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INHOMOGENEOUS, NON-LINEAR AND ANISOTROPIC SYSTEMS WITH RANDOM MAGNETISATION MAIN DIRECTIONS

Part one: The useful form of functional 3D for computation the field created by permanent magnets

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Abstract

The paper treats the *transformation of the concentrated form of the functional* (associated to the finite elements method – FEM) attached to the resolution of the three dimensional (3D) field from the inhomogeneous, non-linear and anisotropic systems that have random main directions of magnetisation. We have got the *useful form of the functional* which is used in the numerical computation of the distribution of the field created by permanent magnets.

Key words: permanent magnets, FEM, inhomogenity, nonlinearity, anisotropy.

1. Introduction

In practice, there appear situations when it is necessary to establish the distribution of the magnetic field in inhomogeneous, non-linear and anisotropic systems (concerning both permanent magnets – field sources – and ferromagnetic yokes existing in the real technical system).

The concentrated form of the functional, valid for the systems with permanent magnets [1, 2], written in Eq(1)

$$F = \int_v \left(\int_0^{\bar{\boldsymbol{H}}} \bar{\boldsymbol{B}} \cdot d\bar{\boldsymbol{H}} \right) dv + \int_{S_N} \bar{\boldsymbol{B}} \cdot \bar{\boldsymbol{n}}\, V_H \, ds \tag{1}$$

must be transformed in order to be used in computations, as it follows. We have made the mention that v is the domain (volume) where the field problem has been analysed, and S_N is its boundary with Neumann conditions. Eq(1) contains three unknown quantities: $\bar{\boldsymbol{B}}$ – flux density, $\bar{\boldsymbol{H}}$ – magnetic field intensity and V_H – magnetic scalar potential. In the process of minimising of the functional it is necessary an expression containing only one variable.

The field sources being permanent magnets, $\bar{\boldsymbol{H}}$ is obtained from the magnetic scalar potential V_H [1, 3]:

$$\bar{\boldsymbol{H}} = -\operatorname{grad} V_H. \tag{2}$$

In Fig.1 we have considered domain (volume) $v = v' \cup v''$, where we have analysed the problem of the magnetic field. The sub-domain v' is without permanent magnetisation ($\bar{\boldsymbol{M}}_p = 0$), and sub-domain v'' is with permanent magnetisation ($\bar{\boldsymbol{M}}_p \neq 0$, the permanent magnet). The boundary of the 3D domain

$$\Sigma = S'_N \cup S'_D \cup S''_N \cup S''_D \tag{3}$$

is a closed surface and it contains – for generality – mixed conditions, for both the area without permanent magnetisation $S' = S'_N \cup S'_D$,

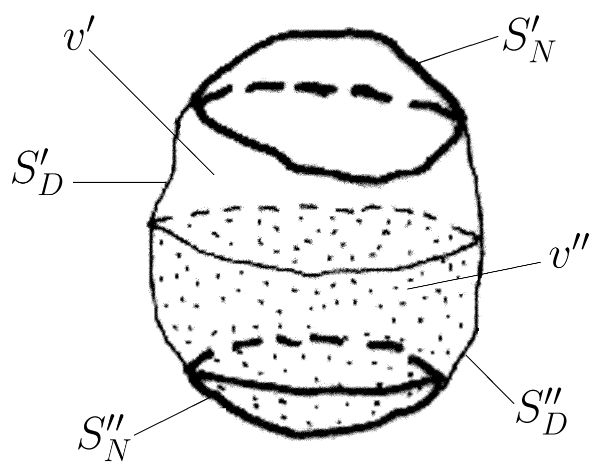

Fig.1. The study domain

and for that with permanent magnet $S'' = S''_N \cup S''_D$. For S'_N and S''_N there have to be known the Neumann conditions and for S'_D and S''_D – the Dirichlet conditions. In a general case there can be more zones with permanent magnetisation, when v'' is a reunion of there and v' is the reunion of all the zones without permanent magnetisation, having a similar meaning as those for boundaries S'' and S'.

2. Anisotropic and non-linear 3D permanent magnets with random main directions of magnetisation

For anisotropic media with permanent magnets, the law of the link between $\bar{\boldsymbol{B}}, \bar{\boldsymbol{H}}$ and $\bar{\boldsymbol{M}}$ has become

$$\bar{\boldsymbol{B}} = \bar{\bar{\mu}}_p \bar{\boldsymbol{H}} + \mu_0 \bar{\boldsymbol{M}}_p, \tag{4}$$

where $\bar{\bar{\mu}}_p$ is tensor of magnetic permeability in the zone with permanent magnet [4]. For the ferromagnetic yokes, considered anisotropic, $\bar{\boldsymbol{B}} = \bar{\bar{\mu}} \bar{\boldsymbol{H}}$, and for the non-ferromagnetic zones (e.q. the zones with air), $\bar{\boldsymbol{B}} = \mu_0 \bar{\boldsymbol{H}}$. It is known that, following the main directions of magnetisation, the components of the flux density are collinear with the components of the magnetic field intensity, namely tensors $\bar{\bar{\mu}}_p$ and $\bar{\bar{\mu}}$ have diagonal matrices [4, 5]. Taking into consideration a 3D domain where the main directions of magnetisation, in random disposition, are noted with $(u, v, w)'$ in the zones with $\bar{\boldsymbol{M}}_p = 0$, and with $(u, v, w)''$ in the permanent magnet, the linking relations between $\bar{\boldsymbol{B}}$ and $\bar{\boldsymbol{H}}$ after there directions are

$$B_\nu = \mu_\nu H_\nu, \quad \nu = u', v', w' \tag{5}$$

for ferromagnetic yokes, and

$$B_\nu = \mu_{p\nu} H_\nu + \mu_0 M_{p\nu}, \quad \nu = u'', v'', w'' \tag{6}$$

for the zones with permanent magnetisation. For the considered case, tensors $\bar{\bar{\mu}}_p$ and $\bar{\bar{\mu}}$ have got non-linear components, depending on $\bar{\boldsymbol{H}}$. Eq(5) show that a component of the flux density following one of the main directions of magnetisation depends only on the components of the magnetic field intensity following the same direction, namely after the main directions of magnetisation $\bar{\boldsymbol{B}}$ and $\bar{\boldsymbol{H}}$ components are collinear. In other words, the components following the main directions of magnetisation do not depend one on another. For zones with permanent magnets (Eq.6) there also intervene the components of permanent magnetisation, which have to be known for a given magnet.

Within the finite element method (FEM), the domain of 3D analysis is divided in m finite elements (e.q. tetrahedra), of sufficiently small elements, so that the calculus accuracy be good. In this case, functional (1) can be written

$$F = \sum_{\lambda=1}^{m} F_\lambda. \tag{7}$$

Taking into consideration Eq(7) and the mentions we have previously done, the functional becomes

$$\begin{aligned} F = {} & \sum_{\lambda=1}^{m'} \left\{ \int_{v'_\lambda} \left(\int_0^{\bar{\boldsymbol{H}}_\lambda} \bar{\bar{\mu}}_\lambda \bar{\boldsymbol{H}}_\lambda \, d\bar{\boldsymbol{H}}_\lambda \right) dv \right\} + \\ & + \sum_{\lambda=1}^{m''} \left\{ \int_{v''_\lambda} \left[\int_0^{\bar{\boldsymbol{H}}_\lambda} \left(\bar{\bar{\mu}}_{p\lambda} \bar{\boldsymbol{H}}_\lambda + \mu_0 \bar{\boldsymbol{M}}_{p\lambda} \right) d\bar{\boldsymbol{H}}_\lambda \right] dv \right\} + \sum_{\lambda=1}^{m'_N} \left\{ \int_{S'_{N\lambda}} \left(\bar{\bar{\mu}}_\lambda \bar{\boldsymbol{H}}_\lambda \right) \cdot \bar{\boldsymbol{n}} V_{H\lambda} ds \right\} + \\ & + \sum_{\lambda=1}^{m''_N} \left\{ \int_{S''_{N\lambda}} \left(\bar{\bar{\mu}}_{p\lambda} \bar{\boldsymbol{H}}_\lambda + \mu_0 \bar{\boldsymbol{M}}_{p\lambda} \right) \cdot \bar{\boldsymbol{n}} V_{H\lambda} \, ds \right\}, \end{aligned} \tag{8}$$

where m' is the number of finite elements in v', m'' is the number of finite elements in v'' $(m = m' + m'')$, m'_N is the number of finite elements adjacent to boundary S'_N, m''_N is the number of finite elements adjacent to boundary S''_N, and index λ represents the fact that the quantities refer to finite element λ.

For the ferromagnetic materials (non-linear and anisotropic) contained in sub-domain v', there are not known in advance the working points (for each finite element) on the magnetisation curves, curves that have to be known. Therefore, the components of tensors $\bar{\bar{\mu}}_\lambda$ are initially unknown. The process of solving the field problem finally implies solving a system of equations (obtained through minimising the functional (8)) where the unknown quantities are magnetic scalar potentials V_{Hi} $(i = \overline{1, n})$ from the n nodes of the discretization mesh of study domain v. As $\bar{\bar{\mu}}_\lambda$ has got unknown values, solving the system of equations implies more iterations, starting from initially chosen values for the components of the tensor, and correcting them at each iteration. If the process is convergent, there are finally obtained stable values for $\bar{\bar{\mu}}_\lambda$, differing from one finite element to the other. Thus, in an iteration, the resolution of the system obtained from Eq(8) is done for the steady components of tensors $\bar{\bar{\mu}}_\lambda$ (as against $\bar{\boldsymbol{H}}_\lambda$), but different for different λ.

For the non-linear and anisotropic permanent magnets contained in subdomain v'' the problem is alike, but we have to take into consideration $\bar{\bar{\mu}}_{p\lambda}$. The demagnetisation curves of the magnets following each main direction have to be known.

The unit vectors of the random main directions $(u, v, w)'$ of zone v' are considered $(\bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2, \bar{\boldsymbol{e}}_3)'$, and the unit vectors of the random main directions $(u, v, w)''$ of the permanent magnet (zone v'') are considered $(\bar{\boldsymbol{e}}_1, \bar{\boldsymbol{e}}_2, \bar{\boldsymbol{e}}_3)''$ (see Fig.2, where index $'$ is for v' and index $''$ for v''). If we take into consideration Eqs(5) and (6) and the above mentioned observations, the terms of functional (8) can be written as following:

– for zone v'

$$\int_0^{\bar{\boldsymbol{H}}_\lambda} \left(\bar{\bar{\mu}}_\lambda \bar{\boldsymbol{H}}_\lambda d\bar{\boldsymbol{H}}_\lambda\right)' = \int_0^{\bar{\boldsymbol{H}}_\lambda} (\mu_u H_u \bar{\boldsymbol{e}}_1 + \mu_v H_v \bar{\boldsymbol{e}}_2 +$$
$$+ \mu_w H_w \bar{\boldsymbol{e}}_3)'_\lambda \cdot (dH_u \bar{\boldsymbol{e}}_1 + dH_v \bar{\boldsymbol{e}}_2 + dH_w \bar{\boldsymbol{e}}_3)'_\lambda.$$

Taking into consideration that $\cos\varphi_{rs} = \cos\varphi_{sr}$ (where $r, s = 1, 2, 3$) and processing this term, we have got

$$\int_0^{\bar{\boldsymbol{H}}_\lambda} \left(\bar{\bar{\mu}}_\lambda \bar{\boldsymbol{H}}_\lambda d\bar{\boldsymbol{H}}_\lambda\right)' = \left[\tfrac{1}{2}\left(\mu_u H_u^2 + \mu_v H_v^2 + \mu_w H_w^2\right) + \right.$$
$$+ H_u H_v (\mu_u + \mu_v)\cos\varphi_{12} + H_v H_w (\mu_v + \mu_w)\cos\varphi_{23} +$$
$$\left. + \ H_w H_u (\mu_w + \mu_u)\cos\varphi_{31}\right]'_\lambda; \quad (9)$$

– for zone v'', similarly, we have got

$$\int_0^{\bar{\boldsymbol{H}}_\lambda} \left[\left(\bar{\bar{\mu}}_{p\lambda} \bar{\boldsymbol{H}}_\lambda + \mu_0 \bar{\boldsymbol{M}}_{p\lambda}\right) d\bar{\boldsymbol{H}}_\lambda\right]'' = \int_0^{H_\lambda} [(\mu_{pu} H_u + \mu_0 M_{pu})\,\bar{\boldsymbol{e}}_1 + (\mu_{pv} H_v + \mu_0 M_{pv})\,\bar{\boldsymbol{e}}_2 +$$
$$+ (\mu_{pw} H_w + \mu_0 M_{pw})\,\bar{\boldsymbol{e}}_3]''_\lambda \cdot [(dH_u \bar{\boldsymbol{e}}_1 + dH_v \bar{\boldsymbol{e}}_2 + dH_w \bar{\boldsymbol{e}}_3)]''_\lambda =$$
$$= \left\{\tfrac{1}{2}\left(\mu_{pu} H_u^2 + \mu_{pv} H_v^2 + \mu_{pw} H_w^2\right) + H_u H_v (\mu_{pu} + \mu_{pv})\cos\varphi_{12} + \right.$$
$$+ H_v H_w (\mu_{pv} + \mu_{pw})\cos\varphi_{23} + H_w H_u (\mu_{pw} + \mu_{pu})\cos\varphi_{31} +$$
$$+ \mu_0 [H_u (M_{pu} + M_{pv}\cos\varphi_{12} + M_{pw}\cos\varphi_{31}) + H_v (M_{pu}\cos\varphi_{12} +$$
$$\left. + M_{pv} + M_{pw}\cos\varphi_{23}) + H_w (M_{pu}\cos\varphi_{31} + M_{pv}\cos\varphi_{23} + M_{pw})]\right\}''_\lambda. \quad (10)$$

In Eq(8) there are terms having the form $(\bar{\boldsymbol{B}} \cdot \bar{\boldsymbol{n}})_\lambda = (B_n)_\lambda$, namely the normal components of the flux density (Neumann conditions) on boundaries $S'_{N\lambda}$ and $S''_{N\lambda}$ of finite elements λ adjacent to them. In order to write the equations as against a rectangular system of axes, there are going to be expressed the components following the main directions of magnetisation of $\bar{\boldsymbol{H}}$ and $\bar{\boldsymbol{M}}_p$, depending on their components following the axes of the rectangular system.

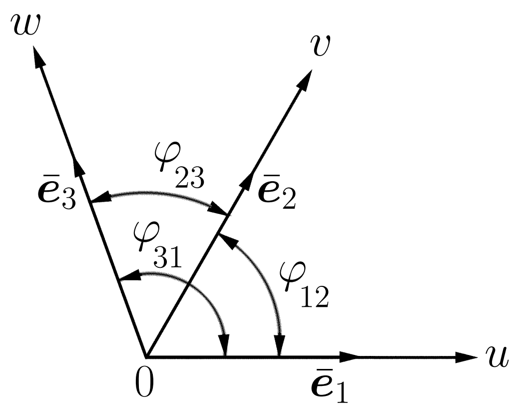

Fig.2. The magnetisation main directions

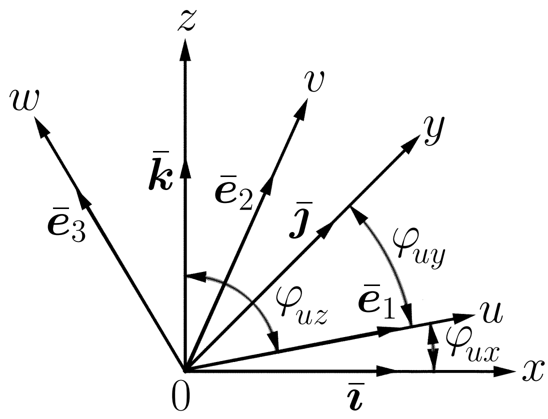

Fig.3. The position of the reference systems

In this context, in Fig.3 there is taken into consideration the general case of relative disposal of the rectangular system (x, y, z) against the system attached to the main directions of magnetisation (u, v, w). The representation has been done by a single drawing for both subdomain v' and v'', having the following meaning: for v', the system of axes $(u, v, w)'$ and angles $(\varphi_{ux}, \varphi_{uy}, \varphi_{uz}, \dots)'$ will wear index $'$; for v'', the axes and angles will be marked by index $''$, namely $(u, v, w)''$ and $(\varphi_{ux}, \varphi_{uy}, \varphi_{uz}, \dots)''$. For reasons of clearness of the drawing, in Fig.3 we have noted only part of the angles, the rest of them being obvious.

In order to obtain the linking relations between the components in the two systems of reference, in a random finite element λ we have expressed $\bar{\boldsymbol{H}}_\lambda$ in system (x, y, z) and then in system (u, v, w). The formulation is similar in subdomains v' and v'', thus the appropriate indices ($'$ and $''$) being required only when the equations are completely written (22 and 29). Thus, on one hand

$$\bar{\boldsymbol{H}}_\lambda = \left(H_x\bar{\boldsymbol{\imath}} + H_y\bar{\boldsymbol{\jmath}} + H_z\bar{\boldsymbol{k}}\right)_\lambda, \tag{11}$$

and on the other

$$\bar{\boldsymbol{H}}_\lambda = (H_u\bar{\boldsymbol{e}}_1 + H_v\bar{\boldsymbol{e}}_2 + H_w\bar{\boldsymbol{e}}_3)_\lambda\,. \tag{12}$$

Through identification, after calculus, from Eqs(11) and (12) there results:

$$\begin{aligned}
(H_x)_\lambda &= (H_u\cos\varphi_{ux} + H_v\cos\varphi_{vx} + H_w\cos\varphi_{wx})_\lambda\,, &(13)\\
(H_y)_\lambda &= (H_u\cos\varphi_{uy} + H_v\cos\varphi_{vy} + H_w\cos\varphi_{wy})_\lambda\,, &(14)\\
(H_z)_\lambda &= (H_u\cos\varphi_{uz} + H_v\cos\varphi_{vz} + H_w\cos\varphi_{wz})_\lambda\,. &(15)
\end{aligned}$$

From (13, 14, 15), applying the rule of Cramer [6], there results $(H_u)_\lambda$, $(H_v)_\lambda$, $(H_w)_\lambda$ expressed in comparison with $(H_x)_\lambda$, $(H_y)_\lambda$, $(H_z)_\lambda$:

$$\begin{aligned}
(H_u)_\lambda &= (k_{ux}H_x + k_{uy}H_y + k_{uz}H_z)_\lambda\,, &(16)\\
(H_v)_\lambda &= (k_{vx}H_x + k_{vy}H_y + k_{vz}H_z)_\lambda\,, &(17)\\
(H_w)_\lambda &= (k_{wx}H_x + k_{wy}H_y + k_{wz}H_z)_\lambda\,, &(18)
\end{aligned}$$

where coefficients k_{rs} $(r = u, v, w$ and $s = x, y, z)$ result trough identification and are steady values to established systems of axes (they depend only on angles φ_{rs} between the two systems of axes).

In zone v'', for $\bar{\boldsymbol{M}}_p$ we have done the same, and similar relations result, but using index $''$:

$$\begin{aligned}
(M_{pu})''_\lambda &= (k_{ux}M_{px} + k_{uy}M_{py} + k_{uz}M_{pz})''_\lambda\,, &(19)\\
(M_{pv})''_\lambda &= (k_{vx}M_{px} + k_{vy}M_{py} + k_{vz}M_{pz})''_\lambda\,, &(20)\\
(M_{pw})''_\lambda &= (k_{wx}M_{px} + k_{wy}M_{py} + k_{wz}M_{pz})''_\lambda\,. &(21)
\end{aligned}$$

If Eqs(16), (17) and (18) are replaced – written with index $'$ – in Eq(9), there results

$$\int_0^{\bar{\boldsymbol{H}}_\lambda} \left(\bar{\bar{\mu}}_\lambda\bar{\boldsymbol{H}}_\lambda d\bar{\boldsymbol{H}}_\lambda\right)' = \big(A_{xx}H_x^2 + A_{yy}H_y^2 + A_{zz}H_z^2 + \\ + A_{xy}H_xH_y + A_{yz}H_yH_z + A_{zx}H_zH_x\big)'_\lambda\,, \tag{22}$$

where the following notations have been made:

$$\begin{aligned}
(A_{xx})'_\lambda &= \left[\tfrac{1}{2}\left(\mu_uk_{ux}^2 + \mu_vk_{vx}^2 + \mu_wk_{wx}^2\right) + k_{ux}k_{vx}\left(\mu_u + \mu_v\right)\cos\varphi_{12} + \right.\\
&\left. + k_{vx}k_{wx}\left(\mu_v + \mu_w\right)\cos\varphi_{23} + k_{wx}k_{ux}\left(\mu_w + \mu_u\right)\cos\varphi_{31}\right]'_\lambda\,, &(23)\\
(A_{yy})'_\lambda &= \left[\tfrac{1}{2}\left(\mu_uk_{uy}^2 + \mu_vk_{vy}^2 + \mu_wk_{wy}^2\right) + k_{uy}k_{vy}\left(\mu_u + \mu_v\right)\cos\varphi_{12} + \right.\\
&\left. + k_{vy}k_{wy}\left(\mu_v + \mu_w\right)\cos\varphi_{23} + k_{wy}k_{uy}\left(\mu_w + \mu_u\right)\cos\varphi_{31}\right]'_\lambda\,, &(24)\\
(A_{zz})'_\lambda &= \left[\tfrac{1}{2}\left(\mu_uk_{uz}^2 + \mu_vk_{vz}^2 + \mu_wk_{wz}^2\right) + k_{uz}k_{vz}\left(\mu_u + \mu_v\right)\cos\varphi_{12} + \right.\\
&\left. + k_{vz}k_{wz}\left(\mu_v + \mu_w\right)\cos\varphi_{23} + k_{wz}k_{uz}\left(\mu_w + \mu_u\right)\cos\varphi_{31}\right]'_\lambda\,, &(25)
\end{aligned}$$

$$\begin{aligned}
(A_{xy})'_\lambda &= [\mu_u k_{ux} k_{uy} + \mu_v k_{vx} k_{vy} + \mu_w k_{wx} k_{wy} + \\
&+ (k_{ux} k_{vy} + k_{vx} k_{uy}) (\mu_u + \mu_v) \cos \varphi_{12} + \\
&+ (k_{vx} k_{wy} + k_{wx} k_{vy}) (\mu_v + \mu_w) \cos \varphi_{23} + \\
&+ (k_{wx} k_{uy} + k_{ux} k_{wy}) (\mu_w + \mu_u) \cos \varphi_{31}]'_\lambda , \quad (26) \\
(A_{yz})'_\lambda &= [\mu_u k_{uy} k_{uz} + \mu_v k_{vy} k_{vz} + \mu_w k_{wy} k_{wz} + \\
&+ (k_{uy} k_{vz} + k_{vy} k_{uz}) (\mu_u + \mu_v) \cos \varphi_{12} + \\
&+ (k_{vy} k_{wz} + k_{wy} k_{vz}) (\mu_v + \mu_w) \cos \varphi_{23} + \\
&+ (k_{wy} k_{uz} + k_{uy} k_{wz}) (\mu_w + \mu_u) \cos \varphi_{31}]'_\lambda , \quad (27) \\
(A_{zx})'_\lambda &= [\mu_u k_{uz} k_{ux} + \mu_v k_{vz} k_{vx} + \mu_w k_{wz} k_{wx} + \\
&+ (k_{uz} k_{vx} + k_{vz} k_{ux}) (\mu_u + \mu_v) \cos \varphi_{12} + \\
&+ (k_{vz} k_{wx} + k_{wz} k_{vx}) (\mu_v + \mu_w) \cos \varphi_{23} + \\
&+ (k_{wz} k_{ux} + k_{uz} k_{wx}) (\mu_w + \mu_u) \cos \varphi_{31}]'_\lambda . \quad (28)
\end{aligned}$$

Similarly, if Eqs (16), (17) and (18) – written with index $''$ – and Eqs (19), (20) and (21) are replaced in (10), we are going to obtain

$$\int_0^{\bar{\boldsymbol{H}}_\lambda} \left[\left(\bar{\bar{\mu}}_{p\lambda} \bar{\boldsymbol{H}}_\lambda + \mu_0 \bar{\boldsymbol{M}}_{p\lambda} \right) d\bar{\boldsymbol{H}}_\lambda \right]'' = \big(A_{xx} H_x^2 + A_{yy} H_y^2 + A_{zz} H_z^2 + A_{xy} H_x H_y + \\ + A_{yz} H_y H_z + A_{zx} H_z H_x \big)''_\lambda + (K_x H_x + K_y H_y + K_z H_z)''_\lambda , \quad (29)$$

where: $(A_{xx})''_\lambda , (A_{yy})''_\lambda , (A_{zz})''_\lambda , (A_{xy})''_\lambda , (A_{yz})''_\lambda , (A_{zx})''_\lambda$ have got similar expression with those in relations $(23 \div 28)$, but – being for v'' – they will contain the elements of matrix $\bar{\bar{\mu}}_p$, and coefficients $(k_{r,s})''_\lambda$ with $r = (u, v, w)''$ and $s = (x, y, z)''$ are written with index $''$;

$$\begin{aligned}
(K_x)''_\lambda &= \mu_0 [M_{pu} (k_{ux} + k_{vx} \cos \varphi_{12} + k_{wx} \cos \varphi_{31}) + \\
&+ M_{pv} (k_{ux} \cos \varphi_{12} + k_{vx} + k_{wx} \cos \varphi_{23}) + \\
&+ M_{pw} (k_{ux} \cos \varphi_{31} + k_{vx} \cos \varphi_{23} + k_{wx})]''_\lambda , \quad (30) \\
(K_y)''_\lambda &= \mu_0 [M_{pu} (k_{uy} + k_{vy} \cos \varphi_{12} + k_{wy} \cos \varphi_{31}) + \\
&+ M_{pv} (k_{uy} \cos \varphi_{12} + k_{vy} + k_{wy} \cos \varphi_{23}) + \\
&+ M_{pw} (k_{uy} \cos \varphi_{31} + k_{vy} \cos \varphi_{23} + k_{wy})]''_\lambda , \quad (31) \\
(K_z)''_\lambda &= \mu_0 [M_{pu} (k_{uz} + k_{vz} \cos \varphi_{12} + k_{wz} \cos \varphi_{31}) + \\
&+ M_{pv} (k_{uz} \cos \varphi_{12} + k_{vz} + k_{wz} \cos \varphi_{23}) + \\
&+ M_{pw} (k_{uz} \cos \varphi_{31} + k_{vz} \cos \varphi_{23} + k_{wz})]''_\lambda . \quad (32)
\end{aligned}$$

Taking into consideration Eqs (2), (22) and (29), the functional (1) can be written as following:

$$F = \sum_{\lambda=1}^{m'} \left\{ \int\limits_{v'_\lambda} \left[A'_{xx} \left(\frac{\partial V_H}{\partial x} \right)^2 + A'_{yy} \left(\frac{\partial V_H}{\partial y} \right)^2 + A'_{zz} \left(\frac{\partial V_H}{\partial z} \right)^2 + \right. \right.$$

$$\left. \left. + A'_{xy} \left(\frac{\partial V_H}{\partial x} \right) \left(\frac{\partial V_H}{\partial y} \right) + A'_{yz} \left(\frac{\partial V_H}{\partial y} \right) \left(\frac{\partial V_H}{\partial z} \right) + A'_{zx} \left(\frac{\partial V_H}{\partial z} \right) \left(\frac{\partial V_H}{\partial x} \right) \right] dv \right\}_\lambda +$$

$$+ \sum_{\lambda=1}^{m''} \left\{ \int\limits_{v''_\lambda} \left[A''_{xx} \left(\frac{\partial V_H}{\partial x} \right)^2 + A''_{yy} \left(\frac{\partial V_H}{\partial y} \right)^2 + A''_{zz} \left(\frac{\partial V_H}{\partial z} \right)^2 + \right. \right.$$

$$+ A''_{xy} \left(\frac{\partial V_H}{\partial x} \right) \left(\frac{\partial V_H}{\partial y} \right) + A''_{yz} \left(\frac{\partial V_H}{\partial y} \right) \left(\frac{\partial V_H}{\partial z} \right) + A''_{zx} \left(\frac{\partial V_H}{\partial z} \right) \left(\frac{\partial V_H}{\partial x} \right) -$$

$$\left. \left. - K''_x \left(\frac{\partial V_H}{\partial x} \right) - K''_y \left(\frac{\partial V_H}{\partial y} \right) - K''_z \left(\frac{\partial V_H}{\partial z} \right) \right] dv \right\}_\lambda +$$

$$+ \sum_{\lambda=1}^{m'_N} \left\{ \int\limits_{S'_{N\lambda}} (B_n V_H) \, ds \right\}_\lambda + \sum_{\lambda=1}^{m''_N} \left\{ \int\limits_{S''_{N\lambda}} (B_n V_H) \, ds \right\}_\lambda . \quad (33)$$

This is the *useful form of the functional* in calculus, the only unknown quantity being magnetic scalar potential V_H.

3. Conclusions

Getting Eq (33) means an important intermediary stage in solving the field problem in inhomogeneous 3D domains, with non-linear and anisotropic permanent magnets and ferromagnetic yokes, having random main direction of magnetisation. In notations “A” and “K” (see Eqs 23-32), there intervene the field sources too (permanent magnetisation $\bar{\boldsymbol{M}}_{p\lambda}$), the parameters of magnetic circuit materials $\bar{\bar{\mu}}_\lambda$ and $\bar{\bar{\mu}}_{p\lambda}$ (obtained from the non-linear magnetisation curves of the ferromagnetic yokes, and those of the demagnetisation of permanent magnets), the relative disposal of the main magnetisation axes in comparison with a chosen rectangular system, as well as the Neumann conditions $(B_n)_\lambda$ on the boundary of the domain. All these have to be known so that, for a system with a given geometry, the solution of field problem be unique.

For the above mentioned case (inhomogeneous, non-linear and anisotropic system having random main direction), the transformation of the functional – in order to obtain a form that can be used in the numerical computations – has got an increased complexity, due to the degree of generality taken into consideration.

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