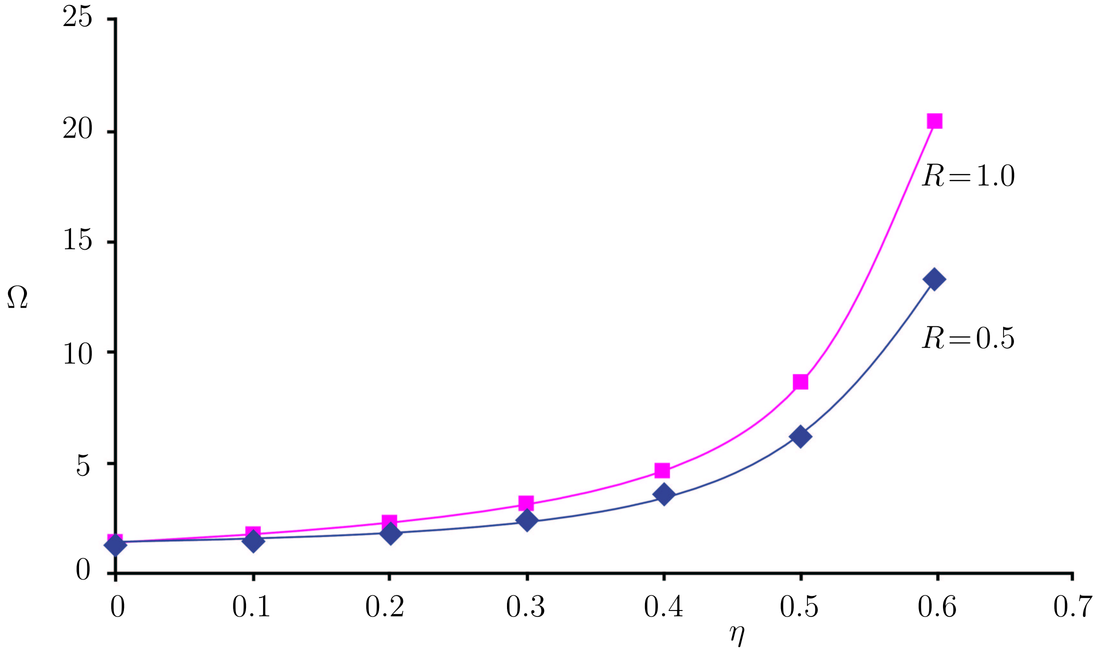

Fig.2

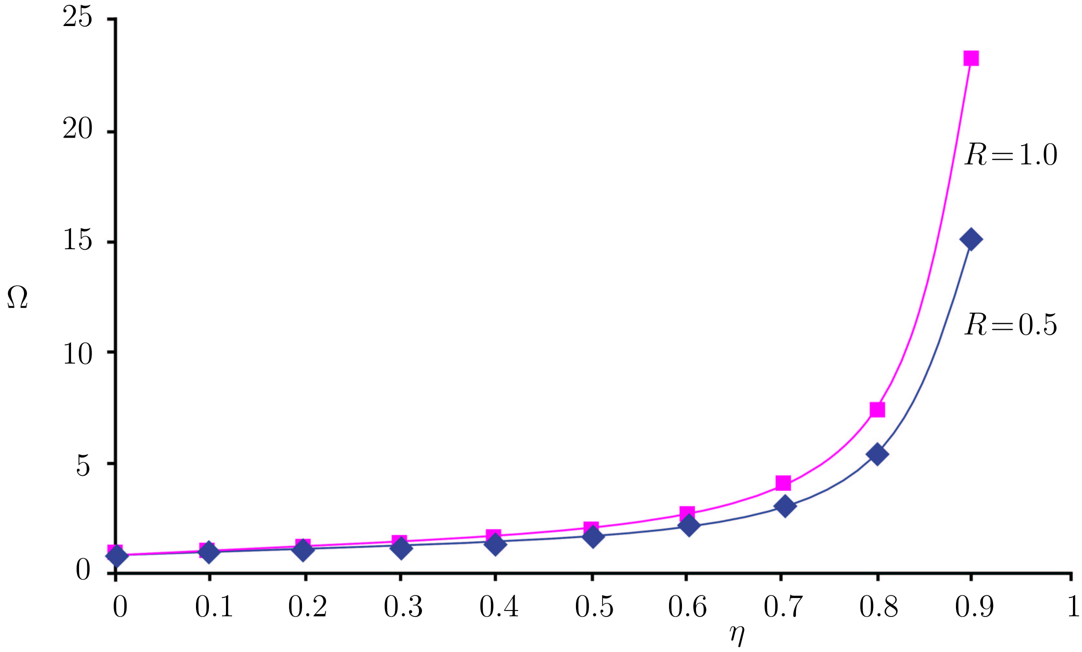

Fig.3

References

[1] Happel, J. and Brenner, H., Low Reynolds Number Hydrodynamics. Noordhoff Int. Publ. The Netherlands(1973).

[2] Bhatt,B.S., Relative motion of liquid and porous spheres. Modelling, Measurement and control B, ASME46,2(1992)55-64.

[3] Bhatt,B.S. and Owen,D., Relative motion of fluid spheres with porous core. To appear in Ensenanza Universitaria (2002).

[4] Neale, G., Epstein, N. and Nadar, W., Creeping flow relative to permeable spheres. Chem. Eng. Sci. 28(1973) 1865-1874.

[5] Verma, P.D. and Bhatt, B.S., Flow past a porous spherical shell using matched asymptotic technique. J. de Mecanique 14(1975)421-434.

[6] Happel, J., Viscous flow in multiparticle system: slow motion of fluids relative to beds of spherical particles. AIChE J. 4(1958)197-201.

[7] Proudman, I. and Pearson, R.R.A., Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder. J. Fluid Mech. 2(1957)237-262.