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*Group analysis of differential equations*

# FACTORIZATION OF FUNCTIONAL DIFFERENTIAL EQUATIONS

L.V.LINCHUK
Russian State Pedagogical University
St.-Petersburg, Russia
E-mail: lidiya@p110.f289.n5030.z2.fidonet.org

**Abstract.**

In the paper we consider applications of the theory of formal operators and the principle of factotization to funcional differential equations. Some examples of second and first order functional differential equations is constructed. These equtions are reduced to a more simple equations. Thus we simplify integration of some classes of functional differential equations.

Let's consider the class of ordinary functional differential equations

$$y^{(n)} = F\Big(x, y(x), y\big(\varphi(x)\big), y'(x), y'\big(\varphi(x)\big), \dots, y^{(n-1)}(x), y^{(n-1)}\big(\varphi(x)\big), y^{(n)}\big(\varphi(x)\big)\Big) = 0, \quad (1)$$

$n \in \mathbb{N}$. In the literature devoted to research of this equations some subclasses of class (1) is also refferred to as differential-delay equations, equations with delay, difference-differential equations.

Functional differential equations (FDEs) have already been found in mathematical papers of XVIII century, for example, in the Euler solution of the problem connected with a search of the general view of a line similar to its evilute. However until 1940 the number of papers devoted to these equations was not too large. Let's specify, that up to 1940s the main theorems of FDEs theory have been not formulated and the initial problem has been not set up [20]. In the middle of the 20th century the rise of numerous applications has regenerated interest in FDEs theory.

It is well known [2, 5, 18] that FDEs arise in the theory of automatic regulations, in the oscillations theory, at the investigation of processes in reactive motors, at the solution of problems of theoretical physics, in some tasks of economy and in different branches of biology [36, 37]. Therefore problem of search of FDEs precise solutions in closed analytical form is actual enough. Although process of their solutions connected with numerous difficulties which are explained by extreme complexity manifolds defined by FDEs.

The fullest review of methods of search of FDEs precise solutions is represented in monographies of E.Pinney [35], L.E.Elsgolts [4] and also in the papers of other authors [1, 3, 6, 7, 10, 19].

Notice that methods based on continuous symmetries of manifolds and offered at the end of XIX century by S.Lie [8, 9, 11, 12, 21, 22] were little use for research of FDEs. Some attempts of application of group analysis it is possible to find in paper of V.R.Petukhov [25–34]. It is appeared that the main difficulty for extending classical results of S.Lie to FDEs class is ambiguity of treatment of a prolongation of infinitesimal operator to the variables such as $\varphi(x)$, $y(\varphi(x))$ and their derivatives. Therefore all offered approaches have not received practical extending and have not made an unified informative theory.

Recently the interest to group methods has increased again. However undetermined differential equations became the main object of research. For example, undetermined differential equations describe control processes At the same time such equations with fixed functional relation between unknowns functions are functional differential equations [14, 16, 38].

In this paper we give results of the research of FDEs symmetries properties based on classical group approach and the theory of formal operators for generalized differential equations [13]. This approach allows to extend the universal principle of a factorization to FDEs and increase considerably the number of equations of this class integrated in closed form or accepting a simplification of its structures (for example, lowing of order) as factor system is a system of embedded in each other equations and one of these equations is solved independently of others.

By generalized differential equation is meant an undetermined differential equations with additional functional or functional-differential relation between unknown functions (thus we include into the consideration class (1)). Then the generalized differential equation appropriate FDE (1) looks like

$$y^{(n)} = F(x, y(x), w(x), y'(x), w'(x), \ldots, y^{(n-1)}(x), w^{(n-1)}(x), w^{(n)}(x)), \tag{2}$$

where $w(x) = y\big(\varphi(x)\big)$ is an additional functional relation between $y$ and $w$. Note that when we use group methods this relation is taken into consideration only after research symmetries of equation (2). Such treatment of FDE allows to solve easy some problems that arise in extending of group analysis to FDEs class. In particular, it is becomes evidently how to construct a prolongation of an operator to differential variables $w, w', \ldots$.

Consider formal operators of the form

$$X = \Phi(x, y, w, y', w', \ldots)\frac{\partial}{\partial y} + \Psi(x, y, w, y', w', \ldots)\frac{\partial}{\partial w}, \tag{3}$$

where functions $\Phi$ and $\Psi$ are coordinates of the operator and depend generally on any order derivatives. The class of operators (2), for example, includes operators descriptive of classical point symmetries (i.e. point operators in canonical form) and operators with coordinates contained nonlocal variables, for example, $\int \zeta\, dx$ (here by integral is meant total integral), in particular, we consider exponential nonlocal operators. As we proved in [13] the universal principle of a factorization for generalized differential equations can be formulated as the following two theorems.

***Theorem 1***. Let a generalized differential equation (2) admits a formal operator (3) with one lowest invariant of the kth order $(k \leqslant n)$

$$z = H\left(x, y, w, \ldots, y^{(k)}, w^{(k)}\right) \in \bar{\mathfrak{I}}_k\big|_{[y^{(n)}=F]}.$$

1) If

$$D_x^{n-k}(z)\big|_{[y^{(n)}=F]} \in \bar{\mathfrak{I}}_{n-1}\big|_{[y^{(n)}=F]}$$

then equation (1) factorized up to the system of two equations

$$\begin{cases} z = H\left(x, y, w, \ldots, y^{(k)}, w^{(k)}\right), \\ z^{(n-k)} = G\left(x, z, \ldots, z^{(n-k-1)}\right). \end{cases} \tag{4}$$

2) If

$$D_x^{n-k}(z)\Big|_{[y^{(n)}=F]} \in \bar{\mathfrak{I}}_n\Big|_{[y^{(n)}=F]} \backslash \bar{\mathfrak{I}}_{n-1}\Big|_{[y^{(n)}=F]}$$

and exists a mapping

$$z^* = H^*\left(x, y, w, \,\ldots\,, y^{(n)}, w^{(n)}\right)$$

such as $z^* \in \mathfrak{I}_n\Big|_{[y^{(n)}=F]} \backslash \mathfrak{I}_{n-1}\Big|_{[y^{(n)}=F]}$ and $x, z^{(0)}, \,\ldots\,, z^{(n-k)}, z^*$ are functionally independent then equation (1) factorized up to the system of three equation

$$\begin{cases} z = H\left(x, y, w, \,\ldots\,, y^{(k)}, w^{(k)}\right), \\ z^* = H^*\left(x, y, w, \,\ldots\,, y^{(n)}, w^{(n)}\right), \\ z^* = G\left(x, z, \,\ldots\,, z^{(n-k)}\right). \end{cases} \tag{5}$$

***Theorem 2***. Let a generalized differential equation (2) admits a formal operator (3) with two lowest invariants of the $k_i$th order ($k_i \leqslant n$)

$$z_i = H_i\left(x, y, w, \,\ldots\,, y^{(k_i)}, w^{(k_i)}\right) \in \bar{\mathfrak{I}}_{k_i}\Big|_{[y^{(n)}=F]}, \quad i = 1, 2.$$

1) If

$$D_x^{n-k_1}(z_1)\Big|_{[y^{(n)}=F]} \in \bar{\mathfrak{I}}_{n-1}\Big|_{[y^{(n)}=F]}$$

then equation (1) factorized up to the system

$$\begin{cases} z_1 = H_1\left(x, y, w, \,\ldots\,, y^{(k_1)}, w^{(k_1)}\right), \\ z_2 = H_2\left(x, y, w, \,\ldots\,, y^{(k_2)}, w^{(k_2)}\right), \\ z_1^{(n-k_1)} = G\left(x, z_1, \,\ldots\,, z_1^{(n-k_1-1)}, z_2, \,\ldots\,, z_2^{(n-k_2-1)}\right) \end{cases} \tag{6}$$

for $k_2 < n$,

$$\begin{cases} z_1 = H_1\left(x, y, w, \,\ldots\,, y^{(k_1)}, w^{(k_1)}\right), \\ z_1^{(n-k_1)} = G\left(x, z_1, \,\ldots\,, z_1^{(n-k_1-1)}\right) \end{cases} \tag{7}$$

for $k_2 = n$.

2) If

$$D_x^{n-k_1}(z_1)\Big|_{[y^{(n)}=F]}, D_x^{n-k_2}(z_2)\Big|_{[y^{(n)}=F]} \in \bar{\mathfrak{I}}_n\Big|_{[y^{(n)}=F]} \backslash \bar{\mathfrak{I}}_{n-1}\Big|_{[y^{(n)}=F]}$$

then equayion (1) factorized up to the system

$$\begin{cases} z_1 = H_1\left(x, y, w, \,\ldots\,, y^{(k_1)}, w^{(k_1)}\right), \\ z_2 = H_2\left(x, y, w, \,\ldots\,, y^{(k_2)}, w^{(k_2)}\right), \\ z_1^{(n-k_1)} = G\left(x, z_1, \,\ldots\,, z_1^{(n-k_1-1)}, z_2, \,\ldots\,, z_2^{(n-k_2)}\right). \end{cases} \tag{8}$$

Having constructed a generalized differential equation (2) corresponding the FDE (1) we can use Theorem 1 and Theorem 2. Then if we factorize FDE, i.e. the corresponding generalized differential equation up to the system with external ordinary differential equation (see (4), (7)), then we shall simplify an input equation, as if we solve the external equation then we shall lower the orde of FDE. It may appear that reducing of the generalized differential equation to a generalized differential with more simple structure (see (6), (8), (5)) is not effective for simplification of integration and researches of the FDE. In this case it is necessary to use the additional condition $w(x) = y\big(\varphi(x)\big)$. If dependent variables of the external equation of the factor system inherit functional relation which exists between variables $y$ and $w$ then the external equation is a functional differential equation. This equation as the equation written in invariants of an admissible operator has more simple structure (as a rule the simplification is a lowering order of the equation). Then knowing its solution we lower the order of the FDE.

Note that we can try to apply group methods to the equation obtained from an initial one by factorization and reduce the process of integration and research of FDE to a sequence of more simple subtasks.

Let's show realization of the formulated group approach for search of symmetries of FDE on several examples.

**Example 1**. The functional differential equation

$$y' = (\alpha y + \beta w + \gamma)w' + \frac{\alpha'}{\alpha}(\alpha y + \beta w)w + \\ + \frac{\alpha'(2\beta + \alpha\gamma) + \alpha(\beta - \beta')}{\alpha^2}w + y + \\ + \frac{\alpha'(2\beta + \alpha\gamma) + \alpha^2(\gamma - \gamma') + \alpha(\beta - \beta')}{\alpha^2}$$

where $w = y(x - \tau),\ \tau \in \mathbb{R},\ \tau > 0\,,\ \alpha = \alpha(x),\ \beta = \beta(x),\ \gamma = \gamma(x)$ factorize up to the system

$$\begin{cases} z = \dfrac{\alpha(\alpha y + \beta w + \gamma) + \beta}{\alpha^2}e^{-\alpha w}, \\ z' = z \end{cases}$$

by point symmetries. Solving the external equation of the factor system we lower the order of the equation and receive the functional equation

$$y(x) = -\frac{\alpha\beta y(x - \tau) + \beta + \alpha\gamma}{\alpha^2} + Ce^{\alpha y(x-\tau)+x}$$

where $C \in \mathbb{R}$. The last equation we can solve, for example, by "step method" [4]. ■

**Example 2**. Consider the second order functional differential equation with arbitrary functional deviation $h = h(x)$

$$y'' = (h')^2 y_h y_h'' + (h')^2[1 + f(x)y_h^2](y_h')^2 - f(x)(y' - 2h'y_h y_h')y' + h'' y_h y_h',$$

$y = y(x)$, $y' = y'(x)$, $y'' = y''(x)$, $y_h = y(h)$, $y_h' = y'(x)|_{x=h}$, $y_h'' = y''(x)|_{x=h}$, $f(x) \not\equiv 0$. The corresponding generalized differential equation has the form

$$y'' = ww'' + [1 + f(x)w^2](w')^2 - f(x)(y' - 2ww')y'$$

where $y = y(x)$, $w = w(x)$ and $w(x) = y(h(x))$. This equation admits the point operator

$$X = w\partial_y + \partial_w.$$

Consequently, it factorized by the universal invariant up to the system

$$\begin{cases} z = 2y - w^2, \\ z'' = \dfrac{1}{2} f(x)(z')^2. \end{cases}$$

The general solution of the external equation is

$$z = -2 \int \frac{dx}{\int f(x)\,dx - C_1} + C_2,$$

$C_1, C_2 \in \mathbb{R}$. Therefore we can lower the order of the functional differential equation and write it as the functional equation

$$y = \frac{1}{2} y_h^2 - \int \frac{dx}{\int f(x)\,dx - C_1} + C_2 \quad ■$$

**Example 3**. Research of symmetries of the second order functional differential equation

$$y''(x) + ay(-x)y'(x) - [ay(x) - c]y'(-x) + acy(x)y(-x) + b = 0$$

where $a, b \in \mathbb{R}$, $c \neq 0$ by searching an admissible point operator for corresponding generalized differential equation

$$y'' + awy' + (ay - c)w' + acyw + b = 0$$

does not lead to any significant result. Generally the equation admits the exponential nonlocal operator

$$X = \exp\left[a\int(y-w)dx\right]\partial_y - \exp\left[a\int(y-w)dx\right]\partial_w.$$

Its invariants $u$ and $v$ allow to factorize the equation up to the system

$$\begin{cases} u = y' + ayw + bc^{-1}, \\ v = -w' + ayw + bc^{-1}, \\ u' + cv = 0. \end{cases}$$

or

$$\begin{cases} u(x) = y'(x) + ay(x)w(x) + bc^{-1}, \\ u'(x) + cu(-x) = 0. \end{cases}$$

The solution of external functional differential equation is

$$u = C[\sin(cx) - \cos(cx)]$$

$C \in \mathbb{R}$. Thus the problem of search of a solution of the input functional differential equation is reduced to an integration of first order functional differential equation

$$y'(x) + ay(x)y(-x) + bc^{-1} = C[\sin(cx) - \cos(cx)].$$

For integration of the resulting equation it may be use the group approach or Sharkovsky method [17, 23, 24]. ■

Recall that that we can not always search an admissible operator in the general form

$$X = \Phi(x, y, w, y', w', \ldots)\frac{\partial}{\partial y} + \Psi(x, y, w, y', w', \ldots)\frac{\partial}{\partial w},$$

$\Phi, \Psi$: $\mathbf{Z}_k \longrightarrow \mathbb{R}$ ( $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$ ), therefore we need consider a subclass of operators. But it is not necessary to be restricted to the class of point or exponential nonlocal operators. In the following example factor system was constructed by a formal operator with coordinate that is linear on nonlocal variable.

**Example 4**. Research of symmetries of the second order functional differential equation

$$yy'' + (h')^2 y_h'' + (y')^2 + (h'y_h')^2 + h'(yy' + y_h)y_h' + h''y_h' = 0,$$

$y = y(x),\ y' = y'(x),\ y'' = y''(x),\ y_h = y(h),\ y'_h = y'(x)|_{x=h},\ y''_h = y''(x)|_{x=h}$, $h = h(x)$ and apply group approach to the corresponding generalized differential equation

$$yy'' + w'' + (y')^2 + (w')^2 + (yy' + w)w' = 0,$$

where $y = y(x)$, $w = w(x)$ and $w(x) = y(h(x))$. Let search an admissible operator in the class of nonlocal operators

$$X = \zeta \left\{ \int \left[\xi_2 (y')^2 + \xi_1 y' + \xi_0\right] dx \right\} \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial w},$$

where $\eta = \eta(x, y, w)$, $\zeta = \zeta(x, y, w)$, $\xi_j = \xi_j(x, y, w, w')$ $(j = 1, 2, 3)$. By using well-known algorithm of search of an admissible operator [13] we find one of admissible operators

$$X = \frac{\partial}{\partial w} - y^{-1} \left\{ \int \left(yy' + w' + w - C_2 e^{-w}\right) dx \right\} \frac{\partial}{\partial y}.$$

This operator has two lowest invariants and consequently the generalized differential equation can be factorized up to the system (see Theorem 2):

$$\begin{cases} z_1 = w', \\ z_2 = e^w (yy' + w' + w - 1) - C_2 w, \\ z'_2 + C_2 z_1 = 0. \end{cases}$$

By integration of the external equation we find the first integral of the generalized differential equation:

$$e^w (yy' + w' + w - 1) = C,$$

where $C \in \mathbb{R}$ and hence the input functional differential equation:

$$e^{y_h} (yy' + h' y'_h + y_h - 1) = C \quad \blacksquare$$

In summary it may be said that considered symmetries approach to research and integration of functional differential equations is universal as representation of functional differential equation in the form of generalized differential equation allows to consider equations with arbitrary deviation. In particular, it is possible to consider a case of dependence on unknown functions and its derivatives. At the same time it is evidently that in solving of concrete problems it is necessary to use the most of all possible approaches as the group approach does not overlap results of application of other methods.

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