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OSCILLATION CRITERIA FOR SECOND ORDER HALF-LINEAR DIFFERENTIAL EQUATIONS WITH DAMPING TERM

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Abstract

By using averaging functions, several new oscillation criteria are established for the half-linear damped differential equation

$$\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+p(t)\varphi\left(|x|^{\alpha-2}x,r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+q(t)|x|^{\alpha-2}x=0,$$

and the more general equation

$$\frac{d}{dt}\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+p(t)\varphi\left(g(x),r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+q(t)g(x)=0,$$

where $p,q,r:[t_0,\infty)\to R$ and $\psi,g: R\to R$ are continuous, $r(t)>0$, $p(t)\geq 0$ and $\psi(x)>0$, $xg(x)>0$ for $x\neq 0$, $\alpha>1$ a fixed real number. Our results generalize and extend some known oscillation criteria in the literature.

1. INTRODUCTION

We are concerned with the oscillation of solutions of second order differential equations with damping terms of the following form

$$\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+p(t)\varphi\left(|x|^{\alpha-2}x, r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+q(t)|x|^{\alpha-2}x=0, \qquad (1.1)$$

and the more general equation

$$\frac{d}{dt}\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+p(t)\varphi\left(g(x), r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+q(t)g(x)=0, \qquad (1.2)$$

where $r\in C[[t_0,\infty),R^+]$, $p\in C[[t_0,\infty),[0,\infty)]$, $q\in C[[t_0,\infty),R]$, $\psi\in C[R,R^+]$ and $g\in C^1[R,R]$ such that $xg(x)>0$ for $x\neq 0$ and $g'(x)>0$ for $x\neq 0$. φ is defined and continuous on $R\times R-\{0\}$ with $u\varphi(u,v)>0$ for $uv\neq 0$ and $\varphi(\lambda u, \lambda v)=\lambda\varphi(u,v)$ for $0<\lambda<\infty$ and $(u,v)\in R\times R-\{0\}$.

By oscillation of equation (1.1)[(1.2)], we mean a function $x\in C^1([T_x,\infty),R)$ for some $T_x\geq t_0$, which has the property that $\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)\in C^1([T_x,\infty),R)$ and satisfies equation (1.1)[(1.2)] on $[T_x,\infty)$.

A solution of equation (1.1)[(1.2)] is called oscillatory if it has arbitrarily large zeros otherwise, it is called nonoscillatory. Finally, equation (1.1)[(1.2)] is called oscillatory if all its solutions are oscillatory.

In Section 2 we provide sufficient conditions for the oscillation of all solutions of (1.1). Several particular cases of (1.1) have been discussed in the literature. To cite a few examples, the differential equation

$$\frac{d}{dt}\left(r(t)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+q(t)|x|^{\alpha-1}x=0, \qquad (1.3)$$

has been studied by [5]-[12]. A more general equation than (1.3)

$$\frac{d}{dt}\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+q(t)|x|^{\alpha-1}x=0, \qquad (1.4)$$

has been considered by [2] and [20]. Our results include, as special cases, known oscillation theorems for (1.3), (1.4). In particular, we extend and improve the results obtained in [13], [17], [2] and [14].

In Section 3 we will establish oscillation criteria for equation (1.2). Several particular cases of (1.2) have been discussed in the literature. To cite a few examples, the differential equation

$$\frac{d}{dt}\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+q(t)g(x)=0, \qquad (1.5)$$

established by [16] and [19] considered a special case of this equation as

$$\frac{d}{dt}\left(r(t)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)+q(t)g(x)=0, \qquad (1.6)$$

Our results in this section generalize and improve [17], [1], [3] and [18].

2. OSCILLATION RESULTS FOR (1.1)

In order to discuss our main results, we need the following well-known inequality which is due to Hardy et al. [4].

Lemma 2.1. If X and Y are nonnegative, then

$$X^{\lambda}+(\lambda-1)Y^{\lambda}-\lambda XY^{\lambda-1}\geq 0,\ \ \lambda>1,$$

where equality holds if and only if X=Y.

Theorem 2.1. Suppose, in addition to conditions

$$\varphi(1,z)\geq z \ \text{ for all } z\neq 0, \qquad (2.1)$$

$$0<\psi(x)\leq\gamma \ \text{ for all } x, \qquad (2.2)$$

that there exist differentiable functions

$$k,\rho:[t_0,\infty)\to(0,\infty) \quad \text{with } \dot{\rho}(t)\geq 0$$

and the continuous function

$$H: D\equiv\{(t,s): t\geq s\geq t_0\}\to R \quad \text{and} \quad h: D_0\equiv\{(t,s): t>s\geq t_0\}\to R,$$

and H has a continuous and nonpositive partial derivative on D with respect to the second variable such that

$$H(t,t)=0 \ \text{ for } \ t\geq t_0, \quad H(t,s)>0 \ \text{ for } \ t>s\geq t_0,$$

and

$$-\frac{\delta}{\delta s}(H(t,s)k(s))=h(t,s)(H(t,s)k(s))^{\frac{\alpha-1}{\alpha}} \quad \text{for all } (t,s)\in D_0.$$

Then equation (1.1) is oscillatory if

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\{H(t,s)\rho(s)k(s)q(s)-\frac{\gamma\rho(s)r(s)R^{\alpha}(t,s)}{\alpha^{\alpha}}\}ds=\infty, \qquad (2.3)$$

where

$$R(t,s)=h(t,s)+(H(t,s)k(s))^{1/\alpha}\left(\frac{1}{\rho(s)}\frac{d\rho(s)}{ds}+p(s)\right).$$

Proof. On the contrary we assume that (1.1) has a nonoscillatory solution x(t). We suppose without loss of generality that x(t)>0 for all $t\in[t_0,\infty)$. We define the function ω(t) as

$$\omega(t)=\rho(t)\frac{\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)}{|x|^{\alpha-2}x} \quad \text{for } t\geq t_0.$$

Thus

$$\frac{d\omega(t)}{dt}=\frac{1}{\rho(t)}\frac{d\rho(t)}{dt}\omega(t)+\rho(t)\frac{\frac{d}{dt}\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)}{|x|^{\alpha-2}x}-(\alpha-1)\frac{\rho(t)r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha}}{|x|^{\alpha}}.$$

This and equation (1.1) imply

$$\frac{d\omega(t)}{dt}\leq\frac{1}{\rho(t)}\frac{d\rho(t)}{dt}\omega(t)\text{-}\rho(t)[q(t)+p(t)\varphi(1,\frac{\omega(t)}{\rho(t)})]-(\alpha-1)[\gamma\rho(t)r(t)]^{\frac{-1}{\alpha-1}}|\omega(t)|^{\frac{\alpha}{\alpha-1}}.$$

From (2.1) we obtain

$$\frac{d\omega(t)}{dt}\leq\frac{1}{\rho(t)}\frac{d\rho(t)}{dt}\omega(t)\text{-}\rho(t)q(t)\text{-}p(t)\omega(t)-(\alpha-1)[\gamma\rho(t)r(t)]^{\frac{-1}{\alpha-1}}|\omega(t)|^{\frac{\alpha}{\alpha-1}}.$$

Multiply the above inequality by H(t,s)k(s) and integrate from T to t we obtain

$$\int_T^t H(t,s)k(s)\rho(s)q(s)ds\leq\int_T^t H(t,s)k(s)\frac{1}{\rho(s)}\frac{d\rho(s)}{ds}\omega(s)ds$$

$$\text{-}\int_T^t H(t,s)k(s)p(s)\omega(s)ds\text{-}\int_T^t H(t,s)k(s)\frac{d\omega(s)}{ds}ds-(\alpha-1)\int_T^t\frac{H(t,s)k(s)|\omega(s)|^{\frac{\alpha}{\alpha-1}}}{[\gamma\rho(s)r(s)]^{1/(\alpha-1)}}ds.$$

Since

$$-\int_T^t H(t,s)k(s)\frac{d\omega(s)}{ds}ds=H(t,T)k(T)\omega(T)+\int_T^t\frac{d}{ds}(H(t,s)k(s))\omega(s)ds$$

$$=H(t,T)k(T)\omega(T)\text{-}\int_T^t h(t,s)(H(t,s)k(s))^{\frac{\alpha}{\alpha-1}}\omega(s)ds.$$

The previous inequality becomes

$$\int_T^t H(t,s)k(s)\rho(s)q(s)ds \le H(t,T)k(T)\omega(T)$$

$$+\int_T^t H(t,s)k(s)\frac{1}{\rho(s)}\frac{d\rho(s)}{ds}|\omega(s)|ds+\int_T^t h(t,s)(H(t,s)k(s))^{\frac{\alpha-1}{\alpha}}\omega(s)ds$$

$$+\int_T^t H(t,s)k(s)p(s)|\omega(s)|ds-(\alpha-1)\int_T^t \frac{H(t,s)k(s)|\omega(s)|^{\frac{\alpha}{\alpha-1}}}{[\gamma\rho(s)r(s)]^{1/(\alpha-1)}}ds.$$

Hence we have

$$\int_T^t H(t,s)k(s)\rho(s)q(s)ds \le H(t,T)k(T)\omega(T)$$

$$+\int_T^t R(t,s)((H(t,s)k(s))^{\frac{\alpha-1}{\alpha}}|\omega(s)|ds-(\alpha-1)\int_T^t \frac{H(t,s)k(s)|\omega(s)|^{\frac{\alpha}{\alpha-1}}}{[\gamma\rho(s)r(s)]^{1/(\alpha-1)}}ds. \qquad (2.4)$$

Define

$$X=[\gamma\rho(s)r(s)]^{1/\alpha}[\frac{1}{\alpha}R(t,s)],$$

$$Y=\left(H(t,s)k(s)[\gamma\rho(s)r(s)]^{-1/(\alpha-1)}|\omega(s)|^{\frac{\alpha}{\alpha-1}}\right)^{1/\alpha}.$$

Since $\alpha>1$, then by Lemma 2.1,

$$R(t,s)(H(t,s)k(s))^{(\alpha-1)/\alpha}|\omega(s)|-\frac{(\alpha-1)H(t,s)k(s)}{[\gamma\rho(s)r(s)]^{1/(\alpha-1)}}|\omega(s)|^{\frac{\alpha}{\alpha-1}}\le\frac{\gamma\rho(s)r(s)}{\alpha^\alpha}R^\alpha(t,s),$$

for all $t>s\ge T$. Moreover, by (2.4) we also have for every $t\ge T$,

$$\int_T^t H(t,s)k(s)\rho(s)q(s)ds \le H(t,T)k(T)\omega(T)+\frac{\gamma\rho(s)r(s)R^\alpha(t,s)}{\alpha^\alpha},$$

or

$$\int_T^t\left\{H(t,s)k(s)\rho(s)q(s)-\frac{\gamma\rho(s)r(s)R^\alpha(t,s)}{\alpha^\alpha}\right\}ds\le H(t,T)k(T)\omega(T)$$

$$\le H(t,t_0)k(T)|\omega(T)|. \qquad (2.5)$$

We use the above inequality for $T=T_0$ to obtain

$$\int_{T_0}^{t}\left\{H(t,s)k(s)\rho(s)q(s)-\frac{\gamma\rho(s)r(s)R^{\alpha}(t,s)}{\alpha^{\alpha}}\right\}ds \le H(t,t_0)k(T_0)\left|\omega(T_0)\right|.$$

Therefore,

$$\begin{aligned}&\int_{t_0}^{t}\left\{H(t,s)k(s)\rho(s)q(s)-\frac{\gamma\rho(s)r(s)R^{\alpha}(t,s)}{\alpha^{\alpha}}\right\}ds\\ &=\int_{t_0}^{T_0}\left\{H(t,s)k(s)\rho(s)q(s)-\frac{\gamma\rho(s)r(s)R^{\alpha}(t,s)}{\alpha^{\alpha}}\right\}ds\\ &+\int_{T_0}^{t}\left\{H(t,s)k(s)\rho(s)q(s)-\frac{\gamma\rho(s)r(s)R^{\alpha}(t,s)}{\alpha^{\alpha}}\right\}ds.\\ &\le H(t,t_0)\left\{\int_{t_0}^{T_0}k(s)\rho(s)\left|q(s)\right|ds+k(T_0)\left|\omega(T_0)\right|\right\},\end{aligned}$$

for all $t \ge T_0$. This gives

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left\{H(t,s)k(s)\rho(s)q(s)-\frac{\gamma\rho(s)r(s)R^{\alpha}(t,s)}{\alpha^{\alpha}}\right\}ds$$

$$\le\left\{\int_{t_0}^{T_0}k(s)\rho(s)\left|q(s)\right|ds+k(T_0)\left|\omega(T_0)\right|\right\}<\infty,$$

which contradicts the assumption (2.3). This completes the proof.
Corollary 2.1. If the condition (2.3) is replaced by the conditions

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)k(s)\rho(s)q(s)ds=\infty,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\rho(s)r(s)R^{\alpha}(t,s)ds<\infty,$$

then the conclusion of Theorem 2.1 remains valid.

Theorem 2.2. Suppose that (2.1) and (2.2) are satisfied and let the functions H, h, and k be the same as in Theorem 2.1. Moreover, assume that

$$0<\inf_{t\ge s}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right]\le\infty, \qquad (2.6)$$

and

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\rho(s)r(s)R^{\alpha}(t,s)ds<\infty, \tag{2.7}$$

hold. If there exists a function $\Omega\in C([t_0,\infty),R)$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\Omega_+^{\alpha/(\alpha-1)}(s)}{\left(k(s)\rho(s)r(s)\right)^{1/(\alpha-1)}}=\infty, \tag{2.8}$$

and for every $T\geq t_0$,

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\{H(t,s)\rho(s)k(s)q(s)\text{-}\frac{\gamma\rho(s)r(s)R^{\alpha}(t,s)}{\alpha^{\alpha}}\}ds\geq\Omega(T), \tag{2.9}$$

where $\Omega_+(t)=\max\{\Omega(t),0\}$, then equation (1.1) is oscillatory.

Proof. On the contrary we assume that (1.1) has a nonoscillatory solution x(t). We suppose without loss of generality that x(t)>0 for all $t\in[t_0,\infty)$. Defining $\omega(t)$ as in the proof of Theorem 2.1, we obtain (2.4) then we get

$$\frac{1}{H(t,T)}\int_{T}^{t}\left\{H(t,s)\rho(s)k(s)q(s)\text{-}\frac{\gamma\rho(s)r(s)R^{\alpha}(t,s)}{\alpha^{\alpha}}\right\}ds\leq k(T)\omega(T)\text{-}J(t,T)$$

where

$$J(t,T)=\frac{1}{H(t,T)}\int_{T}^{t}\{\frac{\gamma\rho(s)r(s)R^{\alpha}(t,s)}{\alpha^{\alpha}}-(H(t,s)k(s))^{(\alpha-1)/\alpha}R(t,s)\left|\omega(s)\right|$$
$$+\frac{(\alpha-1)H(t,s)k(s)}{(\gamma\rho(s)r(s))^{1/(\alpha-1)}}\left|\omega(s)\right|^{\frac{\alpha}{(\alpha-1)}}\}ds,$$

for all $T\geq T_0$. Thus, by (2.9), we have

$$\Omega(T)\leq k(T)\omega(T)\quad\text{for all}\quad T\geq T_o \tag{2.10}$$

and

$$\limsup_{t\to\infty}J(t,T)<\infty\qquad\text{for all}\quad T\geq T_o. \tag{2.11}$$

Let

$$F(t)=\frac{1}{H(t,T_0)}\int_{T_0}^{t}R(t,s)(H(t,s)k(s))^{(\alpha-1)/\alpha}\left|\omega(s)\right|ds,$$

$$G(t)=\frac{(\alpha-1)}{H(t,T_0)}\int_{T_0}^{t}H(t,s)k(s)(\gamma\rho(s)r(s))^{\frac{-1}{(\alpha-1)}}\left|\omega(s)\right|^{\frac{\alpha}{\alpha-1}}ds,$$

for $t>T_0$. Then by (2.4) and (2.11) we get that

$$\lim_{t\to\infty} \sup|G(t)-F(t)| \leq \lim_{t\to\infty} \sup \frac{1}{H(t,T_0)} \int_{T_0}^{t} \{ \frac{(\alpha-1)H(t,s)k(s)}{(\gamma\rho(s)r(s))^{1/(\alpha-1)}} |\omega(s)|^{\frac{\alpha}{\alpha-1}}$$

$$-R(t,s)\left(H(t,s)k(s)\right)^{(\alpha-1)/\alpha}\omega|(s)|\}ds$$

$$\leq \lim_{t\to\infty} \sup J(t,T_0)<\infty. \qquad (2.12)$$

Now, we claim that

$$\int_{T_0}^{\infty} k(s)\frac{|\omega(s)|^{\alpha/(\alpha-1)}}{(\rho(s)r(s))^{1/(\alpha-1)}}ds < \infty. \qquad (2.13)$$

Suppose to the contrary that

$$\int_{T_0}^{\infty} k(s)\frac{|\omega(s)|^{\alpha/(\alpha-1)}}{(\rho(s)r(s))^{1/(\alpha-1)}}ds = \infty. \qquad (2.14)$$

By (2.6), there is a positive constant η satisfying

$$\inf_{s\geq t_0}\left[\lim_{t\to\infty} \inf \frac{H(t,s)}{H(t,t_0)}\right] > \eta. \qquad (2.15)$$

On the other hand, by (2.14) for any positive number μ there exists a $T_1>T_0$ such that

$$\int_{T_0}^{t} k(s)\frac{|\omega(s)|^{\alpha/(\alpha-1)}}{(\rho(s)r(s))^{1/(\alpha-1)}}ds \geq \frac{\gamma^{\frac{1}{\alpha-1}}\mu}{(\alpha-1)\eta} \quad \text{for all } t\geq T_1,$$

so for all $t\geq T_1$

$$G(t)=\frac{(\alpha-1)\gamma^{\frac{-1}{\alpha-1}}}{H(t,T_0)}\int_{T_0}^{t} H(t,s)d\left[\int_{T_0}^{s} k(u)\frac{|\omega(u)|^{\alpha/(\alpha-1)}}{(\rho(u)r(u))^{1/(\alpha-1)}}du\right]$$

$$=\frac{(\alpha-1)\gamma^{\frac{-1}{\alpha-1}}}{H(t,T_0)}\int_{T_0}^{t}\left[\frac{-\delta H(t,s)}{\delta s}\right]d\left[\int_{T_0}^{s} k(u)\frac{|\omega(u)|^{\alpha/(\alpha-1)}}{(\rho(u)r(u))^{1/(\alpha-1)}}du\right]ds$$

$$\geq\frac{(\alpha-1)\gamma^{\frac{-1}{\alpha-1}}}{H(t,T_0)}\int_{T_1}^{t}\left[\frac{-\delta H(t,s)}{\delta s}\right]d\left[\int_{T_0}^{s} k(u)\frac{|\omega(u)|^{\alpha/(\alpha-1)}}{(\rho(u)r(u))^{1/(\alpha-1)}}du\right]ds$$

$$\geq\frac{\gamma^{\frac{1}{\alpha-1}}\mu}{(\alpha-1)\eta}\frac{(\alpha-1)\gamma^{\frac{-1}{\alpha-1}}}{H(t,T_0)}\int_{T_1}^{t}\left[\frac{-\delta H(t,s)}{\delta s}\right]ds=\frac{\mu H(t,T_1)}{\eta H(t,T_0)}. \qquad (2.16)$$

From (2.15) we have

$$\lim_{t\to\infty} \inf \frac{H(t,T_1)}{H(t,t_0)} > \eta > 0.$$

So there exists $T_2 \geq T_1$ such that $\frac{H(t,T_1)}{H(t,t_0)} \geq \eta$ for all $t \geq T_2$. Therefore by (2.16) $G(t) \geq \eta$ for all $t \geq T_2$, and since μ is arbitrary constant, we conclude that

$$\lim_{t\to\infty} G(t) = \infty. \qquad (2.17)$$

Next, consider a sequence $\{t_n\}_{n=1}^{\infty}$ in (T_0,∞) with $\lim_{n\to\infty} t_n = \infty$ and such that

$$\lim_{n\to\infty} \left[G(t_n)\text{-}F(t_n)\right] = \lim_{t\to\infty} \sup \left[G(t)\text{-}F(t)\right].$$

In view of (2.12), there exists a constant M such that

$$G(t_n)\text{-}F(t_n) \leq M \quad \text{for all sufficient large n.} \qquad (2.18)$$

It follows from (2.17) that

$$\lim_{n\to\infty} G(t_n) = \infty. \qquad (2.19)$$

This and (2.18) give

$$\lim_{n\to\infty} F(t_n) = \infty. \qquad (2.20)$$

Then, by (2.18) and (2.19),

$$\frac{F(t_n)}{G(t_n)} \text{-}1 \geq \frac{\text{-}M}{G(t_n)} > \frac{\text{-}1}{2} \qquad \text{for n large enough.}$$

Thus,

$$\frac{F(t_n)}{G(t_n)} > \frac{1}{2} \qquad \text{for n large enough.}$$

This and (2.20) imply that

$$\lim_{n\to\infty} \frac{F^{\alpha}(t_n)}{G^{\alpha\text{-}1}(t_n)} = \infty. \qquad (2.21)$$

On the other hand, by the Holder's inequality, we have

$$F(t_n)=\frac{1}{\mathrm{H}(\mathrm{t_n},\mathrm{T_0})}\int_{\mathrm{T_0}}^{t_n}\mathrm{R}(\mathrm{t_n},\mathrm{s})(\mathrm{H}(\mathrm{t_n},\mathrm{s})\mathrm{k}(\mathrm{s}))^{(\alpha-1)/\alpha}\left|\omega(\mathrm{s})\right|\mathrm{ds}$$

$$\le\left\{\frac{\alpha-1}{\mathrm{H}(\mathrm{t_n},\mathrm{T_0})}\int_{\mathrm{T_0}}^{t_n}\mathrm{H}(\mathrm{t_n},\mathrm{s})\mathrm{k}(\mathrm{s})\frac{\left|\omega(\mathrm{s})\right|^{\alpha/(\alpha-1)}}{(\rho(s)r(s))^{1/(\alpha-1)}}\mathrm{ds}\right\}^{(\alpha-1)/\alpha}$$

$$\times\left\{\frac{\gamma}{(\alpha-1)^{\alpha-1}\mathrm{H}(\mathrm{t_n},\mathrm{T_0})}\int_{\mathrm{t_0}}^{t_n}\rho(s)r(s)R^{\alpha}(\mathrm{t_n},\mathrm{s})\mathrm{ds}\right\}^{1/\alpha}$$

$$\le\frac{G^{(\alpha-1)/\alpha}(t_n)}{(\alpha-1)^{(\alpha-1)/\alpha}}\left\{\frac{\gamma}{\mathrm{H}(\mathrm{t_n},\mathrm{T_0})}\int_{\mathrm{t_0}}^{t_n}\rho(s)r(s)R^{\alpha}(\mathrm{t_n},\mathrm{s})\mathrm{ds}\right\}^{1/\alpha},$$

and therefore,

$$\frac{\mathrm{F}^{\alpha}(\mathrm{t_n})}{\mathrm{G}^{\alpha-1}(\mathrm{t_n})}\le\frac{\gamma}{(\alpha-1)^{(\alpha-1)}\mathrm{H}(\mathrm{t_n},\mathrm{T_0})}\int_{\mathrm{t_0}}^{t_n}\rho(s)r(s)R^{\alpha}(\mathrm{t_n},\mathrm{s})\mathrm{ds}$$

$$\le\frac{\gamma}{(\alpha-1)^{(\alpha-1)}\eta\mathrm{H}(\mathrm{t_n},\mathrm{t_0})}\int_{\mathrm{t_0}}^{t_n}\rho(s)r(s)R^{\alpha}(\mathrm{t_n},\mathrm{s})\mathrm{ds}$$

for all large n. It follows from (2.21) that

$$\lim_{\mathrm{n}\to\infty}\frac{1}{\mathrm{H}(\mathrm{t_n},\mathrm{t_0})}\int_{\mathrm{t_0}}^{t_n}\rho(s)r(s)R^{\alpha}(\mathrm{t_n},\mathrm{s})\mathrm{ds}=\infty, \tag{2.22}$$

that is,

$$\lim_{\mathrm{t}\to\infty}\frac{1}{\mathrm{H}(\mathrm{t},\mathrm{t_0})}\int_{\mathrm{T_0}}^{t}\rho(s)r(s)R^{\alpha}(\mathrm{t_n},\mathrm{s})\mathrm{ds}=\infty,$$

which contradicts (2.7). Hence, (2.13) holds. Then, it follows from (2.10) that

$$\int_{\mathrm{T_0}}^{t}k(s)\frac{\Omega_{+}^{\alpha/(\alpha-1)}(s)}{\left(\mathrm{k}(\mathrm{s})\rho(\mathrm{s})\mathrm{r}(\mathrm{s})\right)^{1/(\alpha-1)}}ds\le\int_{\mathrm{T_0}}^{t}k(s)\frac{\left|\omega(s)\right|^{\alpha/(\alpha-1)}}{\left(\rho(\mathrm{s})\mathrm{r}(\mathrm{s})\right)^{1/(\alpha-1)}}ds<\infty,$$

which contradicts (2.8). This completes the proof of Theorem 2.2.

Theorem 2.3. Suppose that (2.1) and (2.2) are satisfied and let the functions H, h, ρ and k be the same as in Theorem 2.1. Moreover, assume that

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)k(s)q(s)ds<\infty, \qquad (2.23)$$

and (2.6) hold. If there exists a function $\Omega \in C([t_0,\infty),R)$ such that (2.8) and (2.9) hold, then equation (1.1) is oscillatory.

Proof. Without loss of generality, we may assume that there exists a solution x(t) of equation (1.1) such that $x(t)\neq 0$ on $[T_0,\infty)$ for some sufficiently large $T_0 \geq t_0$. Define ω(t) as of Theorem 2.1. As in the proofs of Theorem 2.1 and 2.2, we can obtain (2.4), (2.5) and (2.10). From (2.23) it follow that

$$\limsup_{t\to\infty} \left|G(t)-F(t)\right| \leq k(t_0)\omega(t_0)$$

$$-\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)k(s)q(s)ds<\infty, \qquad (2.24)$$

where F(t) and G(t) are defined as in the proof of Theorem 2.2. By (2.9) we have

$$\Omega(t_0) \leq \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)k(s)q(s)ds$$

$$-\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \rho(s)r(s)R^{\alpha}(t,s)ds.$$

This and (2.9) imply that

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \rho(s)r(s)R^{\alpha}(t,s)ds < \infty.$$

Considering a sequence $\{t_n\}_{n=1}^{\infty}$ in (T_0,∞) with $\lim_{n\to\infty} t_n=\infty$ and such that

$$\lim_{n\to\infty} \frac{1}{H(t_n,t_0)} \int_{t_0}^{t_n} \rho(s)r(s)R^{\alpha}(t_n,s)ds$$

$$= \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \rho(s)r(s)R^{\alpha}(t,s)ds<\infty. \qquad (2.25)$$

Now, suppose that (2.14) holds. With the same argument as in Theorem 2.2, we conclude that (2.17) is satisfied. By (2.24), there exists a constant M such that (2.18) is fulfilled. Then, following the procedure of the proof of Theorem 2.2, we see that (2.22) holds, which contradicts (2.25). This contradiction proves that (2.25) fails. The remainder of the proof is similar to that of Theorem 2.2, so we omit the details. This completes the proof of Theorem 2.3.

Theorem 2.4. Suppose that (2.1) and (2.2) are satisfied and let the functions H, h, ρ and k be the same as in Theorem 2.1. Moreover, suppose that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\rho(s)r(s)R^{\alpha}(t,s)ds<\infty, \qquad (2.26)$$

and (2.6) hold. If there exists a function $\Omega\in C([t_0,\infty),R)$ such that (2.8) hold for every $T\geq t_0$ and

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left\{H(t,s)\rho(s)k(s)q(s)-\frac{\gamma\rho(s)r(s)R^{\alpha}(t,s)}{\alpha^{\alpha}}\right\}ds\geq\Omega(T), \quad (2.27)$$

then equation (1.1) is oscillatory.
The proof of Theorem 2.4 can be carried out as the proof of Theorem 2.2 and therefore it will be omitted.

Remark 2.1. If $\alpha=2$ and $p(t)\equiv0$, $r(t)\equiv1$ and $\psi(x)\equiv1$, then Theorem 2.1, 2.2 extend and improve theorem in [17].

Remark 2.2. If $p(t)\equiv0$, $r(t)\equiv1$ and $\psi(x)\equiv1$, then Theorem 2.1, 2.3 and 2.4 extend and improve Theorem 2, 4 and 3 of Li [13], respectively.

Remark 2.3. If $p(t)\equiv0$, then Theorem 2.1-2.4 extend and improve Theorem 2, 4, 6 and 5 in [2], respectively.

Example 2.1. Consider the differential equation

$$\frac{d}{dt}\left(t^{-2}(1+e^{-|x(t)|})\frac{dx}{dt}\right)+t^{-5/2}x(t)=0, \quad t\geq t_0>1.$$

We note that

$$\alpha=2 \text{ and } \psi(x)=1+e^{-|x(t)|}.$$

Let

$$\rho(s)=1,\ k(s)=s^2 \text{ and } H(t,s)=(t-s)^2.$$

Then

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left\{H(t,s)k(s)\rho(s)q(s)-\frac{\gamma\rho(s)r(s)R^{\alpha}(t,s)}{\alpha^{\alpha}}\right\}ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\int_{t_0}^{t}\left(t^2s^{\frac{-1}{2}}-2ts^{\frac{1}{2}}+s^{\frac{3}{2}}-\frac{9t^2}{2s^2}-\frac{6t}{s}-2\right)ds=\infty.$$

Hence, this equation is oscillatory by Theorem 2.1. while, Ayanlar and Tiryaki [2], fails.

Example 2.2. Consider the differential equation

$$\frac{d}{dt}\left(\frac{1}{t}\frac{2+\cos^2 t}{1+3\cos^2 t}\frac{1+3x^2}{2+x^2}\frac{dx}{dt}\right)+\frac{1}{t^2}\frac{dx}{dt}+\frac{1}{t}x=0, \quad t\geq t_0=1.$$

We not that

$$0<\psi(x)=\frac{1+3x^2}{2+x^2}\leq3=\gamma, \quad \alpha=2.$$

If we take $\rho(t)=t$, $k(t)=t$, $H(t,s)=(t-s)^2$, then

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left\{H(t,s)k(s)\rho(s)q(s)-\frac{\gamma\rho(s)r(s)R^{\alpha}(t,s)}{\alpha^{\alpha}}\right\}ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_{1}^{t}\left((t-s)^2 s-\frac{3}{4}\left(\frac{2+\cos^2 s}{1+3\cos^2 s}\right)\left(2\sqrt{s}+\frac{t\sqrt{s}}{s^2}-\frac{\sqrt{s}}{s}\right)^2\right)ds$$

$$\geq\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_{1}^{t}\left((t-s)^2 s-\frac{1}{4}\left(2\sqrt{s}+\frac{t\sqrt{s}}{s^2}-\frac{\sqrt{s}}{s}\right)^2\right)ds=\infty.$$

Hence, this equation is oscillatory. One such solution of this equation is $x(t)=\cos t$.

3. OSCILLATION RESULTS FOR (1.2)

Theorem 3.1. Suppose that (2.1) and

$$\frac{g'(x)}{(\psi(x)|g(x)^{\alpha-2}|)^{1/(\alpha-1)}}\geq\delta>0 \quad \text{for } x\neq 0, \tag{3.1}$$

hold, and let the functions H, h and k be the same as in Theorem 2.1. Moreover, suppose that there exist $\rho\in C^1([t_0,\infty),(0,\infty))$. Then equation (1.2) is oscillatory if

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left\{H(t,s)k(s)\rho(s)q(s)-\frac{\beta^{1-\alpha}\rho(s)r(s)Q^{\alpha}(t,s)}{\alpha^{\alpha}}\right\}ds=\infty,$$

where

$$\beta=\frac{\delta}{\alpha-1} \quad \text{and} \quad Q(t,s)=\left|h(t,s)-\left(\frac{1}{\rho(s)}\frac{d\rho(s)}{ds}-p(s)\right)(H(t,s)k(s))^{1/\alpha}\right|.$$

Proof. Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (1.2) such that $x(t)\neq 0$ on $[T_0,\infty)$ for some sufficiently large $T_0\geq t_0$. Define $\omega(t)$ as

$$\omega(t)=\rho(t)\frac{r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}}{g(x)} \quad \text{for } t\geq t_0.$$

Thus,

$$\frac{d\omega(t)}{dt}=\frac{1}{\rho(t)}\frac{d\rho(t)}{dt}\omega(t)+\rho(t)\frac{\frac{d}{dt}\left(r(t)\psi(x)\left|\frac{dx}{dt}\right|^{\alpha-2}\frac{dx}{dt}\right)}{g(x)}\frac{dx}{dt}$$

$$-\frac{g'(x)}{(\psi(x)|g(x)^{\alpha-2}|)^{1/(\alpha-1)}}\frac{|\omega(t)|^{\frac{\alpha}{\alpha-1}}}{[\rho(t)r(t)]^{1/(\alpha-1)}}.$$

This and equation (1.2) imply

$$\frac{d\omega(t)}{dt}\le\frac{1}{\rho(t)}\frac{d\rho(t)}{dt}\omega(t)-\rho(t)[q(t)+p(t)\varphi(1,\frac{\omega(t)}{\rho(t)})]$$

$$-\frac{g'(x)}{(\psi(x)|g(x)^{\alpha-2}|)^{1/(\alpha-1)}}\frac{|\omega(t)|^{\frac{\alpha}{\alpha-1}}}{[\rho(t)r(t)]^{1/(\alpha-1)}}.$$

From (2.1) and (3.1) we have

$$\frac{d\omega(t)}{dt}\le\frac{1}{\rho(t)}\frac{d\rho(t)}{dt}\omega(t)-\rho(t)q(t)-p(t)\omega(t)-\delta\frac{|\omega(t)|^{\frac{\alpha}{\alpha-1}}}{[\rho(t)r(t)]^{1/(\alpha-1)}}.$$

Multiply the above inequality by H(t,s)k(s) and integrate from T to t we obtain

$$\int_T^t H(t,s)k(s)\rho(s)q(s)ds\le\int_T^t H(t,s)k(s)\left(\frac{1}{\rho(s)}\frac{d\rho(s)}{ds}-p(s)\right)\omega(s)ds$$

$$-\int_T^t H(t,s)k(s)\frac{d\omega(s)}{ds}ds-\delta\int_T^t H(t,s)k(s)[\gamma\rho(s)r(s)]^{\frac{-1}{\alpha-1}}|\omega(s)|^{\frac{\alpha}{\alpha-1}}\frac{d\omega(s)}{ds}ds.$$

Since

$$-\int_T^t H(t,s)k(s)\frac{d\omega(s)}{ds}ds=H(t,T)k(T)\omega(T)+\int_T^t\frac{\delta}{\delta s}(H(t,s)k(s))\omega(s)ds$$

$$=H(t,T)k(T)\omega(T)-\int_T^t h(t,s)(H(t,s)k(s))^{\frac{(\alpha-1)}{\alpha}}\omega(s)ds.$$

The previous inequality becomes

$$\int_T^t H(t,s)k(s)\rho(s)q(s)ds\le H(t,T)k(T)\omega(T)+\int_T^t Q(t,s)(H(t,s)k(s))^{\frac{(\alpha-1)}{\alpha}}\left|\omega(s)\right|ds$$

$$-(\alpha-1)\int_T^t\frac{\beta H(t,s)k(s)\left|\omega(s)\right|^{(\alpha-1)/\alpha}}{[\rho(s)r(s)]^{1/(\alpha-1)}}ds. \qquad (3.2)$$

Define

$$X = \beta^{(1-\alpha)/\alpha}[\rho(s)r(s)]^{1/\alpha}[\frac{1}{\alpha}Q(t,s)],$$

$$Y = \left(\beta^{(1-\alpha)/\alpha}\left[H(t,s)k(s)\right]^{(\alpha-1)/\alpha}[\rho(s)r(s)]^{-1/\alpha}\left|\omega(s)\right|\right)^{1/(\alpha-1)}.$$

Then use the lemma 2.1, we have

$$Q(t,s)(H(t,s)k(s))^{(\alpha-1)/\alpha}\left|\omega(s)\right|-(\alpha-1)\frac{\beta H(t,s)k(s)|\omega(s)|^{\alpha/(\alpha-1)}}{[\rho(s)r(s)]^{1/(\alpha-1)}} \le \frac{\beta^{(1-\alpha)}\rho(s)r(s)Q^{\alpha}(t,s)}{\alpha^{\alpha}}.$$

From (3.2) we have

$$\int_T^t \left\{H(t,s)k(s)\rho(s)q(s)-\frac{\beta^{(1-\alpha)}\rho(s)r(s)Q^{\alpha}(t,s)}{\alpha^{\alpha}}\right\}ds \le H(t,T)k(T)\omega(T).$$

The remainder of the proof proceeds as in the proof of Theorem 2.1. The proof is complete.

Following the procedure of the proof of Theorem 2.2 and 2.3, we can also prove the following theorems.

Theorem 3.2. Suppose that (2.1) and (3.1) hold, and let the functions H, h and k be the same as in Theorem 2.1. If there exist two functions $\rho \in C^1([t_0,\infty),(0,\infty))$ and $\Omega \in C([t_0,\infty),R)$ such that

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t \rho(s)r(s)Q^{\alpha}(t,s)ds<\infty, \qquad (3.3)$$

and that for every $T \ge t_0$,

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \{H(t,s)\rho(s)k(s)q(s)-\frac{\beta^{(1-\alpha)}\rho(s)r(s)Q^{\alpha}(t,s)}{\alpha^{\alpha}}\}ds \ge \Omega(T), \quad (3.4)$$

and (2.8) hold, then every solution of (1.2) is oscillatory.

Theorem 3.3. Suppose that (2.1) and (3.1) hold, and let the functions H, h and k be the same as in Theorem 2.1. If there exist two functions $\rho \in C^1([t_0,\infty),(0,\infty))$ and $\Omega \in C([t_0,\infty),R)$ such that (2.8), (3.4) and

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\rho(s)k(s)q(s)ds<\infty, \qquad (3.5)$$

hold, then every solution of (1.2) is oscillatory.

Theorem 3.4. Suppose that (2.1) and (3.1) are satisfied. Let the functions H, h and k be the same as in Theorem 2.1. If there exist two functions $\rho \in C^1([t_0,\infty),(0,\infty))$ and $\Omega \in C([t_0,\infty),R)$ such that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\rho(s)r(s)Q^{\alpha}(t,s)ds<\infty, \qquad (3.6)$$

and for every $T\geq t_0$,

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left\{H(t,s)\rho(s)k(s)q(s)-\frac{\beta^{1-\alpha}\rho(s)r(s)Q^{\alpha}(t,s)}{\alpha^{\alpha}}\right\}ds\geq\Omega(T), \quad (3.7)$$

and (2.8) hold, then every solution of (1.2) is oscillatory.

Remark 3.1. If $p(t)\equiv0$ and $\alpha=2$, then Theorem 3.2 and 3.4 extend and improve Theorem 4 and 3 of Grace [3].

Remark 3.2. If $p(t)\equiv0$ and $H(t,s)=(t-s)^n$ from Theorem 3.1, we obtain Theorem 2 of Agarwal and Grace [1].

Example 3.1. Consider the differential equation

$$\frac{d}{dt}\left(t^{-2}x^{2}(t)\frac{dx}{dt}\right)+t^{-1}x^{3}(t)=0, \quad t\geq t_0>1.$$

We note that

$$\alpha=2,\ r(t)=t^{-2},\ q(t)=t^{-1} \text{ and } \frac{g'(x)}{\psi(x)}=3.$$

Let

$$\rho(s)=1,\ k(s)=s^2 \text{ and } H(t,s)=(t-s)^2.$$

Then

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left\{H(t,s)k(s)\rho(s)q(s)-\frac{\beta^{1-\alpha}\rho(s)r(s)Q^{\alpha}(t,s)}{\alpha^{\alpha}}\right\}ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\int_{t_0}^{t}\left(t^2s-2ts+s^3-\frac{3t^2}{4s^2}+\frac{t}{s}-\frac{1}{3}\right)ds=\infty.$$

Hence, this equation is oscillatory by Theorem 3.1.

Example 3.2. Consider the differential equation

$$\left(\frac{1}{t}\frac{2+\cos^2 t}{1+3\cos^2 t}\frac{1+3x^2}{2+x^2}\frac{dx}{dt}\right)+\frac{1}{t^2}\frac{dx}{dt}+\frac{1}{t(1+3\cos^2 t)}(x+x^3)=0, \quad t\geq t_0=1.$$ We not that

$$\frac{g'(x)}{\psi(x)}=2+x^2\geq2=\delta, \quad \alpha=2.$$

If we take $\rho(t)=t$, $k(t)=t$, $H(t,s)=(t-s)^2$, then

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left\{H(t,s)k(s)\rho(s)q(s)-\frac{\beta^{1-\alpha}\rho(s)r(s)Q^{\alpha}(t,s)}{\alpha^{\alpha}}\right\}ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_{1}^{t}\left(\frac{(t-s)^2 s}{1+3\cos^2 s}-\frac{1}{2}\left(\frac{2+\cos^2 s}{1+3\cos^2 s}\right)\left(4\sqrt{s}+\frac{t\sqrt{s}}{s^2}-\frac{\sqrt{s}}{s}-\frac{2t}{\sqrt{s}}\right)^2\right)ds$$

$$\geq\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_{1}^{t}\left((t-s)^2\left(\frac{s}{4}\right)-\frac{1}{4}\left(4\sqrt{s}+\frac{t\sqrt{s}}{s^2}-\frac{\sqrt{s}}{s}-\frac{2t}{\sqrt{s}}\right)^2\right)ds=\infty.$$

Hence, this equation is oscillatory. One such solution of this equation is x(t)=cos t.

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