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Group analysis of differential equations

ON INTEGRABILITY OF ONE EVOLUTION SYSTEM

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1 Introduction

Let us consider an evolution system of the following type

$$u_t = u_3 + F(u_2, v_2, u_1, v_1, u, v), \quad v_t = G(u_1, v_1, u, v), \tag{1}$$

where F and G are arbitrary functions, $u_i = \partial^i u/\partial x^i,\ i = 1, 2, 3,\ u_t = \partial u/\partial t$. The symmetry classification of integrable systems of the type (1) was given in the article [1]. We folowed there to ideas of the article [2]. But the canonical conserved densities were obtained by the Chinese algorithm [3]. The list of integrable systems that was presented in [1] consists of 15 systems. But some of the systems are connected by the several contact transformations. We write here the independent integrable systems from the article [1] for completeness:

$$u_t = w_2 + 6c_0 e^u u_1 - 1/2 w^3 + c_1 w + c_2 u_1, \quad w = u_1 - v,$$
$$v_t = 3c_0 e^u u_1^2 + 4c_0 e^u v_1 + c_0 e^u v^2 - 2c_0 c_1 e^u + c_2 v_1, \quad c_0 \neq 0. \tag{2}$$

$$u_t = u_3 + u_1 v + u v_1, \quad v_t = u_1. \tag{3}$$

$$u_t = u_3 + 3u_1 v + 2u v_1 + v_1 v_2 + c_1 v_1 - 2v^2 v_1, \quad v_t = u_1 + v v_1. \tag{4}$$

$$u_t = u_3 + 2u_1 v + u v_1, \quad v_t = 2c u u_1, \quad c \neq 0. \tag{5}$$

$$u_t = u_3 - \frac{u_1 v_2}{v} - \frac{3u_2 v_1}{2v} + \frac{3u_1 v_1^2}{2v^2} + \frac{u^2 u_1}{v} + \frac{c_1 u_1}{2v} + 2c_2 v^2 u_1 + 3c_2 u v v_1, \quad v_t = 2u u_1. \tag{6}$$

$$u_t = u_3 - \frac{u_1 v_2}{v} - \frac{3u_2 v_1}{2v} + \frac{3u_1 v_1^2}{2v^2} + \frac{3c_1 u v_1^2}{2v^2} + \frac{u^2 u_1}{v} - c_1 \frac{u_1 v_1 + u v_2 - u^3}{v} + \frac{c_1^2 u v_1}{2v} - c_1^2 u_1 - c_2 u,$$
$$v_t = 2u u_1 + 2c_1 u^2 - 2c_2 v. \tag{7}$$

$$u_t = u_3 + 3u u_1 + 2c_1 v v_1, \quad v_t = u v_1 + u_1 v, \quad c_1 \neq 0. \tag{8}$$

Let us mention, that the systems (3), (5) are known as the Drinfeld-Sokolov systems [4], [5].

We investigate the system (8) in this article.

2 Zero curvature representation

If we perform the dilatation $v \to v/|c_1|^{1/2}$, then the system (8) takes more simple form

$$u_t = u_{xxx} + 3u u_x + 2\varepsilon v v_x, \quad v_t = (uv)_x, \tag{9}$$

where $\varepsilon = c_1/|c_1| = \pm 1$. Now we consider the zero curvature representation for the system (9)

$$U_t - V_x + [U, V] = 0. \tag{10}$$

If we adopt that the matrix U depend on u, v, and the matrix V depend on u, v, u_1, u_2, then performing the differentiation in (10) one can easyly obtain the U and V matrices

$$U = A_1 + A_2 u + A_3 v^2,$$
$$V = A_2(u_2 + \varepsilon v^2) + A_4 u_1 + A_3 u v^2 + \frac{1}{2}[A_2,\, A_4]\, u^2 + \frac{3}{2} A_2 u^2 + [A_1,\, A_4]\, u + A_5,$$

where A_i are constant matrices satisfying the following algebra

$$\begin{aligned} &[A_2,\, A_3] = 0, \quad [A_1,\, A_5] = 0, \quad [A_3,\, A_5] = -\varepsilon A_4, \quad [A_3,\, A_4] = -A_3, \\ &[A_1,\, [A_1,\, A_4]] + [A_2,\, A_5] = 0, \quad [A_4,\, [A_1,\, A_3]] = 0, \quad [A_1,\, A_2] = A_4, \\ &[A_1,\, [A_2,\, A_4]] = -A_4, \quad [A_2,\, [A_2,\, A_4]] = 0. \end{aligned} \tag{11}$$

Follow to the ideas of the article [6] we set

$$[A_1,\, A_4] = A_6, \quad \text{and } A_3 = \frac{\varepsilon}{\mu} A_2,$$

where μ is a parameter. Then exploiting the Jacobi identity we found the following 5-dimensional Lie algebra L:

$$\begin{aligned} &[A_1,\, A_5] = 0, \quad [A_1,\, A_2] = A_4, \quad [A_1,\, A_6] = \mu A_4, \quad [A_2,\, A_4] = -A_2, \\ &[A_1,\, A_4] = A_6, \quad [A_2,\, A_5] = -\mu A_4, \quad [A_2,\, A_6] = -A_4, \\ &[A_4,\, A_6] = -A_6 + \mu A_2, \quad [A_4,\, A_5] = -\mu A_6, \quad [A_5,\, A_6] = \mu^2 A_4 \end{aligned} \tag{12}$$

Algebra (12) possesses two dimensional center $Z = \{A_5 - \mu A_1,\, A_1 + \mu A_2 - A_6\}$. Therefore setting $A_5 = \mu A_1$, and $A_6 = A_1 + \mu A_2$ we obtain three dimensional algebra

$$[A_1,\, A_2] = A_4, \quad [A_2,\, A_4] = -A_2, \quad [A_1,\, A_4] = A_1 + \mu A_2, \tag{13}$$

isomorphic to the factor algebra $L \backslash Z$. This algebra is $sl(2)$ obviously. Then constructing a representation of the algebra (13), we found the following explicit form of the matrices U and V:

$$\begin{aligned} U &= \begin{pmatrix} 0 & u + (\varepsilon/\mu) v^2 - \frac{1}{2}\mu \\ -\frac{1}{2} & 0 \end{pmatrix}, \\ V &= \begin{pmatrix} \frac{1}{2} u_1 & u_2 + \varepsilon v^2 + (\varepsilon/\mu) u\, v^2 + u^2 + \frac{1}{2}\mu(u - \mu) \\ -\frac{1}{2}(u + \mu) & -\frac{1}{2} u_1 \end{pmatrix}. \end{aligned} \tag{14}$$

3 Recursion operator and Lie-Bäcklund symmetries

The recursion operator maps Lie-Bäcklund algebra of an evolution system into itself. As we found the matrices (21) that satisfy the equation (10) then we used the direct algorithm suggested in [10] and developed in [11] for constructing a recursion operator. The recursion operator of the system (9) takes the following form

$$\Lambda = \begin{pmatrix} D^2 + 2\,u + u_1 D^{-1} & 2\,\varepsilon\, v \\ D\, v\, D^{-1} & 0 \end{pmatrix}. \qquad (15)$$

If we denote $\sigma_0 = \{u_1,\, v_1\}$, $\sigma_1 \equiv K = \{u_3 + 3\,u\,u_1 + 2\,\varepsilon\, v\, v_1,\; u\, v_1 + u_1\, v\}$ then the following formula $\Lambda\sigma_0 = \sigma_1$ can be easily checked. One can find the higher symmetries by the recusrsion formula $\sigma_{n+1} = \Lambda\sigma_n$. For example,

$$\begin{aligned} \sigma_2^u &= u_5 + 5\,u\,u_3 + 10\,u_2\,u_1 + 6\,\varepsilon\, v_2\, v_1 + 2\,\varepsilon\, v\, v_3 + 15/2\, u_1\, u^2 + \\ &\qquad + 3\,u_1\,\varepsilon\, v^2 + 6\,u\,\varepsilon\, v\, v_1, \\ \sigma_2^v &= v\,u_3 + 3\,v\,u\,u_1 + 3\,\varepsilon\, v^2\, v_1 + v_1\, u_2 + 3/2\, v_1\, u^2. \end{aligned} \qquad (16)$$

The direct calculation shows that the system (9) also admits another symmetries:

$$\begin{aligned} \tau_1 &= \begin{pmatrix} 3\,t\,(u_3 + 3\,u\,u_1 + 2\,\varepsilon\, v\, v_1) + 2\,u + x\,u_1 \\ 3\,t\,(u_1 v + u\, v_1) + 2\,v + x\, v_1 \end{pmatrix}, \\ \omega_1 &= \begin{pmatrix} 2\; - \varepsilon\, v^{-2} v_1 \\ v^{-3}(v_3 - 6\,v^{-1} v_1 v_2 + 6\,v^{-2} v_1^3 + 2\,u\,v_1 - v\,u_1) \end{pmatrix}. \end{aligned} \qquad (17)$$

Here τ_1 corresponds to the point symmetry with the following infinitesimal operator

$$X = 3\,t\,\frac{\partial}{\partial t} + x\,\frac{\partial}{\partial x} - 2\,u\,\frac{\partial}{\partial u} - 2\,v\,\frac{\partial}{\partial v}.$$

But ω_1 is nonclassic Lie-Bäcklund symmetry. τ_1 creates the infinite sequence of Lie-Bäcklund symmetries according to the formula $\tau_{n+1} = \Lambda\tau_n$. It can be checked that τ_2 is nonlocal symmetry:

$$\begin{aligned} \tau_2^u &= xu_3 + 4u_2 + 3xuu_1 + u_1 u_{-1} + 4u^2 + 2\varepsilon(2v^2 + xvv_1) + \\ &+3t(u_5 + 5u\,u_3 + 10u_1\,u_2 + 2\varepsilon v\,v_3 + 6\varepsilon v_1\,v_2 + \tfrac{15}{2}u^2\,u_1 + 3\varepsilon v^2\,u_1 + 6\varepsilon u\,v\,v_1), \\ \tau_2^v &= 3t(\,v\,u_3 + v_1\,u_2 + 3\,\varepsilon\,v^2\,v_1 + 3\,v\,u\,u_1 + \tfrac{3}{2}\,v_1\,u^2) + x\,v\,u_1 + 2\,u\,v + \\ &\qquad + x\,u\,v_1 + v_1\,u_{-1}, \end{aligned} \qquad (18)$$

where $u_{-1} = D^{-1}u$. The other symmetries τ_i are nonlocal too obviously.

It is easy to check that $\Lambda\omega_1 = 0$, but $\Lambda^{-1}\omega_1 \equiv \omega_2$ is the local 5th-order Lie-Bäcklund symmetry:

$$\Lambda^{-1} = \begin{pmatrix} 0 & D\,v^{-1}D^{-1} \\ \varepsilon/(2\,v) & -\varepsilon/(2\,v)(D^3 + 2\,u\,D + u_1)\,v^{-1}D^{-1} \end{pmatrix}, \tag{19}$$

$$\begin{aligned} \omega_2^u &= -1/2\,(2\,v_3\,v^2 - 14\,v_1\,v_2\,v + 15\,v_1^3 - 2\,u_1\,v^3 + 6\,u\,v_1 v^2)/v^6, \\ \omega_2^v &= 1/4\,\varepsilon\,(2\,v_5\,v^4 - 30\,v_1\,v_4\,v^3 - 50\,v_2\,v_3\,v^3 + 10\,v_3\,u\,v^4 - 2\,u_3\,v^5 + \\ &\quad +225\,v_1^2\,v_3\,v^2 + 20\,u_1\,v_2\,v^4 + 18\,v_1\,u_2\,v^4 - 1050\,v_1^3\,v_2\,v + 630\,v_1^5 + \\ &\quad +150\,v_1^3\,u\,v^2 + 300\,v_1\,v_2^2\,v^2 - 100\,v_1\,v_2\,u\,v^3 - 6\,u\,u_1\,v^5 + 4\,\varepsilon\,v_1\,v^6 - \\ &\quad -75\,u_1\,v_1^2\,v^3 + 12\,u^2\,v_1\,v^4)/v^9 \end{aligned} \tag{20}$$

Hence our system admits the third sequence of Lie-Bäcklund symmetries: $\omega_{n+1} = \Lambda^{-1}\omega_n$. ω_3 is a local vector function and the other vector functions ω_i will be local probably.

Let us consider the evolution system $(u_t,\ v_t) = \omega_1$. If we set $v = 1/w$ then this system takes the following simple form

$$u_t = 2\,\varepsilon\,w_1, \quad w_t = w^3(w_3 + u_1 w + 2\,u\,w_1). \tag{21}$$

The second equation can be rewritten as the conservation law $(1/w)_t = (-w\,w_2 + w_1^2/2 - u\,w^2)_x$. Hence the following transformation $(t,\,x,\,u,\,w) \to (\tau,\,y,\,u',\,z)$

$$\begin{aligned} t' &= t, \quad dy = w^{-1}\,dx + (-w\,w_2 + w_1^2/2 - u\,w^2)\,dt, \\ u'(t',\,y) &= u(t,\,x), \quad w(t,\,x) = \exp(z(t',\,y)/2). \end{aligned} \tag{22}$$

is possible for (21). Performing it we obtain the following new integrable system

$$u_t = \frac{1}{2}\,u_1\,z_2 - \frac{1}{8}\,u_1\,z_1^2 + u_1\,e^z u + \varepsilon\,z_1, \quad z_t = z_3 - \frac{1}{8}\,z_1^3 + 2\,e^z u_1 + 3\,e^z u\,z_1, \tag{23}$$

where $u_i = \partial^i u/\partial y^i$.

Let us mention in conclusion that the operators (15) and (19) are hereditary operators.

4 Conserved densities and Noether operators

The pair of functions (ρ, θ) is called the conserved current of a partial diffetential system with two independent variables t, x and dependent variables u^α, $\alpha = 1, 2, \dots, m$, if $D_t\rho(t, x, u) = D\,\theta(t, x, u)$ for any solution of the system. The function ρ is said to be the conserved density and θ be the density current. If we consider the evolution system $u_t = K(u)$ then gradient of any conserved density $\gamma = E\,\rho$:

$$\gamma_\alpha = \sum_n (-D)^n \frac{\partial \rho}{\partial u_n^\alpha}$$

solves the following equation

$$(D_t + K'^{+})\gamma = 0,$$

where K'^{+} is the adjoint for K' operator (see [7], [8] or [9]).

Operator Θ satisfying the equation

$$(D_t - K')\Theta = \Theta(D_t - K'^{+}) \tag{24}$$

is called the Noether operator [12].

According to this definition Noether operator of an evolution system maps the set of gradients of conserved densities into the set of Lie-Bäcklund symmetries of the system.

We solved the equation (24) and found two following Noether operators for the system (9):

$$\Theta_1 = \begin{pmatrix} D^3 + D\,u + u\,D & v\,D \\ D\,v & 0 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} D & 0 \\ 0 & (\varepsilon/2)\,D \end{pmatrix}. \tag{25}$$

The direct check shows that Θ_1 and Θ_2 are implectic and compatible (see [12], [13]). In other words Θ_1 and Θ_2 is the Hamiltonian pair. The recursion operator (15) takes the form $\Lambda = \Theta_1\Theta_2^{-1}$ as it can be checked. This formula implies that Λ and Λ^{-1} are hereditary operators.

Solving the equation $D_t\rho = D\,\theta$ one can easyly find the following simplest conserved currents $j_k = [\rho_k,\, \theta_k]$:

$$j_1 = [u,\ u_2 + \frac{3}{2}\,u^2 + \varepsilon\,v^2],\quad j_2 = [v,\ u\,v].$$

$$j_3 = [\frac{1}{2}\,u^2 + \varepsilon\,v^2,\ u\,u_2 - \frac{1}{2}\,u_1^2 + u^3 + 2\,\varepsilon\,u\,v^2],$$

$$j_4 = [-\frac{1}{2}\,u_1^2 + \frac{1}{2}\,u^3 + \varepsilon\,u\,v^2,\ -u_1\,u_3 + \frac{3}{2}\,u^2\,u_2 + \\ +\varepsilon\,v^2\,u_2 + \frac{1}{2}\,u_2^2 - 2\,\varepsilon\,v\,v_1\,u_1 - 3\,u\,u_1^2 + \frac{9}{8}\,u^4 + \frac{5}{2}\,\varepsilon\,v^2\,u^2 + \frac{1}{2}v^4],$$

$$j_5 = [\frac{1}{2}\,v^{-3}\,v_1^2 - v^{-1}\,u,\ -v^{-1}\,u_2 + \frac{1}{2}v^{-3}\,v_1^2\,u - v^{-1}\,u^2 - 2\,\varepsilon\,v].$$

The corresponding gradients $\gamma_k = [\delta\rho_k/\delta u,\, \delta\rho_k/\delta v]$ take the following form

$$\gamma_1 = [1,\, 0],\quad \gamma_2 = [0,\, 1],\quad \gamma_3 = [u,\, 2\,\varepsilon\,v],\quad \gamma_4 = [u_2 + \frac{3}{2}\,u^2 + \varepsilon\,v^2, 2\,\varepsilon\,u\,v],$$
$$\gamma_5 = [-v^{-1}, -v^{-3}\,v_2 + \frac{3}{2}\,v^{-4}\,v_1^2 + u\,v^{-2}].$$

Using these expressions we found that

$$\begin{aligned} &\Theta_1\gamma_1 = \{u_1,\ v_1\} \equiv \sigma_0,\quad \Theta_1\gamma_2 = 0,\quad \Theta_1\gamma_3 = \sigma_1,\quad \Theta_1\gamma_4 = \sigma_2,\quad \Theta_1\gamma_5 = 0,\\ &\Theta_2\gamma_1 = \Theta_2\gamma_2 = 0,\quad \Theta_2\gamma_3 = \sigma_0,\quad \Theta_2\gamma_4 = \sigma_1,\quad \Theta_2\gamma_5 = \omega_1/(2\,\varepsilon). \end{aligned} \tag{26}$$

Hence, the system (9) is the bi-Hamiltonian one:

$$\mathbf{u}_t = \Theta_1 E\rho_3 = \Theta_2 E\rho_4. \tag{27}$$

Here $\mathbf{u} = \{u,\ v\}$.

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