DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 3, 2017
Electronic Journal,
reg. N ΦC77-39410 at 15.04.2010
ISSN 1817-2172

http://www.math.spbu.ru/diffjournal
e-mail: jodiff@mail.ru

Differential equations with randomness

Solvability of a stochastic differential equation with nonlocal and integral conditions

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Abstract

In this paper we are concerned with a stochastic differential equation with nonlocal condition. We study the existence of a unique mean square continuous solution. The continuous dependencies of the solution with respect to the random initial value and deterministic coefficients of the nonlocal condition are shown. A stochastic differential equation with an integral condition is considered as well.
Keywords: Nonlocal condition, unique mean square solution, continuous dependence, random data, nonlocal coefficients, integral condition.

1 Introduction

Problems with nonlocal conditions have been extensively studied by several authors in the last decades for ordinary differential equations. The reader is referred to ([3]-[4]) and ([6]-[8]), and references therein. Also problems of stochastic differential equations have been extensively studied by several authors in the last decades, especially those contain the noisy term of Brownian motion $W(t)$.

The reader is referred to ([1]-[2]) and ([10]-[17]) and references therein.
Here we are concerned with the stochastic differential equation

$$\frac{dX(t)}{dt} = f(t, X(t)) + W(t), \quad t \in (0, T] \tag{1}$$

with the nonlocal random initial condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = X_0, \quad \tau_k \in (0, T), \tag{2}$$

where X_0 is a second order random variable independent of the Brownian motion (or Wiener process) $W(t)$ and a_k are positive real numbers.
The existence of a unique mean square continuous solution will be studied. The continuous dependencies with respect to random initial value X_0 and deterministic coefficients of the nonlocal condition a_k will be established. The problem (1) with the mean square Riemann-Stieltjes integral condition

$$X(0) + \int_0^T X(s)dv(s) = X_0. \tag{3}$$

will be considered.

2 Integral representation

Let $I = [0, T]$, (Ω, F, P) be a fixed probability space, where Ω is a sample space, F is a $\sigma-$algebra and P is a probability measure. We denote by $L_2(\Omega)$ the Banach space of random variables $X : \Omega \to R$ such that

$$\int_\Omega X^2 dP < \infty.$$

Let $X(t;\omega) = \{X(t), t \in I, \omega \in \Omega\}$ be a second order stochastic process, i.e.,

$$E(X^2(t)) < \infty, t \in I.$$

Now let $C = C(I, L_2(\Omega))$ be the class of all mean square continuous second order stochastic processes with the norm

$$\| X \|_C = \sup_{t \in [0,T]} \| X(t) \|_2 = \sup_{t \in [0,T]} \sqrt{E(X(t))^2}.$$

Throughout the paper we assume that the following assumptions hold

(i) The function $f : [0,T] \times L_2(\Omega) \to L_2(\Omega)$ is mean square continuous.

(ii) There exists an integrable deterministic function $k : [0,T] \to R^+$ with

$$\sup_{t\in[0,T]} \int_0^t k(s)ds \le m$$

where m is a positive real number such that the function f satisfies the mean square Lipschitz condition

$$\| f(t, X_1(t)) - f(t, X_2(t)) \|_2 \le k(t) \| X_1(t) - X_2(t) \|_2 .$$

(iii) There exists a positive real number m_1 such that

$$\sup_{t\in[0,T]} \| f(t, 0) \|_2 \le m_1$$

where m_1 is a positive real number.

Now we have the following lemma.

Lemma 1 *The solution of the problem (1)-(2) can be expressed by the stochastic Riemann integral equation*

$$\begin{aligned} X(t) &= a\left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} W(s)ds\right) \\ &+ \int_0^t f(s, X(s))ds + \int_0^t W(s)ds \end{aligned} \tag{4}$$

where $a = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1}$.

Proof. Integrating equation (1), we obtain

$$X(t) = X(0) + \int_0^t f(s, X(s))ds + \int_0^t W(s)ds$$

and

$$X(\tau_k) = X(0) + \int_0^{\tau_k} f(s, X(s))ds + \int_0^{\tau_k} W(s)ds,$$

then

$$\sum_{k=1}^{n} a_k X(\tau_k) = \sum_{k=1}^{n} a_k X(0) + \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds + \sum_{k=1}^{n} a_k \int_0^{\tau_k} W(s)ds$$

$$X_0 - X(0) = \sum_{k=1}^{n} a_k X(0) + \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds + \sum_{k=1}^{n} a_k \int_0^{\tau_k} W(s)ds$$

and

$$\left(1 + \sum_{k=1}^{n} a_k\right) X(0) = X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} W(s)ds,$$

then

$$X(0) = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1} \left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} W(s)ds\right).$$

Hence

$$X(t) = a\left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} W(s)ds\right) + \int_0^t f(s, X(s))ds + \int_0^t W(s)ds,$$

where $a = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1}$. ■

Now define the mapping

$$FX(t) = a\left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} W(s)ds\right) + \int_0^t f(s, X(s))ds + \int_0^t W(s)ds$$

then we can prove the following lemma.

Lemma 2 $F : C \to C$.

Proof. Let $X \in C$, $t_1,\ t_2 \in [0, T]$ such that $\mid t_2 - t_1 \mid < \delta$, then

$$\| FX(t_2) - FX(t_1) \|_2 \leq \int_{t_1}^{t_2} ||f(s, X(s))||_2 \, ds + \int_{t_1}^{t_2} ||W(s)||_2 \, ds.$$

From assumption (ii) we have

$$\| f(t, X(t)) \|_2 - \| f(t, 0) \|_2 \leq \| f(t, X(t)) - f(t, 0) \|_2 \leq k(t) \| X(t) \|_2,$$

then we have

$$\begin{aligned} \| f(t, X(t)) \|_2 &\leq k(t) \| X(t) \|_2 + m_1 \\ &\leq k(t) \| X \|_C + m_1. \end{aligned}$$

So,

$$\| FX(t_2) - FX(t_1) \|_2 \leq \| X \|_C \int_{t_1}^{t_2} k(s)ds + m_1(t_2 - t_1) + \int_{t_1}^{t_2} \sqrt{s}ds,$$

which proves that $F : C \to C$. ■

3 Existence and uniqueness

For the existence of a unique solution $X \in C$ of the problem (1)-(2), we have the following theorem.

Theorem 1 *Let the assumptions (i)-(iii) be satisfied. If $2m < 1$, then the problem (1)-(2) has a unique solution $X \in C$.*

Proof. Let X and $X^* \in C$, then

$$\begin{aligned} \| FX(t) - FX^*(t) \|_2 &\leq \int_0^t ||f(s, X(s)) - f(s, X^*(s))||_2 \, ds \\ &+ a \sum_{k=1}^n a_k \int_0^{\tau_k} ||f(s, X(s)) - f(s, X^*(s))||_2 \, ds \\ &\leq \| X - X^* \|_C \int_0^t k(s)ds \\ &+ a \sum_{k=1}^n a_k \| X - X^* \|_C \int_0^{\tau_k} k(s)ds. \end{aligned}$$

Then

$$\begin{aligned}\| FX - FX^* \|_C &\le m\left(1 + a\sum_{k=1}^{n} a_k\right) \| X - X^* \|_C \, .\\ &\le 2m \| X - X^* \|_C \, .\end{aligned}$$

Now if $2m < 1$, then the operator $F : C \to C$ is contraction.
Then by Banach's fixed point theorem [2], there exists a unique solution $X \in C$ of the problem (1)-(2) can be expressed by (4). ■

4 Continuous dependence

Consider the random differential equation (1) with the nonlocal condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = \tilde{X}_0 \qquad , \tau_k \in (0, T) \tag{5}$$

Definition 1 *The solution $X \in C$ of the nonlocal problem (1)-(2) is continuously dependent (on the random initial value X_0) if $\forall \epsilon > 0,\ \exists \delta > 0$ such that $\| X_0 - \tilde{X}_0 \|_2 \le \delta$ implies that $\| X - \tilde{X} \|_C \le \epsilon$.*

Consider now the two problems (1)-(2) and (1),(5).
Here, we study the continuous dependence (on the random initial value X_0) of the solution of the random differential equation (1) and (2).

Theorem 2 *Let the assumptions (i)-(iii) be satisfied. Then the solution of the nonlocal problem (1)-(2) is continuously dependent on the random initial value X_0.*

Proof. Let $X(t)$ as defined in equation (4) be the solution of the nonlocal problem (1)-(2) and

$$\begin{aligned}\tilde{X}(t) &= a\left(\tilde{X}_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, \tilde{X}(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} W(s)ds\right)\\ &+ \int_0^t f(s, \tilde{X}(s))ds + \int_0^t W(s)ds\end{aligned}$$

be the solution of the nonlocal problem (1) and (5). Then

$$X(t) - \tilde{X}(t) = a[X_0 - \tilde{X}_0] - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} [f(s, X(s)) - f(s, \tilde{X}(s))]ds + \int_0^t [f(s, X(s)) - f(s, \tilde{X}(s))]ds.$$

Using our assumptions, we get

$$\begin{aligned} \| X(t) - \tilde{X}(t) \|_2 &\leq a \| X_0 - \tilde{X}_0 \|_2 \\ &+ a\sum_{k=1}^{n} a_k \int_0^{\tau_k} \| f(s, X(s)) - f(s, \tilde{X}(s)) \|_2 \, ds \\ &+ \int_0^t \| f(s, X(s)) - f(s, \tilde{X}(s)) \|_2 \, ds \\ &\leq a\delta + 2m \| X - \tilde{X} \|_C, \end{aligned}$$

then

$$\| X - \tilde{X} \|_C \leq \frac{a\delta}{1 - 2m} = \epsilon.$$

This completes the proof. ■

Now consider the random differential equation (1) with the nonlocal condition

$$X(0) + \sum_{k=1}^{n} \tilde{a}_k X(\tau_k) = X_0 \qquad , \tau_k \in (0, T) \tag{6}$$

Definition 2 *The solution $X \in C$ of the nonlocal problem (1)-(2) is continuously dependent (on the coefficients a_k of the nonlocal condition) if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $| a_k - \tilde{a}_k | \leq \delta$ implies that $\| X - \tilde{X} \|_C \leq \epsilon$.*

Consider now the two problems (1)-(2) and (1),(6).

Here, we study the continuous dependence (on the coefficients a_k of the nonlocal condition) of the solution of the random differential equation (1) and (2).

Theorem 3 *Let the assumptions (i)-(iii) be satisfied. Then the solution of the nonlocal problem (1)-(2) is continuously dependent on the coefficients a_k of the nonlocal condition.*

Proof.Let $X(t)$ as defined in equation (4) be the solution of the nonlocal problem (1)-(2) and

$$\begin{aligned}\tilde{X}(t) &= \tilde{a}\left(X_0 - \sum_{k=1}^{n}\tilde{a}_k\int\limits_0^{\tau_k} f(s,\tilde{X}(s))ds - \sum_{k=1}^{n}\tilde{a}_k\int\limits_0^{\tau_k} W(s)ds\right)\\ &+ \int\limits_0^t f(s,\tilde{X}(s))ds + \int\limits_0^t W(s)ds\end{aligned}$$

be the solution of the nonlocal problem (1) and (6). Then

$$\begin{aligned}X(t) - \tilde{X}(t) &= [a-\tilde{a}]X_0 + \int\limits_0^t [f(s,X(s)) - f(s,\tilde{X}(s))]ds\\ &- a\sum_{k=1}^{n}a_k\int\limits_0^{\tau_k} f(s,X(s))ds + \tilde{a}\sum_{k=1}^{n}\tilde{a}_k\int\limits_0^{\tau_k} f(s,\tilde{X}(s))ds.\\ &- \left[a\sum_{k=1}^{n}a_k - \tilde{a}\sum_{k=1}^{n}\tilde{a}_k\right]\int\limits_0^{\tau_k} W(s)ds.\end{aligned}$$

Now

$$\begin{aligned}\mid a-\tilde{a}\mid &= \left|\frac{1}{1+\sum\limits_{k=1}^{n}a_k} - \frac{1}{1+\sum\limits_{k=1}^{n}\tilde{a}_k}\right|\\ &= \left|\frac{\sum\limits_{k=1}^{n}(\tilde{a}_k - a_k)}{\left(1+\sum\limits_{k=1}^{n}a_k\right)\left(1+\sum\limits_{k=1}^{n}\tilde{a}_k\right)}\right|\\ &\leq \left|\sum_{k=1}^{n}(\tilde{a}_k - a_k)\right| \leq n\delta,\end{aligned}$$

and

$$\begin{aligned}\left|a\sum_{k=1}^{n}a_k-\tilde{a}\sum_{k=1}^{n}\tilde{a}_k\right| &= \left|\frac{\sum\limits_{k=1}^{n}a_k}{1+\sum\limits_{k=1}^{n}a_k}-\frac{\sum\limits_{k=1}^{n}\tilde{a}_k}{1+\sum\limits_{k=1}^{n}\tilde{a}_k}\right| \\ &= \left|\frac{\sum\limits_{k=1}^{n}(a_k-\tilde{a}_k)}{\left(1+\sum\limits_{k=1}^{n}a_k\right)\left(1+\sum\limits_{k=1}^{n}\tilde{a}_k\right)}\right| \\ &\leq \left|\sum_{k=1}^{n}(a_k-\tilde{a}_k)\right|\leq n\delta,\end{aligned}$$

also

$$\begin{aligned}& \tilde{a}\sum_{k=1}^{n}\tilde{a}_k\int\limits_0^{\tau_k}f(s,\tilde{X}s))ds-a\sum_{k=1}^{n}a_k\int\limits_0^{\tau_k}f(s,X(s))ds \\ = & \ \tilde{a}\left(1+\sum_{k=1}^{n}\tilde{a}_k\right)\int\limits_0^{\tau_k}f(s,\tilde{X}(s))ds-a\left(1+\sum_{k=1}^{n}a_k\right)\int\limits_0^{\tau_k}f(s,X(s))ds \\ - & \ \tilde{a}\int\limits_0^{\tau_k}f(s,\tilde{X}(s))ds+a\int\limits_0^{\tau_k}f(s,X(s))ds \\ = & \ \tilde{a}(\tilde{a}^{-1})\int\limits_0^{\tau_k}f(s,\tilde{X}(s))ds-a(a^{-1})\int\limits_0^{\tau_k}f(s,X(s))ds \\ - & \ \tilde{a}\int\limits_0^{\tau_k}f(s,\tilde{X}(s))ds+a\int\limits_0^{\tau_k}f(s,X(s))ds \\ = & \ -\int\limits_0^{\tau_k}[f(s,X(s))-f(s,\tilde{X}(s))]ds+a\int\limits_0^{\tau_k}f(s,X(s))ds \\ - & \ \tilde{a}\int\limits_0^{\tau_k}f(s,\tilde{X}(s))ds-\tilde{a}\int\limits_0^{\tau_k}f(s,X(s))ds+\tilde{a}\int\limits_0^{\tau_k}f(s,X(s))ds\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^{\tau_k}[f(s,X(s))-f(s,\tilde{X}(s))]ds+[a-\tilde{a}]\int_0^{\tau_k}f(s,X(s))ds \\
&+ \tilde{a}\int_0^{\tau_k}[f(s,X(s))-f(s,\tilde{X}(s))]ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
\| X(t)-\tilde{X}(t)\|_2 &\leq n\delta \| X_0 \|_2 + \int_{\tau_k}^{t} \| f(s,X(s))-f(s,\tilde{X}(s)) \|_2 \, ds \\
&+ n\delta[m \| X \|_C + m_1 T] n\delta \int_0^{\tau_k} \| W(s) \|_2 \, ds \\
&+ \tilde{a}\int_0^{\tau_k} \| f(s,X(s))-f(s,\tilde{X}(s)) \|_2 \, ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\| X-\tilde{X} \|_C &\leq n\delta \| X_0 \|_2 + m \| X-\tilde{X} \|_C + n\delta[m \| X \|_C + m_1 T] \\
&+ \tilde{a}m \| X-\tilde{X} \|_C + n\delta \int_0^{\tau_k} \sqrt{s}ds,
\end{aligned}
$$

then

$$
\begin{aligned}
\| X-\widetilde{X} \|_C &\leq n\delta\left[\| X_0 \|_2 + m \| X \|_C + m_1 T + \frac{2}{3}\tau_k^{\frac{3}{2}}\right] \\
&+ (1+\tilde{a})m \| X - X^* \|_C \\
&\leq n\delta\left[\| X_0 \|_2 + m \| X \|_C + m_1 T + \frac{2}{3}T^{\frac{3}{2}}\right] \\
&+ 2m \| X - X^* \|_C .
\end{aligned}
$$

Hence

$$
\| X-\widetilde{X} \|_C \leq \frac{n\delta\left(\| X_0 \|_2 + m \| X \|_C + m_1 T + \frac{2}{3}T^{\frac{3}{2}}\right)}{1-2m} = \epsilon.
$$

This completes the proof. ■

5 Nonlocal integral condition

Let $v : [0,T] \to [0,T]$ be nondecreasing function such that

$$a_k = v(t_k) - v(t_{k-1}),\ \tau_k \in (t_{k-1}, t_k)$$

where $(0 < \ t_1 \ < \ t_2 \ < \ t_3 \ < .... < T)$.

Then, the nonlocal condition (2) will be in the form

$$X(0) + \sum_{k=1}^{n} X(\tau_k)(v(t_k) - v(t_{k-1})) = X_0.$$

From the mean square continuity of the solution of the nonlocal problem (1)-(2), we obtain from [15]

$$\underset{n\to\infty}{l.i.m} \sum_{k=1}^{n} X(\tau_k)(v(t_k) - v(t_{k-1})) = \int_0^T X(s)dv(s).$$

That is, the nonlocal conditions (2) is transformed to the mean square Riemann-Stieltjes integral condition

$$X(0) + \int_0^T X(s)dg(s) = X_0,$$

Now, we have the following corollary.

Corollary 1 *Let the assumptions (i)-(iii) be satisfied, then the random differential equation (1) with the nonlocal integral condition (3) has a unique solution given by the solution of*

$$\begin{aligned} X(t) &= a^\star \left(X_0 - \int_0^T \int_0^s f(\theta, X(\theta))d\theta dv(s) - \int_0^T \int_0^s W(\theta)d\theta dv(s) \right) \\ &+ \int_0^t f(\theta, X(\theta))d\theta + \int_0^t W(\theta)d\theta, \end{aligned}$$

where $a^\star = (1 + v(T) - v(0))^{-1}$.

Proof. Taking the limit of equation (4) we get the proof. ∎

References

[1] P. Balasubramaniam, J.Y. Parkand and A.V.A. Kumar, Existence of solutions for semilinear neutral stochastic functional differential equations with nonlocal conditions, *Nonlinear Anal.*, 71(2009), 1049-1058.

[2] A. T. Bharucha-Reid, Fixed point theorems in probabilistic analysis, *Bulletin of the American Mathematical Society* , 82, 5(1976).

[3] A. Boucherif and Radu Precup, On the nonlocal initial value problem for first order differential equations, *Fixed Point Theory*, 4, 2(2003), 205-212.

[4] L.Byszewski and V.Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Applicable analysis*, 40(1991), 11-19.

[5] P. Chen and Y. Li, Existence and uniqueness of strong solutions for nonlocal evolution equations, *Electronic Journal of Differential Equations*, 2014(2014), 1-9.

[6] A.M. A. El-Sayed, R. O. Abd El-Rahman and M. El-Gendy, Uniformly stable solution of a nonlocal problem of coupled system of differential equations, *Differ. Equ. Appl.*, 5, 3(2013), 355-365.

[7] A.M. A. El-Sayed, R. O. Abd El-Rahman and M. El-Gendy, Existence of solution of a coupled system of differential equation with nonlocal conditions, *Malaya Journal Of Matematik*, 2, 4(2014), 345-351.

[8] A. M. A. EL-Sayed and E. O. Bin-Tahir, An arbitraty fractional order differential equation with internal nonlocal and integral conditions, *advances in Pure Mathematics*, 1, 3(2011), 59-62.

[9] D. Gordeziani and G. Avalishvili, Investigation of the nonlocal initial boundary value problems for some hyperbolic equations, *Hiroshima Math. J.*, 31(2001), 345-366.

[10] D. Isaacson, Stochastic integrals and derivatives, *The Annals of Mathematical Statistics*, 40, 5(1969), 1610-1616.

[11] S. Itoh, Random fixed point theorems with an application to random differential equations in Banach spaces, *Journal Of Mathematical Analysis And Applications*, 67(1979), 261-273.

[12] B. Oksendal, Stochastic differential equations (An introduction with applications), *Springer-Verlag Berlin Heidelberg New York*, (2013).

[13] M. Rockner, R. Zhu and X. Zhu, Existence and uniqueness of solutions to stochastic functional differential equations in infinite dimensions, *Nonlinear Analysis: Theory, Methods and Applications*, 125(2015), 358-397.

[14] R. Sakthivel, P. Revathi and Y. Ren, Existence of solutions for nonlinear fractional stochastic differential equations, *Nonlilear Anal.*, 81(2013), 70-86.

[15] T. T. Soong, Random differential equations in science and engineering, *Mathematics in Science and Engineering*, 103, (1973).

[16] D. W. Stroock, Topics in stochastic differential equations, *Tata Institute of Fundamental Research Bombay*, (1982).

[17] S. Watanabe and T. Yamada, On uniqueness of solutions of stochastic differential equations, *J.Math. Kyoto Univ.*, (1971), 155-167 and 553-563.