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**Coupled systems of Hammerstein and Urysohn integral equations in reflexive Banach spaces**

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**Abstract:** We present existence theorems for at least one weak solution for coupled systems of integral equations of Hammerstein type and of Uryshon type in a reflexive Banach spaces relative to the weak topology.

**Keywords:** Weak solution; Hammerstein integral equation; Urysohn integral equation; Coupled systems.

# 1 Introduction and Preliminaries

Systems occur in various problems of applied nature, for instance, see ([1]-[3] and [10]-[13]). Recently, Su [18] discussed a two-point boundary value problem for a coupled system of fractional differential equations. Gafiychuk et al. [12] analyzed the solutions of coupled nonlinear fractional reaction-diffusion equations.
Let $L_1(I)$ be the space of Lebesgue integrable functions defined on the interval $I = [0, 1]$. Let $E$ be a reflexive Banach space with the norm $||.||$ and its dual $E^*$ and denote by $C[I, E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm $||.||_0$.

The existence of weak solutions of the integral equations studied by many authors such as [4], [5], [9] and [14]-[17].

The existence of weak solutions to the Hammerstein integral equation

$$x(t) \; = \; h(t) \; + \; \int_0^1 \; k(t,s) \; f(s,x(s)) \; ds, \qquad t \in I$$

was proved by O'Regan [15] where $x$ takes values in reflexive Banach spaces and $f$ is weakly-weakly continuous.

Recently, the existence of weak solution of the nonlinear fractional-order integral equation

$$x(t) \; = \; g(t) \; + \; \lambda \; I^{\alpha} \; f(t,x(t)), \quad t \in I, \quad 0 < \alpha < 1 \tag{1}$$

was proved in [17] where $x$ takes values in reflexive Banach spaces and $f$ is weakly measurable in $t$ and weakly sequentially continuous in $x$ .

An existence result for (1), in the case $E = R$ found in [8] where the real-valued function $f$ satisfies Carathéodory condition.

Also, The authors [9] proved the existence of solution $x \in C[I,E]$ of the Hammerstein integral equation

$$x(t) \; = \; a(t) \; + \; \int_0^1 \; k(t,s) \; f(s,x(s)) \; ds, \qquad t \in I,$$

where $x$ takes values in reflexive Banach spaces and $f$ is weakly measurable in $t$ and weakly sequentially continuous in $x$ .

and the Urysohn integral equation

$$x(t) \; = \; a(t) \; + \; \int_0^1 \; u(t,s,x(s)) \; ds, \qquad t \in I,$$

where $x$ takes values in reflexive Banach spaces and $u$ is weakly measurable in $s$ and weakly sequentially continuous in $x$ .

It well known that the existence of weak solutions of the Hammerstein integral equation has been considered for the first time, by M. Cichon, I. Kubiaczyk [5].

In this paper, we study the existence of a weak solution for the coupled systems

$$\begin{aligned} x(t) \; &= \; g_1(t) \; + \; \int_0^1 \; k_1(t,s) \; f_1(s,y(s)) \; ds, \qquad t \in [0,1], \\ y(t) \; &= \; g_2(t) \; + \; \int_0^1 \; k_2(t,s) \; f_2(s,x(s)) \; ds, \qquad t \in [0,1]. \end{aligned} \tag{2}$$

and

$$\begin{aligned} x(t) \; &= \; g_1(t) \; + \; \int_0^1 \; u_1(t,s,y(s)) \; ds, \qquad t \in [0,1], \\ y(t) \; &= \; g_2(t) \; + \; \int_0^1 \; u_2(t,s,x(s)) \; ds, \qquad t \in [0,1]. \end{aligned} \tag{3}$$

Now, we shall present some auxiliary results that will be need in this work. Let $E$ be a Banach space (need not be reflexive) and let $x: I \to E$, then

(1) $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in I$ if for every $\phi \in E^*$, $\phi(x(.))$ is continuous (measurable) at $t_0$.

(2) A function $h : E \to E$ is said to be weakly sequentially continuous if $h$ maps weakly convergent sequences in $E$ to weakly convergent sequences in $E$.

If $x$ is weakly continuous on $I$, then $x$ is strongly measurable and hence weakly measurable (see[7] and [6]). Note that in reflexive Banach space weakly measurable functions are Pettis integrable if and only if $\phi(x(.))$ is Lebesgue integrable on $I$ for every $\phi \in E^*$ (see[7] pp. 78).
While it is not always possible to show that a given mapping between Banach spaces is weakly continuous, quite often its weak sequential continuity and weakly sequentially continuous offers no problem. A "sequential" concept of continuity is more general than the continuity and moreover more useful (for example the Lebesgue's dominated convergence theorem is valid for sequence but not for nets) so we shall state a fixed point theorem and some propositions which will be used in the sequel (see[16]).

**Theorem 1** *Let $E$ be a Banach space and let $Q$ be a nonempty, bounded, closed and convex subset of the space $E$ and let $T : Q \to Q$ be a weakly sequentially continuous and assume that $TQ(t)$ is relatively weakly compact in $E$ for each $t \in [0, 1]$ . Then, $T$ has a fixed point in the set $Q$.*

**Proposition 1** *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

**Proposition 2** *Let $E$ be a normed space with $y \neq 0$. Then there exists a $\phi \in E^*$ with $||\phi|| = 1$ and $||y|| = \phi(y)$.*

## 2 Hammerstein Coupled System

This section deals with the existence of weak solutions for the coupled system of Hammerstein type(2).
Let $E$ be a reflexive Banach space and $D \subset E$. Consider the following assumptions:

(1:) $g_i \in C[I, E], \quad i = 1, 2;$

(2:) $f_i : I \times D \to E, \quad i = 1, 2$ satisfy the following:

- (i) For each $t \in I, \quad f_{i_t} = f_i(t, .)$ are weakly sequentially continuous;
- (ii) For each $x \in D, \quad f_i(., x(.))$ are weakly measurable on $I$ ;
- (iii) The weak closure of the range of $f_i(I \times D)$ are weakly compact in $E$
(or equivalently: there exist $M_i$ such that $||f_i(t, x)|| \leq M_i$
$(t, x) \in I \times D$;)

(3:) $k_i : I \times I \to R_+$ are integrable in $s$ and continuous in $t$, the operators

$$\int_0^1 k_i(t, s)\ y(s)\ ds$$

map $L_1(I)$ into $L_1(I)$ and $\int_0^1 k_i(t, s) < A_i$.

**Definition 1** *By a weak solution for the coupled system (2), we mean the pair of functions* $(x, y) \in C[I, E] \times C[I, E]$ *such that*

$$\phi(x(t)) = \phi(g_1(t)) + \int_0^1 k_1(t, s)\ \phi(f_1(s, y(s)))\ ds, \quad t \in [0, 1],$$

$$\phi(y(t)) = \phi(g_2(t)) + \int_0^1 k_2(t, s)\ \phi(f_2(s, x(s)))\ ds, \quad t \in [0, 1]$$

*for all* $\phi \in E^*$.

**Theorem 2** *Let the assumptions (1:)-(3:) be satisfied. Then the coupled system (2) has at least one weak solution* $(x, y) \in C[I, E] \times C[I, E]$.

**Proof:** Define the operators $T_1, T_2$ by

$$T_1 y(t) = g_1(t) + \int_0^1 k_1(t, s)\ f_1(s, y(s))\ ds, \quad t \in I$$

$$T_2x(t) \;=\; g_2(t) \;+\; \int_0^1 \; k_2(t,s)\; f_2(s,x(s))\; ds, \qquad t \in I.$$

Then the coupled system (2) may be written as:

$$x(t) \;=\; T_1y(t)$$

$$y(t) \;=\; T_2x(t).$$

Define the operator $T$ by

$$T(x,y)(t) \;=\; (T_1y(t),\; T_2x(t)).$$

For any $y \in C[I,E]$, since $f_1(.,y(.))$ is weakly measurable on $I$ and $||f_1(t,y)|| \leq M_1$, then $\phi(f_1(.,y(.)))$ is Lebesgue integrable on $I$ $\forall \phi \in E^*$ and since $k_1(t,.)$ is Lebesgue integrable on $I$, then we have $\phi(k_1(t,.)\; f_1(.,y(.))) = k_1(t,.)\; \phi(f_1(.,y(.)))$ is Lebesgue integrable on $I$ $\forall \phi \in E^*$, then $k_1(t,.)\; f_1(.,y(.))$ is Pettis integrable on $I$. Thus $T_1$ is well defined.

Now, we shall prove that $T_1 : C[I,E] \to C[I,E]$.

Let $t_1,\; t_2 \in I$ and ( without loss of generality assume that $T_1y(t_2) - T_1y(t_1) \neq 0$)

$$T_1y(t_2) - T_1y(t_1) \;=\; g_1(t_2) \;-\; g_1(t_1)$$

$$+ \int_0^1 k_1(t_2,s)\; f_1(s,y(s))\; ds \;-\; \int_0^1 k_1(t_1,s)\; f_1(s,y(s))\; ds$$

$$= \; g_1(t_2) \;-\; g_1(t_1) \;+\; \int_0^1 [k_1(t_2,s) \;-\; k_1(t_1,s)]\; f_1(s,y(s))\; ds.$$

Therefore as a consequence of Proposition 2, we obtain

$$||T_1y(t_2) - T_1y(t_1)|| \;=\; \phi(T_1y(t_2) - T_1y(t_1))$$

$$= \; \phi(g_1(t_2) \;-\; g_1(t_1)) \;+\; \int_0^1 |k_1(t_2,s) \;-\; k_1(t_1,s)|\; \phi(f_1(s,y(s)))\; ds$$

$$= \; ||g_1(t_2) \;-\; g_1(t_1)|| \;+\; \int_0^1 |k_1(t_2,s) \;-\; k_1(t_1,s)|\; ||f_1(s,y(s))||\; ds$$

$$\leq \; ||g_1(t_2) \;-\; g_1(t_1)|| \;+\; M_1 \int_0^1 |k_1(t_2,s) \;-\; k_1(t_1,s)|\; ds.$$

As done above we can show that

$$||T_2x(t_2) - T_2x(t_1)|| \;\leq\; ||g_2(t_2) \;-\; g_2(t_1)|| \;+\; M_2 \int_0^1 |k_2(t_2,s) \;-\; k_2(t_1,s)|\; ds.$$

Now, we shall prove that $T : C[I,E] \times C[I,E] \to C[I,E] \times C[I,E]$

$$Tu(t_2) \;-\; Tu(t_1) \;=\; T(x,y)(t_2) \;-\; T(x,y)(t_1)$$

$$= (T_1y(t_2),\ T_2x(t_2)) - (T_1y(t_1),\ T_2x(t_1)) =$$
$$(T_1y(t_2) - T_1y(t_1),\ T_2x(t_2) - T_2x(t_1)),$$

then we have

$$|| \ Tu(t_2) - Tu(t_1) \ || \leq ||T_1y(t_2) - T_1y(t_1)|| + ||T_2x(t_2) - T_2x(t_1)||$$
$$\leq ||g_1(t_2) - g_1(t_1)|| + M_1 \int_0^1 |k_1(t_2,s) - k_1(t_1,s)| \, ds$$
$$+ ||g_2(t_2) - g_2(t_1)|| + M_2 \int_0^1 |k_2(t_2,s) - k_2(t_1,s)| \, ds,$$

and

$$||T_1 \ y(t)|| = \phi(T_1y(t)) = \phi(g_1(t)) + \int_0^1 k_1(t,s) \ \phi(f_1(s,y(s))) \, ds$$
$$= ||g_1|| + \int_0^1 k_1(t,s) \ ||f_1(s,y(s))|| \, ds$$
$$\leq ||g_1|| + M_1 \int_0^1 k_1(t,s) \, ds$$
$$\leq ||g_1|| + A_1 \ M_1.$$

By a similar way as done above we can prove that

$$||T_2x(t)|| \leq ||g_2|| + A_2 \ M_2.$$

Then, $T_1,\ T_2$ are well defined on the sets
$Q_1 = \{ \ y \in C[I,E] : \ ||y|| \leq \ M_1 \ \}, \quad M_1 = ||g_1|| + A_1$
and $Q_2 = \{ \ x \in C[I,E] : \ ||x|| \leq \ M_2 \ \}, \quad M_2 = ||g_2|| + A_2$ respectively.
Now, define the set $Q$ by

$$Q = \{ \ u = (x,y) \in C[I,E] \times C[I,E] : \ ||u|| \leq \ ||g_1|| + A_1 \ M_1 + ||g_2|| + A_2 \ M_2 \ \}.$$

Then, for any $u \in Q$ we have

$$||Tu(t)|| = ||T(x,y)(t)|| = ||(T_1y(t), T_2x(t))|| \leq ||T_1y(t)|| + ||T_2x(t)||$$
$$\leq ||g_1|| + A_1 \ M_1 + ||g_2|| + A_2 \ M_2.$$

i.e. $\forall u \in Q \Rightarrow Tu \in Q \Rightarrow TQ \subset Q$. Thus $T : Q \to Q$.
Then $Q$ is nonempty, uniformly bounded and strongly equi-continuous subset of $C[I,E] \times C[I,E]$. Also, it can be shown that $Q$ is convex and closed. As a consequence of Proposition 1, then $TQ$ is relatively weakly compact. It remains to prove that $T$ is weakly sequentially continuous.
Let $\{y_n(t)\}$ and $\{x_n(t)\}$ be two sequences in $Q_1,\ Q_2$ converge weakly

to $x(t)$, $y(t)$ respectively $\forall t \in I$. Since $f_1(t, y(t))$ and $f_2(t, x(t))$ are weakly sequentially continuous in the second argument, then $f_1(t, y_n(t))$ and $f_2(t, x_n(t))$ converge weakly to $f_1(t, y(t))$ and $f_2(t, x(t))$ respectively and hence $\phi(f_1(t, y_n(t)))$ and $\phi(f_2(t, x_n(t)))$ converge strongly to $\phi(f_1(t, y(t)))$ and $\phi(f_2(t, x(t)))$ respectively.
Using assumption (iii)) and applying Lebesgue Dominated Convergence Theorem for Pettis integral, then we get

$$\phi(\int_0^1 k_1(t,s)\ f_1(s, y_n(s))\ ds) \ = \ \int_0^1 k_1(t,s)\ \phi(f_1(s, y_n(s)))\ ds$$
$$\to \int_0^1 k_1(t,s)\ \phi(f_1(s, y(s)))\ ds \qquad \forall \phi \in E^*, \quad t \in I,$$

and

$$\phi(\int_0^1 k_2(t,s)\ f_2(s, x_n(s))\ ds) \ = \ \int_0^1 k_2(t,s)\ \phi(f_2(s, x_n(s)))\ ds$$
$$\to \int_0^1 k_2(t,s)\ \phi(f_2(s, x(s)))\ ds \qquad \forall \phi \in E^*, \quad t \in I.$$

Then $T$ is weakly sequentially continuous. Since all conditions of Theorem 1 are satisfied, then the operator $T$ has at least one fixed point $u \in Q$ which competes the proof. ■

## 3 Urysohn Coupled System

This section deals with the existence of weak solutions for the coupled system of Urysohn type(3).

Let $E$ be a reflexive Banach space and $D \subset E$. Consider the following assumptions:

(1$^*$:) $g_i \in C[I, E]$, $i = 1, 2$;

(2$^*$ :) $u_i :\ I \times I \times D \to E$, $i = 1, 2$ satisfy the following:

- ($i^*$) For each $t, s \in I \times I$, $u_i(t, s, .)$ are weakly sequentially continuous;
- ($ii^*$) For each $x \in D$ and $t \in I$ $u_i(t, ., x(.))$ are weakly measurable on $I$ ;
- ($iii^*$) For each $x \in D$ and $s \in I$ $u_i(., s, x(s))$ are continuous on $I$ ;

$(3^* :)$ $||u_i(t,s,x(s))|| \le k_i(t,s)$, $i=1,2$ $k_i : I \times I \to R_+$ are integrable in $s$ and continuous in $t$, the operators

$$\int_0^1 k_i(t,s)\, z(s)\, ds$$

maps $L_1(I)$ into $L_1(I)$ and $\int_0^1 k_i(t,s) ds < A_i$, $t \in I$.

**Definition 2** *By a weak solution for the coupled system (3), we mean the pair of functions* $(x,y) \in C[I,E] \times C[I,E]$ *such that*

$$\phi(x(t)) = \phi(g_1(t)) + \int_0^1 \phi(u_1(t,s,y(s)))\, ds, \qquad t \in [0,1],$$

$$\phi(y(t)) = \phi(g_2(t)) + \int_0^1 \phi(u_2(t,s,x(s)))\, ds, \qquad t \in [0,1]$$

*for all* $\phi \in E^*$.

**Theorem 3** *Let the assumptions* $(1^* :)$*-*$(3^* :)$ *be satisfied. Then the coupled system (3) has at least one weak solution* $(x,y) \in C[I,E] \times C[I,E]$.

**Proof:**
Define the operators $T_1, T_2$ by

$$T_1 y(t) = g_1(t) + \int_0^1 u_1(t,s,y(s))\, ds, \qquad t \in I$$

$$T_2 x(t) = g_2(t) + \int_0^1 u_2(t,s,x(s))\, ds, \qquad t \in I.$$

Then the coupled system (3) may be written as:

$$x(t) = T_1 y(t)$$

$$y(t) = T_2 x(t).$$

Define the operator $T$ by

$$T(x,y)(t) = (T_1 y(t),\ T_2 x(t)).$$

For any $y \in C[I,E]$ and since $u_1(t,.,y(.))$ is weakly measurable on $I$, then $\phi(u_1(t,.,y(.)))$ is strongly measurable on $I$ $\forall \phi \in E^*$ and since $||u_1(t,s,y)|| \le k_1(t,s)$, then $\phi(u_1(t,.,y(.)))$ is Lebesgue integrable on $I$ $\forall \phi \in E^*$ and hence $u_1(t,.,y(.))$ Pettis integrable on $I$. Thus $T_1$ is

well defined.
Now, we shall prove that $T_1 : \ C[I, E] \to C[I, E]$.
Let $t_1, \ t_2 \in I$ and ( without loss of generality assume that $T_1 y(t_2) - T_1 y(t_1) \neq 0$)

$$T_1 y(t_2) - T_1 y(t_1) \ =$$

$$= \ g_1(t_2) \ - \ g_1(t_1) \ + \ \int_0^1 \ u_1(t_2, s, y(s)) \ ds \ - \ \int_0^1 \ u_1(t_1, s, y(s)) \ ds$$

$$= \ g_1(t_2) \ - \ g_1(t_1) \ + \ \int_0^1 \ [u_1(t_2, s, y(s)) \ - \ u_1(t_1, s, y(s)) \ ] \ ds$$

Therefore as a consequence of Proposition 2, we obtain

$$||T_1 y(t_2) - T_1 y(t_1)|| \ = \ \phi(T_1 y(t_2) - T_1 y(t_1))$$

$$= \ \phi(g_1(t_2) \ - \ g_1(t_1)) \ + \ \int_0^1 \ \phi[u_1(t_2, s, y(s)) \ - \ u_1(t_1, s, y(s)) \ ] \ ds$$

$$\leq \ ||g_1(t_2) \ - \ g_1(t_1)|| \ + \ \int_0^1 \ ||u_1(t_2, s, y(s)) \ - \ u_1(t_1, s, y(s))|| \ ds \qquad (4)$$

As done above we can show that

$$||T_2 x(t_2) - T_2 x(t_1)|| \ \leq$$

$$\leq \ ||g_2(t_2) \ - \ g_2(t_1)|| \ + \ \int_0^1 \ ||u_2(t_2, s, x(s)) \ - \ u_2(t_1, s, x(s))|| \ ds.$$

Now, we shall prove that $T : \ C[I, E] \times C[I, E] \to C[I, E] \times C[I, E]$

$$Tv(t_2) \ - \ Tv(t_1) \ = \ T(x, y)(t_2) \ - \ T(x, y)(t_1), \quad v(t) = (x, y)(t)$$

$$= \ (T_1 y(t_2), \ T_2 x(t_2)) \ - \ (T_1 y(t_1), \ T_2 x(t_1)) \ =$$

$$= \ (T_1 y(t_2) \ - \ T_1 y(t_1), \ T_2 x(t_2) \ - \ T_2 x(t_1)),$$

then we have

$$|| \ Tv(t_2) \ - \ Tv(t_1) \ || \ \leq \ ||T_1 y(t_2) \ - \ T_1 y(t_1)|| \ + \ ||T_2 x(t_2) \ - \ T_2 x(t_1)||$$

$$\leq \ ||g_1(t_2) \ - \ g_1(t_1)|| \ + \ \int_0^1 \ ||u_1(t_2, s, y(s)) \ - \ u_1(t_1, s, y(s))|| \ ds$$

$$+ \ ||g_2(t_2) \ - \ g_2(t_1)|| \ + \ \int_0^1 \ ||u_2(t_2, s, x(s)) \ - \ u_2(t_1, s, x(s))|| \ ds,$$

and

$$||T_1 \ y(t)|| \ = \ \phi(T_1 y(t)) \ = \ \phi(g_1(t)) \ + \ \int_0^1 \ \phi(u_1(t, s, y(s))) \ ds$$

$$= \ \phi(g_1(t)) \ + \ \int_0^1 \ ||u_1(t, s, y(s))|| \ ds \ \leq \ ||g_1|| \ + \ \int_0^1 \ k_1(t, s) \ ds$$

$$\le \ ||g_1|| \ + \ A_1.$$

By a similar way as done above we can prove that

$$||T_2x(t)|| \ \le \ ||g_2|| \ + \ A_2.$$

Then, $T_1$, $T_2$ are well defined on the sets
$Q_1 \ = \ \{ \ y \in C[I,E] : \ ||y|| \le \ M_1 \ \}, \quad M_1 = ||g_1|| \ + \ A_1$
and $Q_2 \ = \ \{ \ x \in C[I,E] : \ ||x|| \le \ M_2 \ \}, \quad M_2 = ||g_2|| \ + \ A_2$ respectively.
Now, let the set $Q$ be defined as

$$Q \ = \ \{ \ v = (x,y) \in C[I,E] \times C[I,E] : \ ||v|| \le \ ||g_1|| \ + \ A_1 \ + \ ||g_2|| \ + \ A_2 \ \}.$$

Then, for any $v \in Q$ we have

$$||Tv(t)|| \ = \ ||T(x,y)(t)|| \ = \ ||(T_1y(t), T_2x(t))|| \ \le \ ||T_1y(t)|| \ + \ ||T_2x(t)||$$

$$\le \ ||g_1|| \ + \ A_1 \ + \ ||g_2|| \ + \ A_2.$$

i.e. $\forall v \in Q \Rightarrow Tv \in Q \Rightarrow TQ \subset Q$. Thus $T : Q \to Q$.
Then $Q$ is nonempty, uniformly bounded and strongly equi-continuous subset of $C[I,E] \times C[I,E]$. Also, it can be shown that $Q$ is convex and closed. As a consequence of Proposition 1, then $TQ$ is relatively weakly compact. It remains to prove that $T$ is weakly sequentially continuous.
Let $\{y_n(t)\}$ and $\{x_n(t)\}$ be two sequences in $Q_1$, $Q_2$ converge weakly to $x(t)$, $y(t)$ respectively $\forall t \in I$. Since $u_1(t,s,y(s))$ and $u_2(t,,s,x(s))$ are weakly sequentially continuous in the third argument, then $u_1(t,s,y_n(s))$ and $u_2(t,s,x_n(s))$ converge weakly to $u_1(t,s,y(s))$ and $u_2(t,s,x(s))$ respectively and hence $\phi(u_1(t,s,y_n(s)))$ and $\phi(u_2(t,,s,x_n(s)))$ converge strongly to $\phi(u_1(t,s,y(s)))$ and $\phi(u_2(t,s,x(s)))$ respectively.
Using assumption $(iii^*)$) and applying Lebesgue Dominated Convergence Theorem for Pettis integral, then we get

$$\phi(\int_0^1 \ u_1(t,s,y_n(s)) \ ds) \ = \ \int_0^1 \ \phi(u_1(t,s,y_n(s))) \ ds$$

$$\to \int_0^1 \ \phi(u_1(t,s,y(s))) \ ds \qquad \forall \phi \in E^*, \quad t \in I$$

and

$$\phi(\int_0^1 \ u_2(t,s,x_n(s)) \ ds) \ = \ \int_0^1 \ \phi(u_2(t,s,x_n(s))) \ ds$$

$$\to \int_0^1 \ \phi(u_2(t,s,x(s))) \ ds \qquad \forall \phi \in E^*, \quad t \in I.$$

Then $T$ is weakly sequentially continuous.
Since all conditions of Theorem 1 are satisfied, then the operator $T$ has at

least one fixed point $v = (x, y) \in Q$ , which completes the proof. ∎

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