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Functional integral inclusions

Existence of integrable solutions for a functional integral inclusion

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Abstract

In this paper we concern with the nonlinear functional integral inclusion in the real line R. The existence of integrable solutions will be study under the assumptions that the set-valued function F has L^1-Caratheodory selection and measurable selection. We reformulate the functional integral inclusion according to these selections and study two cases of such problem.

Keywords : Set-valued function, integrable solution, functional integral inclusion, selection of the set-valued function, L^1-Caratheodory selection, measurable selection.

1 Introduction

Let R denote the real line. Let $I = [0, T]$ and let $L^1(I)$ be the class of all Lebesgue integrable functions defined on the interval I, with the norm

$$\|x\| = \int_0^T |x(t)|dt.$$

The topic of differential and integral inclusions is of much interest in the subject of set-valued analysis.

The existence theorems for the inclusions problems are generally obtained under the assumptions that the set-valued function is either lower or upper semicontinuous on the domain of its definitions (see [1] and [16]) and for the discontinuity of the set-valued function (see [8]).
The integral inclusions have been studied by B.C. Dhage and D. O'Regan (see [7] and [16]) for the existence results under Caratheodory condition of F.

Let $m : [0, T] \to [0, T]$ be continuous and nondecreasing function.
In this paper we study the existence of integrable solution $x \in L^1[0, T]$ of the functional integral inclusion

$$x(t) \in F(t, \int_0^t g(s, x(m(s)))ds), \quad t \in [0, T] \tag{1}$$

where $F : [0, T] \times R \to P(R)$ is a nonlinear set-valued mapping and $P(R)$ denote the family of nonempty subsets of R under a set of several suitable assumptions on the function F.

Our study is based on the selections of the set-valued function F on which we have a functional integral equation, such a type has been studied in several papers (see [5], [12]-[13] and [17]).

We study two approaches, the first approach we study the existence of integrable solution $x \in L^1[0, T]$ by reformulating the functional integral inclusion (1) into a coupled system under the assumption that the set-valued function F has L^1-Caratheodory selection and the second approach study the integrable solution $x \in L^1[0, T]$ under the assumption that the set-valued function F has measurable selection, .

2 Preliminaries

Now, we present some definitions and results that will be used in this work.

Definition 1 [*11*] *Let (T, Σ) be a measurable space and X be a topological space, a multivalued function $F : T \to X$ is measurable if for each open set A in X the set*

$$F^{-1}(A) = \{t \in T : F(t) \cap A \neq \phi\}$$

is measurable. (I.e. $F^{-1}(A) \in \Sigma$).

Definition 2 [*10*] *A set-valued function $F : I \times R \to P(R)$ is called L^1-Caratheodory if:*
(1) $t \to F(t,x)$ is measurable in $t \in I$ for all $x \in R$, and
(2) $x \to F(t,x)$ is upper semicontinuous in $x \in R$ for almost all $t \in I$.
(3) There exists $h \in L^1[I,R]$ such that

$$|F(t,x)| = sup\{|f| : \ f \in F(t,x)\} \leq h(t)$$

for almost all $t \in I$.

Definition 3 [*10*] *A single-valued function $f : I \times R \to R$ is called L^1-Caratheodory if:*
(1) $t \to f(t,x)$ is measurable in $t \in I$ for all $x \in R$, and
(2) $x \to f(t,x)$ is continuous in $x \in R$ for almost all $t \in I$.
(3) There exists $h \in L^1[I,R]$ such that

$$|f(t,x)| \leq h(t)$$

for almost all $t \in I$.

Definition 4 [*10*] *The set*

$$S^1_{F(.,x(.))} = \{f \in L^1(I,R) : \ f(t,x) \in F(t,x(t)) \ for \ a.e. \ t \in I\}$$

is called the set of selections of the set-valued function F that belongs to L^1.

Theorem 1 [*10*] *Let $F : I \times R \to P(R)$ be an L^1-Caratheodory multifunction, the set $S^1_{F(.,x(.))}$ is nonempty (i.e. there exists a selector f of F which belongs to L^1).*

Theorem 2 [*6*] *Let $F : I \times R \to P(R)$ be an multifunction. Assume that the multifunction F satisfies the following assumptions*
(1) $F(t,x)$ is nonempty, closed and convex for all $(t,x) \in I \times R$,
(2) $F(t,.)$ is lower semicontinuous from R into R,
(3) $F(.,.)$ is measurable,
Then there exists a measurable selection for $F(.,x(.))$. This selection is integrable if

$$|F(t,x)| \leq a(t) + b(t)|x|$$

for each $t \in (0,1)$ and $x \in R$, where $a(.) \in L^1(0,1)$ and $b(.)$ is measurable and bounded.

Theorem 3 [*9*] *"Kolmogorov Compactness Criterion"*
Let $\Omega \subseteq L^p(0,1),\ 1 \leq p \leq \infty$. *If*
(i) Ω *is bounded in* $L^p(0,1)$, *and*
(ii) $x_h \to x$ *as* $h \to 0$ *uniformly with respect to* $x \in \Omega$, *then* Ω *is relatively compact in* $L^p(0,1)$, *where*

$$x_h(t) = \frac{1}{h}\int_t^{t+h} x(s)ds$$

Theorem 4 [*10*] *"Schauder fixed point theorem"*
Let Q *be a convex subset of a Banach space* X, $\ T : Q \to Q$ *be a compact, continuous map. Then* T *has at least one fixed point in* Q.

3 Existence of integrable solution

In this section, we present our main result by proving the existence of at least one integrable solution $x \in L^1[0,T]$ of the functional integral inclusion (1).

3.1 Coupled system approach

Let $F : I \times R \to R$ satisfy the following assumptions:
(H1) The set $F(t,x)$ is nonempty, closed and convex for all $(t,x) \in I \times R$.
(H2) $F(t,.)$ is upper semicontinuous in $x \in R$ for each $t \in I$.
(H3) $F(.,x)$ is measurable in $t \in I$ for each $x \in R$.
(H4) There exists an integrable function $h(t) \in L^1[I,R]$ such that

$$|F(t,x)| = sup\{|f| :\ f \in F(t,x)\} \leq h(t)$$

for almost all $t \in [0,T]$.
(H5) The function $g : [0,T] \times R \to R$ satisfies Caratheodory condition, i.e $g(t,.)$ is continuous in $x \in R$ for each $t \in I$ and $g(.,x)$ is measurable in $t \in I$ for each $x \in R$.
(H6)There exists an integrable function $a \in L^1[I,R]$ and a positive constant $b > 0$ such that

$$|g(t,x)| \leq |a(t)| + b|x|, \quad \forall\ t \in I,\ x \in R.$$

(H7) There exists $\beta > 0$ such that $m^{'}(t) > \beta$, for every $t \in I$.
For the application of these assumptions (see [2], [4], [6], [9] and [15]).

Now, let

$$y(t)=\int_0^t g(s,x(m(s)))ds, \quad t\in[0,T].$$

Then the nonlinear functional integral inclusion (1) can written in the form of the coupled system of functional inclusion and functional integral equation.

$$x(t)\in F(t,y(t)), \qquad t\in[0,T] \tag{2}$$

$$y(t)=\int_0^t g(s,x(m(s))ds, \quad t\in[0,T] \tag{3}$$

Definition 5 *Let X be the class of all ordered pairs (u,v), $u,v\in C[I,R]$, with the norm $\|(u,v)\|_X=\|u\|+\|v\|$.*

Definition 6 *By a solution of the coupled system (2), (3) we mean the functions $x,y\in L^1[0,T]$ satisfying (2), (3).*

Now for the existence of integrable solution $U=(x,y)$, $x,y\in L^1[0,T]$ of the coupled system (2), (3) we have the following theorem.

Theorem 5 *Let the assumptions (H1)-(H7) be satisfied. Then there exists at least one integrable solution $U=(x,y)$, $x,y\in L^1[0,T]$ of the coupled system (2), (3).*

Proof. Let the set-valued function F satisfy the assumptions (H1)-(H4), then from Theorem 2.1, we deduce that there exists a selection $f\in F$, this selection is L^1-Caratheodory,
i.e. f satisfy the following assumptions:
(I) $f(t,.)$ is continuous in $x\in R$ for each $t\in I$.
(II) $f(.,x)$ is measurable in $t\in I$ for each $x\in R$.
(III) There exists an integrable function $h(t)\in L^1[I,R]$ such that

$$|f(t,x)|\le h(t)$$

for almost all $t\in[0,T]$.
And

$$x(t)=f(t,y(t)), \quad t\in[0,T] \tag{4}$$

Now consider the coupled system (3), (4).
Let

$$U(t)=(x(t),y(t))=(f(t,y(t)),\ \int_0^t g(s,x(m(s)))ds), \quad t\in[0,T]$$

.

Let A be any operator defined by

$$AU(t) = A(x(t), y(t)) = (A_1 y(t), A_2 x(t))$$

where

$$A_1 y(t) = f(t, y(t)), \quad t \in [0, T]$$

$$A_2 x(t) = \int_0^t g(s, x(m(s)))ds, \quad t \in [0, T]$$

Let the set Q_r be defined as

$$Q_r = \{U = (x, y) \in X : x, y \in L^1[I, R], \|U\| \le r\}, \; r = \|h\|_{L^1} + \|a\|_{L^1} T + \frac{b}{\beta}\|x\|_{L^1} T.$$

Then, it is clear that it is nonempty, bounded, closed and convex set.
Let $U \in Q_r$ be an arbitrary ordered pair, then

$$|A_1 y(t)| = |f(t, y(t))|, \quad t \in [0, T]$$

from the properties (1) and (3), of Definition 2.3, and by integration, we get

$$\begin{aligned} \int_0^t |A_1 y(s)| ds &= \int_0^t |f(s, y(s))| ds \\ &\le \int_0^t h(s) ds. \end{aligned}$$

Then

$$\|A_1 y\|_{L^1} \le \|h\|_{L^1}.$$

And

$$\begin{aligned} |A_2 x(t)| &= |\int_0^t g(s, x(m(s)))ds| \\ &\le \int_0^t |g(s, x(m(s)))| ds \\ &\le \int_0^t \{|a(s)| + b|x(m(s))|\} ds \\ &\le \int_0^t |a(s)| ds + b \int_0^t |x(m(s))| ds \end{aligned}$$

taking $m(s) = u$ and $ds = \frac{du}{m'(s)}$, then

$$\begin{aligned}|A_2x(t)| &\leq \int_0^t |a(s)|ds + b\int_{m(0)}^{m(t)} |x(u)|\frac{du}{m'(s)} \\ &\leq \int_0^t |a(s)|ds + \frac{b}{\beta}\int_0^t |x(u)|du \\ &\leq \|a\|_{L^1} + \frac{b}{\beta}\|x\|_{L^1}\end{aligned}$$

taking integration over t, we get

$$\|A_2x\|_{L^1} \leq \|a\|_{L^1}T + \frac{b}{\beta}\|x\|_{L^1}T.$$

Now

$$\begin{aligned}\|AU\|_X &= \|A_1y\| + \|A_2x\| \\ &\leq \|h\|_{L^1} + \|a\|_{L^1}T + \frac{b}{\beta}\|x\|_{L^1}T = r.\end{aligned}$$

Hence $AU \in Q_r$, which proves that $AQ_r \subset Q_r$, *i.e.* $A : Q_r \to Q_r$.
Now, let us observe that the assumptions (I)-(III) imply that A_1 is continuous on the set Q_r (see [3]), and from the assumption (H5)-(H6) the operator A_2 is continuous on the set Q_r (see [4] and [15]).
Hence we deduce that the operator A is continuous on Q_r.
Finally, we will show that A is compact, to prove this we will apply Kolmogorov compactness criterion.
Let Ω be a subset of the set Q_r, then $(A\Omega)$ is bounded in L^1, i.e. condition (i) of theorem 2.3 is satisfied. It remains to show that $(AU)_h \to (AU)$ in L^1 as $h \to 0$ uniformly with respect to $AU \in \Omega$, we have the following.
Let $U \in \Omega \subset Q_r$, that is $y, x \in \Omega \subset Q_r$, $\{A_1\Omega\}$, $\{A_2\Omega\} \subset Q_r \subset L^1[0,T]$, then

$$\begin{aligned}A_1y_h(t) - A_1y(t) &= \frac{1}{h}\int_t^{t+h} A_1y(s)ds - A_1y(t) \\ &= \frac{1}{h}\int_t^{t+h} (A_1y(s) - A_1y(t))ds\end{aligned}$$

and

$$|A_1y_h(t) - A_1y(t)| \leq \frac{1}{h}\int_t^{t+h} |A_1y(s) - A_1y(t)|ds$$

then

$$\begin{aligned}\|A_1y_h - A_1y\|_{L^1} &= \int_0^T |A_1y_h(t) - A_1y(t)|dt \\ &\leq \int_0^T \frac{1}{h}\int_t^{t+h} |A_1y(s) - A_1y(t)|dsdt \\ &\leq \int_0^T \frac{1}{h}\int_t^{t+h} |f(s,y(s)) - f(t,y(t))|dsdt\end{aligned}$$

since $f \in L^1[0,T]$, then

$$\frac{1}{h}\int_t^{t+h} |f(s,y(s)) - f(t,y(t))|ds \to 0 \quad as \quad h \to 0 \quad for \quad t \in I.$$

Therefore

$$(A_1y)_h \to (A_1y), \quad uniformly \quad as \quad h \to 0$$

And

$$|A_2x_h(t) - A_2x(t)| \leq \frac{1}{h}\int_t^{t+h} |A_2x(s) - A_2x(t)|ds$$

then

$$\begin{aligned}\|A_2x_h - A_2x\|_{L^1} &= \int_0^T |A_2x_h(t) - A_2x(t)|dt \\ &\leq \int_0^T \frac{1}{h}\int_t^{t+h} |A_2x(s) - A_2x(t)|dsdt \\ &\leq \int_0^T \frac{1}{h}\int_t^{t+h} |\int_0^s g(v,x(m(v)))dv - \int_0^t g(w,x(m(w)))dw|dsdt\end{aligned}$$

since $g \in L^1[0,T]$, then

$$\frac{1}{h}\int_t^{t+h} |\int_0^s g(v,x(m(v)))dv - \int_0^t g(w,x(m(w)))dw|ds \to 0 \ as \ h \to 0 \ for \ t \in I.$$

Therefore

$$(A_2x)_h \to (A_2x), \quad uniformly \quad as \quad h \to 0.$$

Hence

$$(AU)_h \to (AU), \quad uniformly \quad as \quad h \to 0.$$

Then, by theorem 2.3, we have that $(A\Omega)$ is relatively compact, that is A is compact operator.
According to Schauder fixed point theorem, there exists at least one fixed point

$U \in Q_r$, and then the system (3), (4) and consequently the system (2), (3) has at least one integrable solution $U = (x, y) \in Q_r, \;\; x, y \in L^1[0, T]$.
Hence, there exists at least one integrable solution of the functional integral inclusion (1).

3.2 Functional integral inclusion approach

Consider now the functional integral inclusion (1) under the assumptions (H5)-(H7) and the following assumptions
(I) The set $F(t, x)$ is nonempty, closed and convex for all $(t, x) \in I \times R$,
(II) $F(t, .)$ is lower semicontinuous from R into R,
(III) $F(., .)$ is measurable in $t \in I$ for each $x \in R$.
(IV) There exists an integrable real function $a_1 \in L^1[I, R]$ and a measurable bounded function b_1, such that

$$|F(t, x)| \leq |a_1(t)| + b_1(t)|x|, \quad \forall\, t \in I, \; x \in R.$$

Definition 7 *By a solution of the functional integral inclusion (1) we mean the function $x(.) \in L^1[0, T]$ satisfying (1).*

Now for the existence of integrable solution $x \in L^1[0, T]$ of the functional integral inclusion (1) we have the following theorem.

Theorem 6 *Let the assumptions (I)-(IV) and (H5)-(H7) be satisfied, then there exists at least one integrable solution $x \in L^1[0, T]$ of the functional integral inclusion (1).*

Proof. Let the set-valued function F satisfy the assumptions (I)-(IV), then from theorem 2.2, we deduce that there exists a selection $f \in F, \; f : I \times R \to R$, which satisfies:
(i) $f(t, .)$ is continuous in $x \in R$ for each $t \in I$.
(ii) $f(., x)$ is measurable in $t \in I$ for each $x \in R$.
(iii) There exists an integrable real function $a_1 \in L^1[I, R]$ and a measurable bounded function b_1, such that

$$|f(t, x)| \leq |a_1(t)| + b_1(t)|x|, \quad \forall\, t \in I, \; x \in R.$$

And f satisfy the functional integral equation

$$x(t) = f(t, \int_0^t g(s, x(m(s)))ds), \quad t \in I \tag{5}$$

Define the operator A by

$$Ax(t) = f(t, \int_0^t g(s, x(m(s)))ds), \quad t \in I \tag{6}$$

Let the set Q_r be defined as

$$Q_r = \{x \in R : \ \|x\|_{L^1} \leq r\}; \quad r = \frac{\|a_1\|_{L^1} + b_1\|a\|_{L^1}T}{1 - \frac{b_1 b}{\beta}T}.$$

Then, it is clear that it is nonempty, bounded, closed and convex set. Let $x \in Q_r$ be an arbitrary element, then

$$\begin{aligned}
|Ax(t)| &= |f(t, \int_0^t g(s, x(m(s)))ds)| \\
&\leq |a_1(t)| + |b_1(t)|| \int_0^t g(s, x(m(s)))ds| \\
&\leq |a_1(t)| + |b_1(t)| \int_0^t |g(s, x(m(s)))|ds \\
&\leq |a_1(t)| + |b_1(t)| \int_0^t \{|a(s)| + b|x(m(s))|\}ds \\
&\leq |a_1(t)| + |b_1(t)| \int_0^t |a(s)|ds + |b_1(t)|b \int_0^t |x(m(s))|ds
\end{aligned}$$

taking $m(s) = u$ and $ds = \frac{du}{m'(s)}$, then

$$\begin{aligned}
|Ax(t)| &\leq |a_1(t)| + |b_1(t)| \int_0^t |a(s)|ds + |b_1(t)b| \int_{m(0)}^{m(t)} |x(u)| \frac{du}{m'(s)} \\
&\leq |a_1(t)| + |b_1(t)| \int_0^t |a(s)|ds + \frac{|b_1(t)|b}{\beta} \int_0^t |x(u)|du \\
&\leq |a_1(t)| + |b_1(t)| \int_0^t |a(s)|ds + \frac{|b_1(t)|b}{\beta} \int_0^t |x(u)|du \\
&\leq |a_1(t)| + |b_1(t)|\|a\|_{L^1} + \frac{|b_1(t)|b}{\beta}\|x\|_{L^1}.
\end{aligned}$$

Taking integration over t, we get

$$\begin{aligned}
\|Ax\|_{L^1} &\leq \int_0^t |a_1(s)|ds + \int_0^t |b_1(s)|\|a\|_{L^1} \\
&+ \frac{b}{\beta}\int_0^t |b_1(s)|\|x\|_{L^1} \\
&\leq \|a_1\|_{L^1} + b_1\|a\|_{L^1}T + \frac{b_1 b}{\beta}\|x\|_{L^1}T \\
&\leq \|a_1\|_{L^1} + b_1\|a\|_{L^1}T + \frac{b_1 b}{\beta}rT = r,
\end{aligned}$$

then

$$\|Ax\|_{L^1} \leq r.$$

Which prove that $A : Q_r \to Q_r$.

Now, we will show that A is continuous.
To achieve this goal we will apply the property of continuity of the function f condition (i). Let $\{x_n\} \subset Q_r$, x_n converges to x_0 in Q_r.
Now

$$\begin{aligned}
|g(t, x_n(m(t)))| &\leq |a(t)| + b|x_n(m(t))| \\
&\leq |a(t)| + br,
\end{aligned}$$

and $x_n \to x_0$, then $g(t, x_n) \to g(t, x_0)$.
Then applying Lebesgue dominated convergence theorem

$$Ax_n(t) = f(t, \int_0^t g(s, x_n(m(s)))ds) \quad t \in I$$

Take limit for both sides, we have

$$\begin{aligned}
\lim_{n\to\infty} Ax_n(t) &= \lim_{n\to\infty} f(t, \int_0^t g(s, x_n(m(s)))ds) \\
&= f(t, \lim_{n\to\infty} \int_0^t g(s, x_n(m(s)))ds) \\
&= f(t, \int_0^t \lim_{n\to\infty} g(s, x_n(m(s)))ds) \\
&= f(t, \int_0^t g(s, x_0(m(s)))ds) \\
&= Ax_0.
\end{aligned}$$

Hence A is continuous.
Finally, we will show that A is compact, to prove this we will apply Kolmogorov compactness criterion.
Let Ω be a subset of the set Q_r, then $A(\Omega)$ is bounded in L^1, i.e. condition (i) of theorem 2.3 is satisfied.
It remains to show that $(Ax)_h \to Ax$ in L^1 as $h \to 0$ uniformly with respect to $Ax \in \Omega$. we have the following:
Let $x \in \Omega \subset Q_r, \quad \{A\Omega\} \subset Q_r \subset L^1[o, T]$, then

$$\begin{aligned} Ax_h(t) - Ax(t) &= \frac{1}{h}\int_t^{t+h} Ax(s)ds - Ax(t) \\ &= \frac{1}{h}\int_t^{t+h} (Ax(s) - Ax(t))ds \end{aligned}$$

and

$$|Ax_h(t) - Ax(t)| = \frac{1}{h}\int_t^{t+h} |Ax(s) - Ax(t)|ds.$$

Then

$$\begin{aligned} \|Ax_h - Ax\|_{L^1} &= \int_0^T |Ax_h(t) - Ax(t)|dt \\ &\leq \int_0^T \frac{1}{h}\int_t^{t+h} |Ax(s) - Ax(t)|dsdt. \end{aligned}$$

Since $f \in L^1[0, T]$, then

$$\frac{1}{h}\int_t^{t+h} |Ax(s) - Ax(t)|ds \to 0 \quad as \quad h \to 0.$$

Therefore, by theorem 2.3 we have that $A(\Omega)$ is relatively compact, that is A is compact operator.
According to Schauder fixed point theorem, there exists at least one integrable solution of the functional integral equation (5), hence, there exists at least one integrable solution of the functional integral inclusion (1).

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