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Applications to physics, electrotechnics, and electronics

SLOW EXTENSIONAL BOUNDED FLOWS

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Abstract

Slow extensional flows past a porous sphere, liquid sphere and a solid smooth sphere with slip at the surface have been obtained when all the spheres are bounded by a sphere whose surface is stretching radially. Mass transfer to the fluid due to the inner sphere has been evaluated in each case.

1. Introduction

Steady flows of Newtonian fluid past a solid sphere, liquid drop and smooth spheres at low Reynolds numbers have been studied extensively (see Happel and Brenner [1]). Recently Bhatt [2] has discussed these problems when the flow at infinity is given by extensional field. Kawase and Moo-Young [3] obtained approximate solutions of power law fluid flow past a solid sphere at small Reynolds numbers when the flow at infinity was given by the extensional field.

In the present problem we have extended the results of Bhatt [2] when the spheres are bounded by a sphere whose surface is stretching radially. Using the Stokes's approximation we get exact solutions in each case. The mass transfer due to the presence of these spheres has been evaluated in each case. The results of Bhatt [2] are derived as a particular case.

2. Extensional flow past a porous sphere

We cosider a porous sphere of radius a and permeability k surrounded by a sphere of radius b whose surface is stretching such that

$$u_x = -\frac{\alpha}{2}x, \quad u_y = -\frac{\alpha}{2}y, \quad u_z = \alpha z, \tag{2.1}$$

where α is constant.

The flow in the free fluid region ($a < r \leq b$) for steady motion is given by

$$\nabla p = \rho\nu\nabla^2 v \tag{2.2}$$

and

$$\nabla . v = 0, \tag{2.3}$$

where ν is the kinematic viscosity of the fluid, ρ is the density, p is the pressure and v is the velocity vector. In spherical polar coordinates (r, θ, ϕ), v has components $(u_r, u_\theta, 0)$. We define the Stokes stream function ψ to satisfy (2.3) by

$$u_r = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \tag{2.4}$$

and (2.2) gives

$$E^4\psi = 0, \tag{2.5}$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}. \tag{2.6}$$

The flow in the porous region($0 \leq r < a$) is given by

$$U = -\frac{k}{\rho\nu}\nabla P \tag{2.7}$$

and

$$\nabla . U = 0 \tag{2.8}$$

where U is the velocity vector and P is the pressure in the porous region. Stokes stream function Ψ can be defined similar to ψ in the porous region with velocity components U_r, U_θ as

$$U_r = -\frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}, \quad U_\theta = \frac{1}{r \sin\theta}\frac{\partial \Psi}{\partial r} \tag{2.9}$$

which satisfy (2.7) and (2.8) and we get

$$E^2 \Psi = 0, \tag{2.10}$$

where E^2 is defined earlier.

The boundary conditions to be satisfied are:
At $r = 0$, the solution should be finite. (2.11)
At $r = a$,

$$\begin{aligned} u_r &= U_r, \\ e_{r\theta} &= \tfrac{\sigma}{\sqrt{k}}(u_\theta - U_\theta), \\ p &= P. \end{aligned} \tag{2.12}$$

where $e_{r\theta}$ is the strain tensor, σ is the slip parameter (the second condition in equation (2.12) is that of Jones [4]).
At $r = b$,

$$\begin{aligned} u_r &= \tfrac{\alpha b}{2}(2\cos^2\theta - \sin^2\theta), \\ u_\theta &= -\tfrac{3\alpha b}{2}\sin\theta\cos\theta. \end{aligned} \tag{2.13}$$

The solutions of (2.5) and (2.10) can be obtained as

$$u_r = -(A_1 r^{-2} + A_2 r + A_3 r^3 + A_4 r^{-4})(2\cos^2\theta - \sin^2\theta), \tag{2.14}$$

$$u_\theta = (3A_2 r + 5A_3 r^3 - 2A_4 r^{-4})\sin\theta\cos\theta, \tag{2.15}$$

$$p = -\mu(2A_1 r^{-3} + 7A_3 r^2)(2\cos^2\theta - \sin^2\theta), \tag{2.16}$$

$$U_r = -(B_1 r + B_2 r^{-4})(2\cos^2\theta - \sin^2\theta), \tag{2.17}$$

$$U_\theta = (3B_1 r - 2B_2 r^{-4})\sin\theta\cos\theta, \tag{2.18}$$

$$P = \frac{\mu}{k}\left(\frac{B_1}{2}r^2 - \frac{B_2}{3}r^{-3}\right)(2\cos^2\theta - \sin^2\theta), \tag{2.19}$$

Using the boundary conditions (2.11)-(2.13) we get:

$$
\begin{aligned}
A_1 &= \frac{1}{3}(2A_3b^5 - 5A_4b^{-2}),\\
A_2 &= -\frac{\alpha}{2} - \frac{5}{3}A_3b^2 + \frac{2}{3}A_4b^{-5},\\
B_1 &= -2\lambda^2(2A_1a^{-3} + 7A_3a^2),\\
B_2 &= 0,\\
A_3 &= \left[\frac{3\alpha}{2} - A_4\{-5b^{-2}a^{-3}(1+4\lambda^4) + 2b^{-5} + 3a^{-5}\}\right] \times\\
&\quad \{2b^5a^{-3}(1+4\lambda^2) - 5b^2 + 3a^2(1+14\lambda^2)\}^{-1},\\
A_4 &= \frac{3\alpha}{2}\left[2b^5a^{-2}(\sigma - 8\lambda^3) - 2a^3(\sigma - 5\lambda + 42\lambda^3)\right] \times\\
&\quad [-42a^{-2}(5\lambda^2\sigma + \sigma + 12\lambda^3) + 25b^{-2}(-2\lambda + 4\lambda^3 + 10\lambda^2\sigma + \sigma) +\\
&\quad 25b^2a^{-4}(\sigma + 2\lambda) - 4b^5a^{-7}(\sigma + 5\lambda + 10\sigma\lambda^2 + 32\lambda^3) -\\
&\quad 4b^{-5}a^3(\sigma - 5\lambda + 42\lambda^3)]^{-1}
\end{aligned}
$$

where $\lambda = \dfrac{\sqrt{k}}{a}$.

Following Bhatt [2] , the mass transfer (M) from a porous sphere at high Schmidt numbers can be obtained as

$$
\begin{aligned}
u_\theta' &= -\frac{15}{2}\alpha M \sin\theta\cos\theta\\
M &= 2[-4(\sigma + 6\sigma\lambda^2 + \lambda + 8\sigma\lambda^4 + 4\lambda^3) + 10\eta^3(\sigma + \lambda + 2\sigma\lambda^2) -\\
&\quad 2\eta^5(14\lambda^3 + 41\sigma\lambda^2 - 4\sigma + 168\sigma\lambda^4 - 4\lambda) +\\
&\quad 10\eta^7(-\sigma - \lambda + \sigma\lambda^2 + 10\lambda^3 + 20\sigma\lambda^4 + 56\lambda^5) +\\
&\quad 35\eta^8(3\sigma\lambda^2 - \sigma - \lambda) + 2\eta^{10}(23\sigma + 53\sigma\lambda^2 - 28\lambda^3 + 23\lambda - 441\sigma\lambda^4) +\\
&\quad 15\eta^{12}(196\lambda^5 + 70\sigma\lambda^4 - 9\sigma\lambda^2 - \lambda - \sigma)] \times\\
&\quad [2(1+4\lambda^2) - 5\eta^3 - 3\eta^5(1+14\lambda^2)]^{-1} \times\\
&\quad [-4(\sigma + 10\sigma\lambda^2 + 5\lambda + 32\lambda^3) + 25\eta^3(\sigma + 2\lambda) - 42\eta^5(\sigma + 12\lambda^3 + 5\sigma\lambda^2) +\\
&\quad 25\eta^7(-2\lambda + 4\lambda^3 + 10\sigma\lambda^2 + \sigma) + 4\eta^{10}(-\sigma + 5\lambda - 42\lambda^3)]^{-1}
\end{aligned}
$$

For $\eta \to 0$, the value of M agrees with Bhatt [2]. The values of M for various values of σ and λ have been given in Table 1. The first entry in each block corresponds to $\eta = \dfrac{1}{2}$ and second entry corresponds to $\eta = 1$. M increases as λ decreases or as σ increases (similar to Bhatt [2]).

Table 1

λ	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$
0	1.66399	1.71728	1.74393
	1.0061479	1.0061482	1.006148
2	0.020886	0.0345009	0569277
	0.022689	0.0358339	0.0574771
4	0.0084211	0.015755	0.0289445
	0.0095417	0.0108176	0.0292896
6	0.0049393	0.0099828	0.0193588
	0.0596707	0.0108176	0.0198333
8	0.0033196	0.00716597	0.0144419
	0.0043298	0.00802823	0.0150234
10	0.0023856	0.0054949	0.01143986
	0.0033948	0.00638415	0.0120993

3. Extensional flow past a liquid sphere

For liquid sphere of radius a if we take $\mu_i (i = 1, 2)$ be the viscosities of the fluid inside and outside of the sphere, the stream functions Ψ_i , are given by

$$\Psi_i = (A_i + B_i r^3 + C_i r^5 + D_i r^{-2}) \sin^2\theta \cos\theta, \tag{3.1}$$

$$u_{ri} = (A_i r^{-2} + B_i r + C_i r^3 + D_i r^{-4})(\sin^2\theta - 2\cos^2\theta), \tag{3.2}$$

$$u_{\theta i} = (3B_i r + 5C_i r^3 - 2D_i r^{-4}) \sin\theta \cos\theta. \tag{3.3}$$

The boundary conditions are:
At $r = b$,

$$\begin{aligned} u_{r1} &= \tfrac{1}{2}\alpha b \left(2\cos^2\theta - \sin^2\theta\right), \\ u_{\theta 1} &= -\tfrac{3}{2}\alpha b \sin\theta \cos\theta. \end{aligned} \tag{3.4}$$

At $r = a$,

$$u_{r1} = u_{r2} = 0,$$
$$\mu_1 \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \Psi_1}{\partial r}\right) = \mu_2 \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \Psi_2}{\partial r}\right). \tag{3.5}$$
$$u_{\theta 1} = u_{\theta 2},$$

At $r = 0$,

$$u_{r2} \text{ and } u_{\theta 2} \ \text{ should be finite.} \tag{3.6}$$

Applying the boundary conditions (3.4)-(3.6) on (3.2) and (3.3), we get

$$A_2 = D_2 = 0, \tag{3.7}$$
$$A_1 = -(B_1 a^3 + C_1 a^5 + D_1 a^{-2}), \tag{3.8}$$
$$B_1 = -\frac{1}{2}\alpha - \frac{5}{3}C_1 b^2 + \frac{2}{3}D_1 b^{-5}, \tag{3.9}$$
$$C_1 = \left[-\frac{3}{2}\alpha a^3 - D_1(5b^{-2} - 2b^{-5}a^3 - 3a^{-2})\right][-2b^5 + 5b^2a^3 - 3a^5]^{-1}, \tag{3.10}$$
$$B_2 = -\frac{a^2}{\sigma}\left(C_1 + \frac{D_1}{a^7}\right), \tag{3.11}$$
$$C_2 = \frac{1}{\sigma}\left(C_1 + \frac{D_1}{a^7}\right), \tag{3.12}$$
$$D_1 = [3\alpha a\{a^5(\sigma - 1) - b^5\sigma\}][42a\sigma - 5(5\sigma + 2)b^5a^{-1} + 5(2 - 5\sigma)a^3b^{-2} + 4(\sigma - 1)a^6b^{-5} + 4(1 + \sigma)a^{-4}b^5]^{-1}. \tag{3.13}$$

where $\sigma = \dfrac{\mu_2}{\mu_1}$. Here we obtain

$$\frac{u'_{\theta 1}}{\sin\theta\cos\theta} = [-3\alpha(5\sigma + 1)\{46a^6 - 35a^4b^2 - 15a^8b^{-2} + 8ab^5 + 10a^{-1}b^7 - 10a^3b^3 - 4a^{-4}b^{10}\}][(-2b^5 + 5a^3b^2 - 3a^5) \times (42a\sigma - 5(5\sigma + 2)a^{-1}b^2 + 5(2 - 5\sigma)a^3b^{-2} + 4(\sigma - 1)a^6b^{-5} + 4(1 + \sigma)a^{-4}b^5)]^{-1}, \tag{3.14}$$

which gives

$$M = \frac{2(1+5\sigma)}{5(2-5\eta^3+3\eta^5)}[12\eta^{12} - 46\eta^{10} + 35\eta^8 + 10\eta^7 - 8\eta^5 - 10\eta^3 + 4] \times$$
$$[42\eta^5\sigma - 5\eta^3(5\sigma+2) - 5\eta^7(5\sigma-2) + 4\eta^{10}(\sigma-1) + 4(1+\sigma)]^{-1}. \quad (3.15)$$

The behaviour of M with σ and η has been given in figure 1. M increases with σ and η.

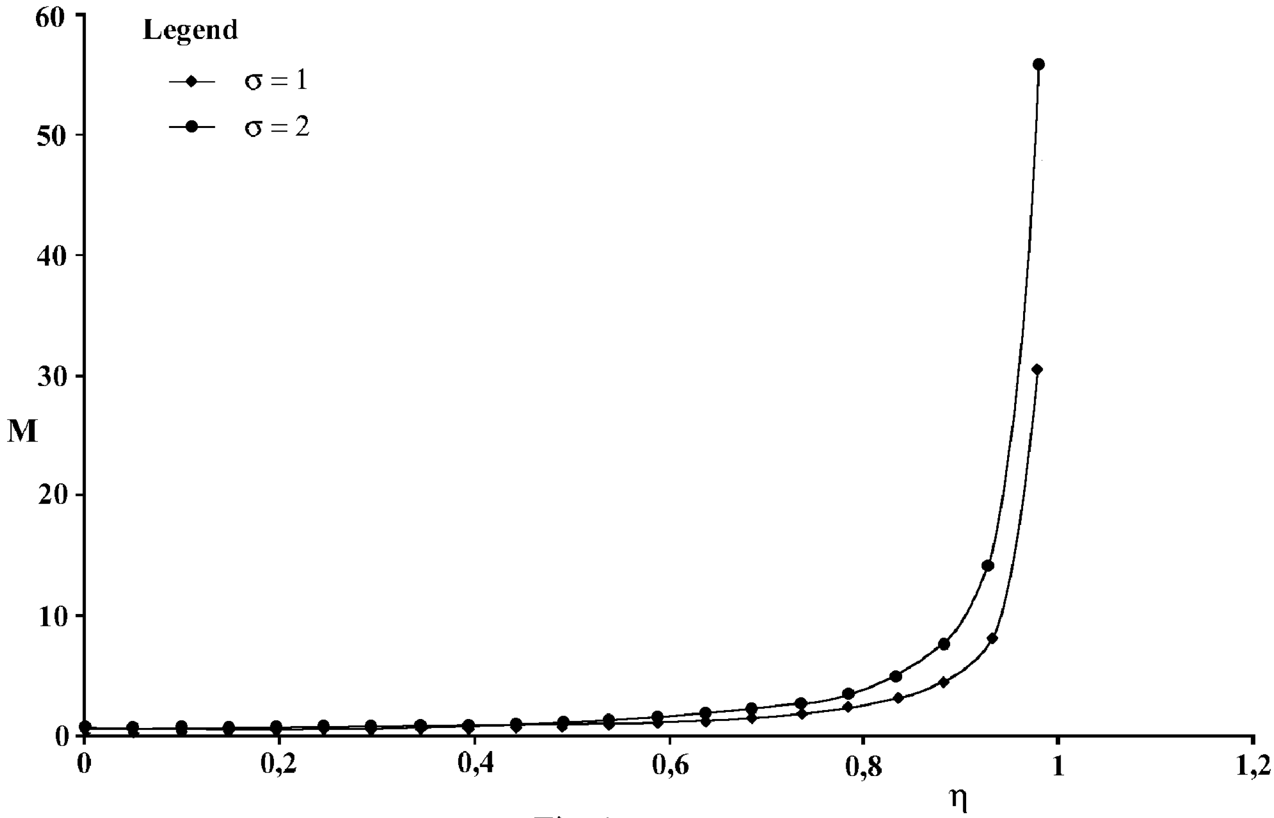

Fig. 1

4. Extensional flow past a solid smooth sphere

Here we consider a solid smooth sphere of radius a placed in an extesional flow of fluid of viscosity μ. The solution is given by

$$
\begin{aligned}
u_r &= -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, \\
\psi &= (A + Br^3 + Cr^5 + Dr^{-5}) \sin^2\theta \cos\theta, && (4.1) \\
u_r &= (Ar^{-2} + Br + Cr^3 + Dr^{-4})(\sin^2\theta - 2\cos^2\theta), && (4.2) \\
u_\theta &= (3Br + 5Cr^3 - 2Dr^{-4}) \sin\theta\cos\theta. && (4.3)
\end{aligned}
$$

The boundary conditions to be satisfied are

At $r = b$,

$$
\begin{aligned}
u_r &= \frac{\alpha b}{2}\left(2\cos^2\theta - \sin^2\theta\right), \\
u_\theta &= -\frac{3}{2}\alpha b \sin\theta\cos\theta. && (4.4)
\end{aligned}
$$

At $r = a$,

$$
\psi = 0, \quad \beta\frac{1}{r}\left(\frac{\partial\psi}{\partial r}\right) = \mu r\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi}{\partial r}\right), \qquad (4.5)
$$

where β is a slip parameter.

Using the boundary conditions we get

$$
\begin{aligned}
A &= -Bb^3 - Cb^5 - Db^{-2} - \frac{\alpha}{2}b^3, && (4.6) \\
B &= -\frac{\alpha}{2} - \frac{5}{3}Cb^2 + \frac{2}{3}Db^{-5}, && (4.7) \\
C &= \left[\frac{3}{2}\alpha a^3 - D(3a^{-2} + 2a^3 b{-}5 - 5b^{-2})\right][3a^5 - 5a^3b^2 + 2b^5\ ]^{-1}, && (4.8) \\
D &= 3\alpha a^6 b^5(-\beta a^5 + \beta b^5 + 5a^4\mu)[a\beta(25a^3b^7 + 25a^7b^3 - 42a^5b^5 \\
&\quad -4a^{10} - 4b^{10}) + 10\mu(2a^{10} - 5a^7b^3 + 5a^3b^7 - 2b^{10})]^{-1}. && (4.9)
\end{aligned}
$$

Again here M can be obtained as

$$
\begin{aligned}
M = 2(a\beta + \mu)(5\eta^7 - 7\eta^5 + 2)[a\beta(4\eta^{10} - 25\eta^7 \\
+ 42\eta^5 - 25\eta^3 + 4) - 10\mu(2\eta^{10} - 5\eta^7 + 5\eta^3 - 2)]^{-1}, \quad (4.10)
\end{aligned}
$$

where $\eta = \dfrac{a}{b}$.

The behaviour of M with η, $a\beta$ and μ is given in figure 2. Here M is a monotonic increasing function of η and $a\beta$ but a monotonic decreasing function of μ. For $\eta \to 0$ The results of M given in all the three cases agree with Bhatt [2].

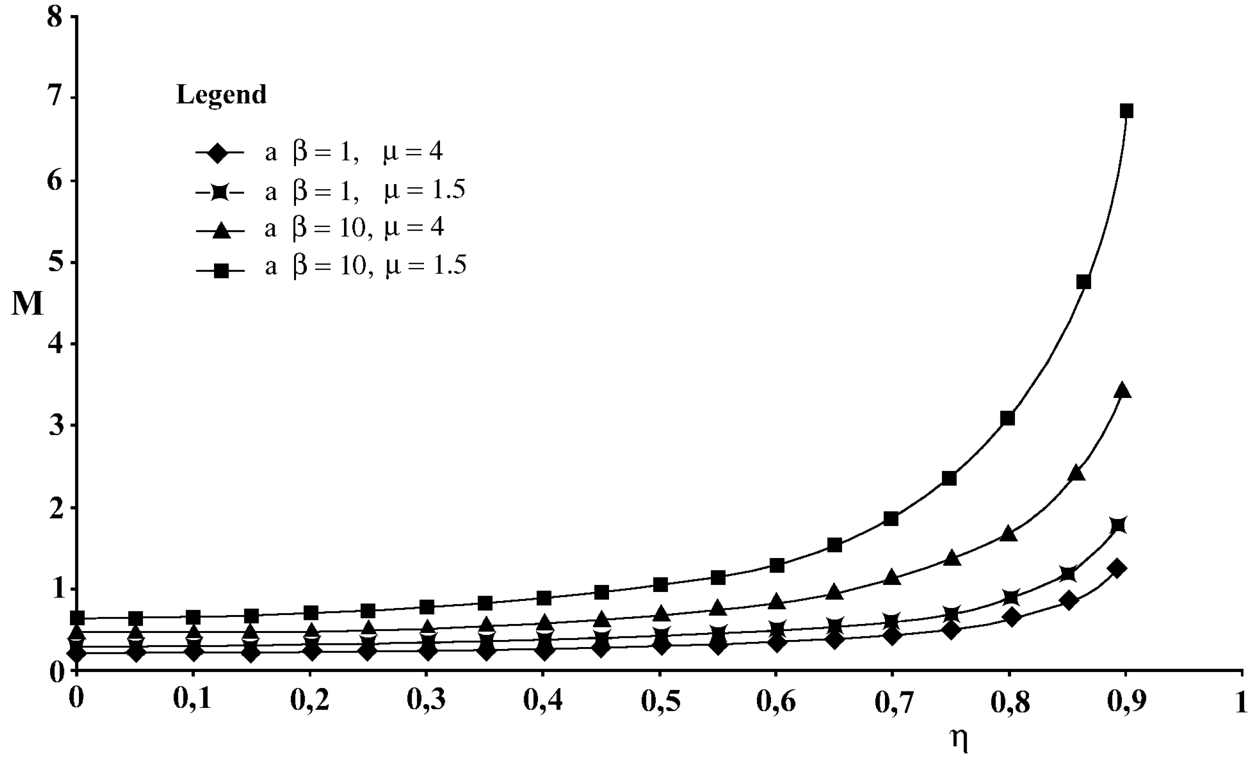

Fig. 2

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