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Unsteady Boundary Layer Flow of a Radiating Fluid under the Influence of an Applied Magnetic Field

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Abstract

We consider the unsteady free convection flow of an electrically conducting fluid near a vertical plate undergoing impulsive motion. The fluid is assumed to be radiating, and subjected to an externally applied magnetic field. Using the well-known Rosseland approximation for the radiative heat transfer, the boundary layer equations governing the flow have been solved analytically. The solutions have been obtained corresponding to constant temperature as well as constant heat flux conditions at the plate. There arise four non-dimensional parameters, whose influence on the velocity profiles have been discussed.

1 Introduction

Hydromagnetic boundary layer flows are known to have applications in various industrial and technological fields. Accordingly, a number of analytical studies has been carried out in the past to understand the dynamics of electrically conducting viscous fluids. One of the key aspects of such flows stems from the likely response of the boundary layer to externally applied forces due to gravity and magnetic fields. Such forces are known to play dominant roles in the flow control and design of equipments. In the case of unsteady free convection flows of electrically conducting fluids near an infinite vertical plate, several analytical investigations have been reported in the literature [1–5] to account for the influence of viscous, bouyancy and magnetic forces, subject to a variety of boundary conditions for velocity and surface temperature.

It is known that the inclusion of radiation effects, due to specific fluid properties, is not only essential but also quite important in applications such as high temperature processing and space technology. In this regard, some studies have been carried out in literature to take account of radiative heat transfer in the fluid dynamical equations connecting velocity and temperature of a viscous fluid [6–9]. These studies do not take into account the interaction between applied magnetic field and radiation on the boundary layer flow of conducting viscous fluids. In the present work, we have thus discussed developing free convection flow of an electrically conducting and radiating fluid past an infinite vertical rigid plate due to impulsive motion of the boundary. To facilitate the analytical treatment of the governing equations, we have resorted to the usual Boussinesq approximation for density variation, and Rosseland approximation [10, 11] for the radiative heat transfer. In this paper, the solutions corresponding to two cases of thermal conditions at the vertical plate have been obtained. The conditions considered are: (i) uniform plate temperature, and (ii) constant heat flux at the plate. The effects of radiation as well as other processes parametrised by the Prandtl, Grashoff and Hartmann numbers, have been brought out in our study.

2 Governing equations

The flow situation corresponds to unsteady two-dimensional flow of an infinite extent of incompressible, electrically conducting and radiating fluid past an infinite vertical rigid plate which is assumed to be non-conducting. With respect to an arbitrary origin O, the x'-axis is taken along the wall in the upward direction, and the y'-axis is taken normal to it into the fluid. The flow takes place due to impulsive motion of the plate in its own plane, and it is assumed that an external magnetic field of constant strength, B_y, is applied in the y'-

direction. Our analysis in this work pertains to the case of magnetic field being fixed relative to the fluid, rather than the plate. Under the assumptions of Boussinesq and Rosseland approximations, the unsteady flow of a radiative fluid will be governed by the equations

$$\frac{\partial u'}{\partial t'} = \nu \frac{\partial^2 u'}{\partial y'^2} + g\beta(T' - T'_\infty) - \frac{\sigma B_y^2}{\rho} u' \tag{1}$$

$$\frac{\partial T'}{\partial t'} = \frac{k}{\rho c_p} \frac{\partial^2 T'}{\partial y'^2} + \frac{16 a_R}{3\alpha_R \rho c_p} \frac{\partial}{\partial y'} \left(T'^3 \frac{\partial T'}{\partial y'} \right) \tag{2}$$

where u' is the only non-zero velocity component in the x' direction, T' is the fluid temperature, T'_∞ the free-stream temperature, g the acceleration due to gravity, β the volumetric coefficient, ν the kinematic viscosity, ρ the density, k the thermal conductivity, c_p the specific heat of the fluid at constant pressure, σ the electrical conductivity, a_R the Stefan-Boltzmann constant, and α_R is the mean absorption coefficient. In the absence of radiation term, *i.e.*, when the last term on the right side of equation (2) is absent, the solutions of equations (1) and (2) have been discussed in literature for different boundary conditions [1, 3–5]. On the other hand, in the absence of applied magnetic field, the effects of radiation, under the present physical setting, have been discussed by Ganesan *et al.* [9], for isothermal boundary conditions. Extending the above works, and based on the analytical solutions of equations (1) and (2), we have studied here the combined effects of thermal radiation, external magnetic field and the fluid properties on the flow near the vertical plate.

We shall obtain analytical solutions of the free convection problem corresponding to the following two cases:

Case (i) Bounding wall subject to impulsive motion and maintained at constant temperature:

In this case, the initial and boundary conditions are

$$\begin{aligned} t' \le 0: &\quad u' = 0,\ T' = T'_\infty \ \text{ for all } \ y' \ge 0 \\ t' > 0: &\quad u' = U,\ T' = T'_0 \ \text{ at } \ y' = 0 \\ &\quad u' \to 0,\ T' \to T'_\infty \ \text{ as } \ y' \to \infty \end{aligned} \tag{3}$$

where U is the constant impulsive velocity and T'_0 is the constant temperature of the plate.

Case (ii) Bounding wall subject to impulsive motion and uniform heat flux:

In this case, the initial and boundary conditions will be

$$\begin{aligned} t' \le 0 : \quad & u' = 0, \;\; T' = T'_\infty \;\text{ for all }\; y' \ge 0 \\ t' > 0 : \quad & u' = U, \;\; \frac{\partial T'}{\partial y'} = -\frac{q}{k} \;\text{ at }\; y' = 0 \\ & u' \to 0, \;\; T' \to T'_\infty \;\text{ as }\; y' \to \infty \end{aligned} \tag{4}$$

where q is the uniform heat flux per unit area at the plate.

3 Solutions

We shall obtain analytical solutions of equations (1) and (2) corresponding to the physical situations as given in cases (i) and (ii) above, after transforming the equations to non-dimensional forms. This would, in turn, facilitate the introduction of certain well-known parameters relating the dynamical and thermodynamical features of the fluid flow.

3.1 Case (i)

We first introduce the non-dimensional quantities

$$\begin{aligned} y &= \frac{Uy'}{\nu}, \quad u = \frac{u'}{U}, \quad t = \frac{U^2 t'}{\nu}, \quad T = \frac{T' - T'_\infty}{T'_0 - T'_\infty} \\ \mathrm{Pr} &= \frac{\rho \nu c_p}{k}, \quad G_1 = \frac{\nu g \beta (T'_0 - T'_\infty)}{U^3} \\ R &= \frac{k \alpha_R}{4 a_R T'^3_\infty}, \quad m = \frac{\nu \sigma B_y^2}{\rho U^2} \end{aligned} \tag{5}$$

In the above, G_1 and R are, respectively, the buoyancy and radiation parameters, $\sqrt{m}$ is the Hartmann number and Pr is the Prandtl number.

Using the new variables given in equation (5), the governing partial differential equations (1) and (2) can be easily shown to transform, respectively, to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + G_1 T - mu \tag{6}$$

$$\frac{\partial T}{\partial t} = \frac{3R+4}{3R\,\mathrm{Pr}} \frac{\partial^2 T}{\partial y^2} \tag{7}$$

while the initial and boundary conditions (3) will become

$$
\begin{aligned}
t \le 0 : \quad & u = 0,\ \ T = 0 \ \text{ for } \ y \ge 0 \\
t > 0 : \quad & u = 1,\ \ T = 1 \ \text{ at } \ y = 0 \\
& u \to 0,\ \ T \to 0 \ \text{ as } \ y \to \infty
\end{aligned}
\tag{8}
$$

Using Laplace transforms and writing

$$
\bar{T}(y,s) = \int_0^\infty T(y,t)e^{-st}\,\mathrm{d}t, \quad \bar{u}(y,s) = \int_0^\infty u(y,t)e^{-st}\,\mathrm{d}t \tag{9}
$$

we can first solve equation (7) with the corresponding initial and boundary conditions, whereupon $u(y,t)$ can be obtained from equations (6) and (8). The detailed expressions for $\bar{T}(y,s)$, $\bar{u}(y,s)$ and the algebraic manipulations to get their inverses are not included here, for brevity. However, one can finally express the solutions for temperature and velocity as

$$
T(y,t) = \operatorname{erfc}\left(\frac{\sqrt{\lambda}\,y}{2\sqrt{t}}\right) \tag{10}
$$

$$
u(y,t) = \sum_{i=1}^{5} u_i(y,t) \tag{11}
$$

where

$$
\begin{aligned}
u_1(y,t) &= 0.5\left[\exp(-y\sqrt{m})\operatorname{erfc}(\phi_1) + \exp(y\sqrt{m})\operatorname{erfc}(\phi_2)\right] \\
u_2(y,t) &= (G_1/m)\operatorname{erfc}\left(0.5y\sqrt{\lambda}/\sqrt{t}\right), \quad u_3(y,t) = -(G_1/m)u_1(y,t) \\
u_4(y,t) &= (0.5G_1/m)\exp(-at)\left[\exp(-iby)\operatorname{erfc}(\phi_3) + \exp(iby)\operatorname{erfc}(\phi_4)\right] \\
u_5(y,t) &= -(0.5G_1/m)\exp(-at)\left[\exp(-iby)\operatorname{erfc}(\phi_5) + \exp(iby)\operatorname{erfc}(\phi_6)\right] \\
\phi_1 &= \frac{y}{2\sqrt{t}} - \sqrt{mt}, \quad \phi_2 = \frac{y}{2\sqrt{t}} + \sqrt{mt} \\
\phi_3 &= \frac{y}{2\sqrt{t}} - i\sqrt{a\lambda t}, \quad \phi_4 = \frac{y}{2\sqrt{t}} + i\sqrt{a\lambda t} \\
\phi_5 &= \frac{y\sqrt{\lambda}}{2\sqrt{t}} - i\sqrt{at}, \quad \phi_6 = \frac{y\sqrt{\lambda}}{2\sqrt{t}} + i\sqrt{at} \\
a &= \frac{m}{1-\lambda}, \quad b = i\sqrt{a\lambda}, \quad \lambda = \frac{3R\,\mathrm{Pr}}{3R+4}, \quad (\lambda \ne 1)
\end{aligned}
$$

and $\operatorname{erfc}(x)$ is the complementary error function defined by

$$
\operatorname{erfc}(x) = 1 - \operatorname{erf}(x), \quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-\eta^2\right)\,\mathrm{d}\eta
$$

In the absence of magnetic field ($m = 0$), it can be shown that the velocity variable in a radiating medium near an isothermal plate can be expressed as

$$\begin{aligned} u(y,t) &= \operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right) + \frac{G_1}{1-\lambda}\left[(0.5\lambda y^2 + t)\operatorname{erfc}\left(\frac{y\sqrt{\lambda}}{2\sqrt{t}}\right)\right. \\ &\quad - (0.5y^2+t)\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right) - y\sqrt{\lambda t/\pi}\,\exp\left(-\frac{\lambda y^2}{4t}\right) \\ &\quad \left. + y\sqrt{t/\pi}\,\exp\left(-\frac{y^2}{4t}\right)\right] \end{aligned} \tag{12}$$

Comparing equation (12) with equation (11), we observe that the presence of magnetic field in the same flow field couples the solution considerably. It may, however, be noted that equation (12) for $m = 0$ does not follow as a special case of equation (11), and is therefore to be derived afresh from the governing equations. Another special case for which equation (11) ceases to be valid is when $\lambda = 1$. The case $\lambda = 1$ corresponds to $R = (4/3)(\mathrm{Pr}-1)^{-1}$, $(\mathrm{Pr} > 1)$, and in this case the expression of u is

$$\begin{aligned} u(y,t) &= (G_1/m)\operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right) \\ &\quad + 0.5(1-G_1/m)\,\left[\exp(-y\sqrt{m})\operatorname{erfc}(\phi_1)\right. \\ &\qquad \left. + \exp(y\sqrt{m})\operatorname{erfc}(\phi_2)\right] \end{aligned} \tag{13}$$

3.2 Case (ii)

As in the case (i) above, we shall solve equations (1) and (2) by first non-dimensionalising them, together with the set of conditions (4). Here, we define the quantities

$$\begin{aligned} y &= \frac{Uy'}{\nu}, \quad u = \frac{u'}{U}, \quad t = \frac{U^2t'}{\nu}, \quad \theta = \frac{kU(T'-T'_\infty)}{\nu q} \\ \mathrm{Pr} &= \frac{\rho\nu c_p}{k}, \quad G_2 = \frac{qg\beta\nu^2}{kU^4} \\ R &= \frac{k\alpha_R}{4a_R T'^3_\infty}, \quad m = \frac{\nu\sigma B_y^2}{\rho U^2} \end{aligned} \tag{14}$$

It may be noted that in the above non-dimensionalisation, the Grashoff number G_2 indicates a different physical mechanism as compared to the Grashoff number G_1 in Case (i). Using equation (14), the governing equations and the initial and boundary conditions become

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + G_2\theta - mu \tag{15}$$

$$\frac{\partial \theta}{\partial t} = \frac{3R+4}{3R\,\mathrm{Pr}}\frac{\partial^2 \theta}{\partial y^2} \tag{16}$$

$$\begin{aligned} t \le 0 : &\quad u = 0,\ \ \theta = 0 \ \text{ for } \ y \ge 0 \\ t > 0 : &\quad u = 1,\ \ \frac{\partial \theta}{\partial y} = -1 \ \text{ at } \ y = 0 \\ &\quad u \to 0,\ \ \theta \to 0 \ \text{ as } \ y \to \infty \end{aligned} \tag{17}$$

Following the same solution method as before, we can obtain solutions similar to the corresponding non-radiating free convection flow [12]. These are given by

$$\theta(y,t) = 2\sqrt{\frac{t}{\pi\lambda}}\,\exp\left(-\frac{\lambda y^2}{4t}\right) - y\,\mathrm{erfc}\left(\frac{\sqrt{\lambda}\,y}{2\sqrt{t}}\right) \tag{18}$$

$$u(y,t) = u_1(y,t) + u_6(y,t) + u_7(y,t) \tag{19}$$

where

$$u_6(y,t) = -\frac{G_2}{(1-\lambda)\sqrt{\pi\lambda}}\int_0^t \sqrt{t-\eta}\,F(y,\eta)\,\mathrm{d}\eta$$

$$u_7(y,t) = \frac{G_2}{(1-\lambda)\sqrt{\pi\lambda}}\int_0^t \sqrt{t-\eta}\,G(y,\eta)\,\mathrm{d}\eta$$

$$F(y,t) = \mathrm{Re}\left[\exp(by - at)\,\mathrm{erfc}(\phi_4)\right]$$

$$G(y,t) = \mathrm{Re}\left[\exp(by - at)\,\mathrm{erfc}(\phi_6)\right]$$

and “Re” denotes the real part.

The special cases $m = 0$ and $\lambda = 1$ for the heat flux boundary condition can also be dealt with as in Case (i), and are not presented here.

4 Skin friction

The expressions for the shear stress at the boundary, in both cases, can be obtained using $\tau = -\partial u/\partial y$ evaluated at the boundary. Denoting them by τ_1

and τ_2 for the cases (i) and (ii), respectively, we obtain

$$\begin{aligned}\tau_1 \;=\;& \frac{G_1}{m}\sqrt{\frac{\lambda}{\pi t}} + \left(1 - \frac{G_1}{m}\right)\left[\frac{\exp(-mt)}{\sqrt{\pi t}} + \sqrt{m}\,\mathrm{erf}\sqrt{mt}\right] \\ & - \frac{G_1\ \exp(-at)}{m}\mathrm{Re}\left[i\sqrt{a\lambda}\,\mathrm{erfc}(i\sqrt{a\lambda t}) + i\sqrt{a\lambda}\,\mathrm{erfc}(-i\sqrt{at})\right] \\ & - \frac{G_1\ \exp(-at)}{m\sqrt{\pi t}}\left[\sqrt{\lambda}\ \exp(at) - \exp(a\lambda t)\right]\end{aligned} \tag{20}$$

$$\begin{aligned}\tau_2 \;=\;& \frac{\exp(-mt)}{\sqrt{\pi t}} + \sqrt{m}\left[1 - \mathrm{erfc}\sqrt{mt}\right] \\ & + \frac{2G_2}{(1-\lambda)\sqrt{\pi\lambda}}\int_0^t \sqrt{t-\eta}\ f(\eta)\,\mathrm{d}\eta\end{aligned} \tag{21}$$

where

$$\begin{aligned}f(t) \;=\;& \frac{1}{\sqrt{\pi t}}\left[\sqrt{\lambda} - \exp(-mt)\right] \\ & + i\sqrt{a\lambda}\ \exp(-at)\left[\mathrm{erfc}(i\sqrt{a\lambda t}) - \mathrm{erfc}(i\sqrt{at})\right]\end{aligned}$$

In the non-magnetic case, τ_1 and τ_2 are given by

$$\tau_1 = \frac{1}{\sqrt{\pi t}} - \frac{2G_1}{1+\lambda}\sqrt{\frac{t}{\pi}} \tag{22}$$

$$\tau_2 = \frac{1}{\sqrt{\pi t}} - \frac{G_2 t}{\sqrt{\lambda}(1+\sqrt{\lambda})} \tag{23}$$

5 Results

In the unsteady free convection near a vertical plate considered here, it is obvious that the velocity variations in the boundary layer are coupled to the temperature variations. In this section, we have presented graphically the variations of velocity in the boundary layer, with special emphasis on the effects of the governing non-dimensional parameters. The plots correspond to the variations of the non-dimensional velocity u against the non-dimensional space variable y. In the Figures 1–5, we have shown the effects of parameters – namely, R, t, m, Pr and G_1 – on the velocity distribution when the bounding vertical plate has been kept at a uniform temperature [Case (i)]. Figure 1 shows the variation of velocity profiles with the radiation parameter R, for fixed values of other parameters. It is apparent that the profiles are qualitatively similar, and are less sensitive to higher values of R (> 5). Also, as R increases, *i.e.*, as the radiation effects decrease, the fluid velocity decreases in the boundary layer. This must be expected, as radiation effects become significant for processes taking place at high temperatures, and this in turn induces larger convection near the boundary. In Figure 2, we have included the temporal profiles of the velocity. It can be noted that for small values of t ($\lesssim 0.1$), the velocity decreases steadily to its free-stream value. However, as t increases, the velocity attains a maximum value before dropping monotonically to its free-stream value. In the next figure (Fig. 3), the effect of the magnetic parameter m on the velocity profiles has been shown. As has been generally observed in literature in the case of non-radiating hydromagnetic free convection flows for the geometrical configuration considered here, the applied magnetic field in our study has also shown a stabilizing effect on the boundary layer flow. In order to gauge the effect of fluid properties on the flow pattern, we have shown, in Figures 4 and 5, respectively, the influence of Prandtl and Grashoff numbers on the radiating flow. These two parameters have opposite effects on the velocity variations. It has been seen that the velocity decreases with increasing Prandtl number (Fig. 4), while it increases with Grashoff number (Fig. 5).

The fluid flow features corresponding to the uniform heat flux condition at the bounding vertical plate [Case (ii)] are seen to be qualitatively similar to the Case (i) profiles. Accordingly, we have given only two plots showing, respectively, in Figures 6 and 7, the effects of R and t. The curves in these figures are self-explanatory.

Before concluding, we find it instructive to present an example illustrating the comparative effects of the two types of thermal conditions at the plate, in a special flow situation for which the Grashoff numbers are equal. This is shown in our last figure (Fig. 8). It is seen that the fluid velocity corresponding to the isothermal boundary condition case is significantly larger than that for the unform heat flux case.

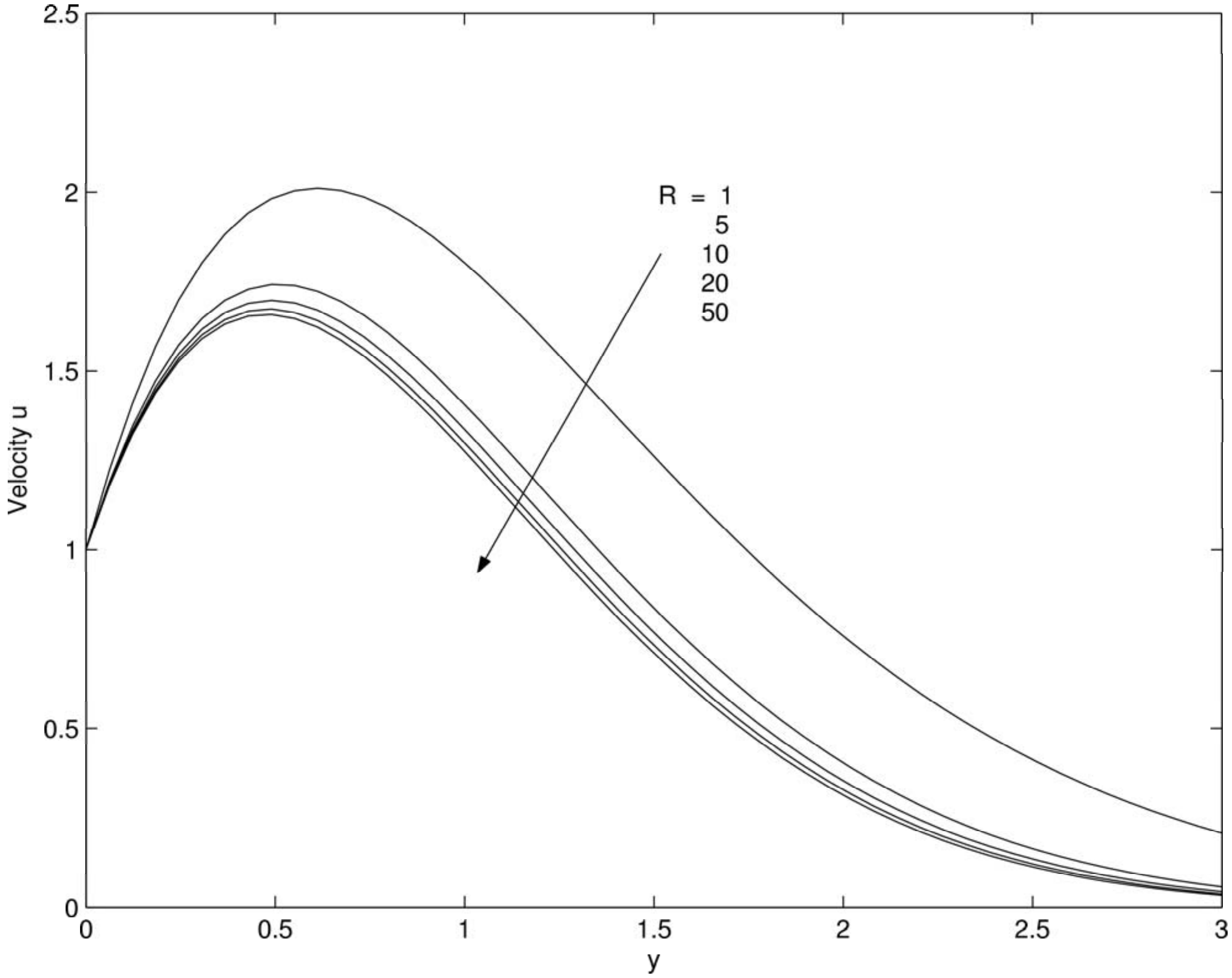

Figure 1: Variation of velocity u [Case (i)]. Effect of R.
$(G_1 = 10,\ m = 0.5,\ \Pr = 0.7,\ t = 0.5)$

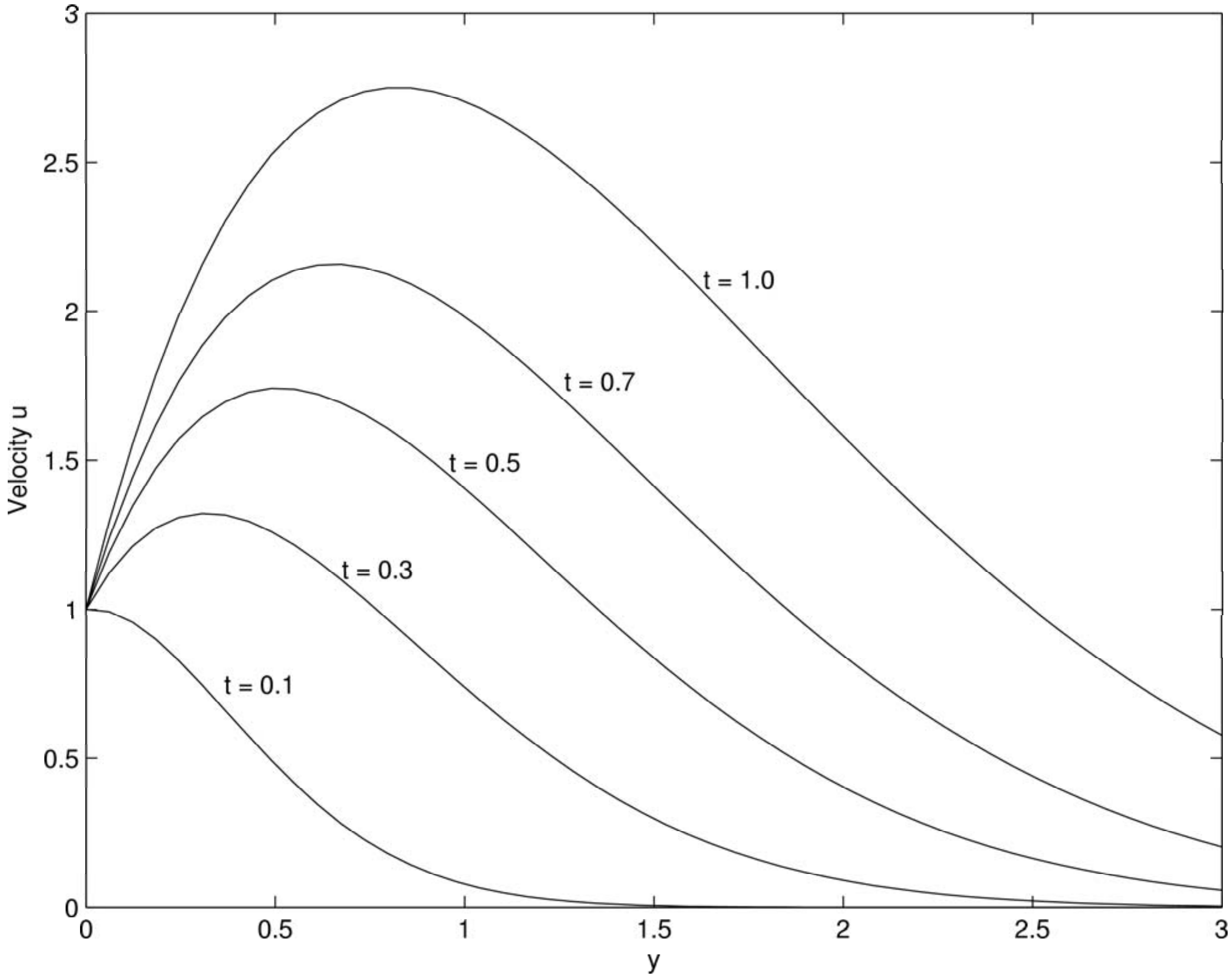

Figure 2: Variation of velocity u [Case (i)]. Effect of t.
($G_1 = 10,\ m = 0.5,\ \mathrm{Pr} = 0.7,\ R = 5$)

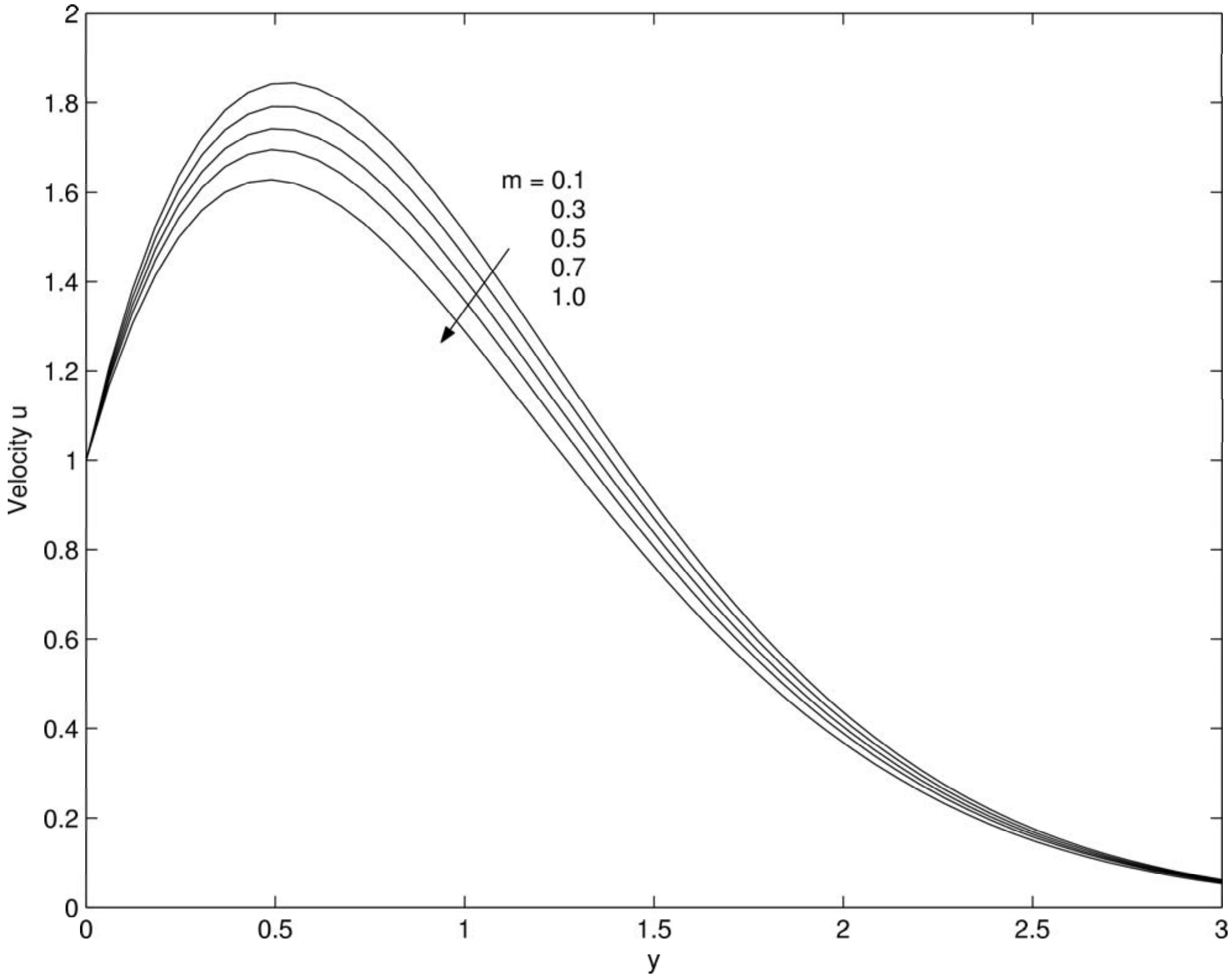

Figure 3: Variation of velocity u [Case (i)]. Effect of m.
$(G_1 = 10,\ \mathrm{Pr} = 0.7,\ R = 5,\ t = 0.5)$

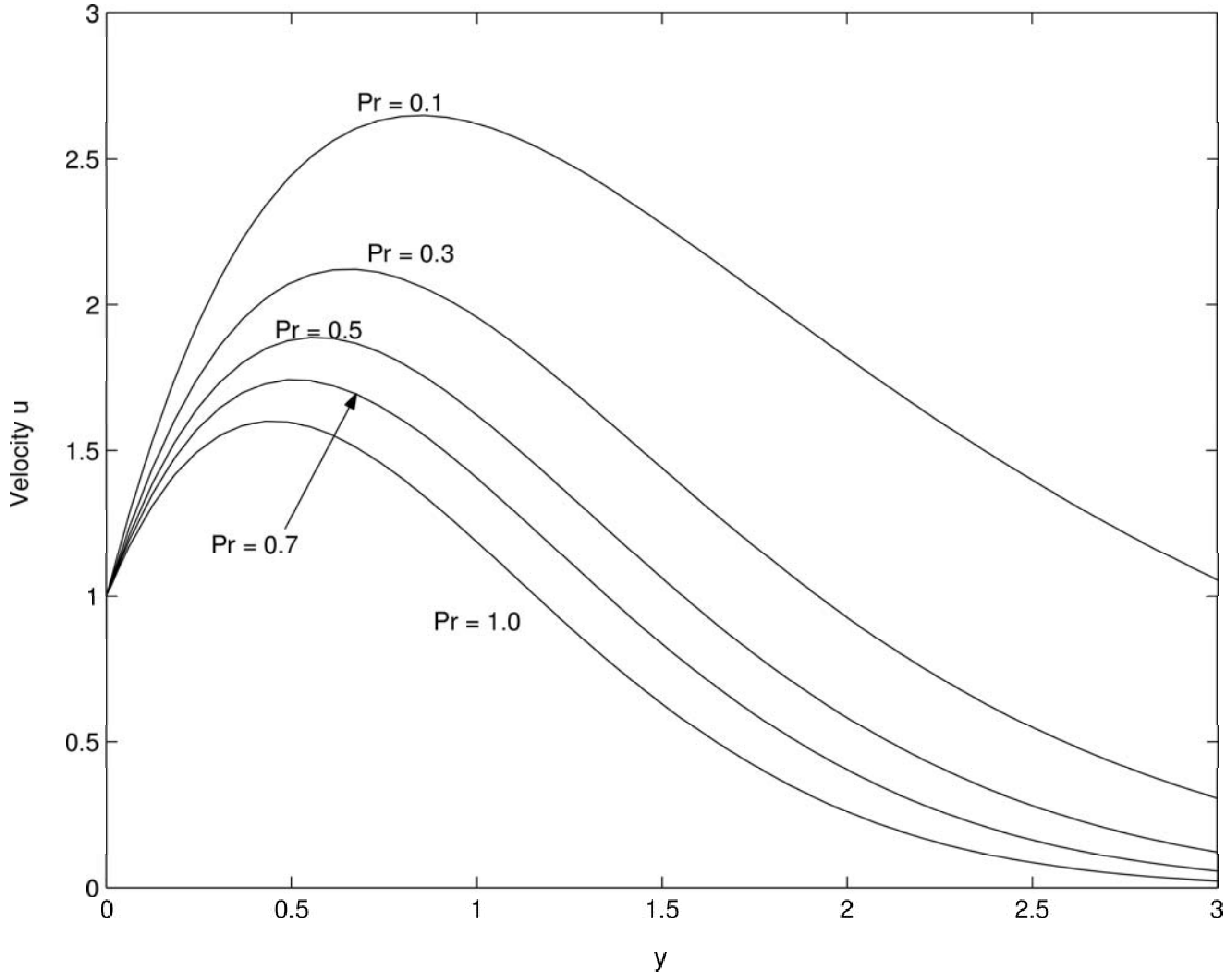

Figure 4: Variation of velocity u [Case (i)]. Effect of Pr.
$(G_1 = 10,\ m = 0.5,\ R = 5,\ t = 0.5)$

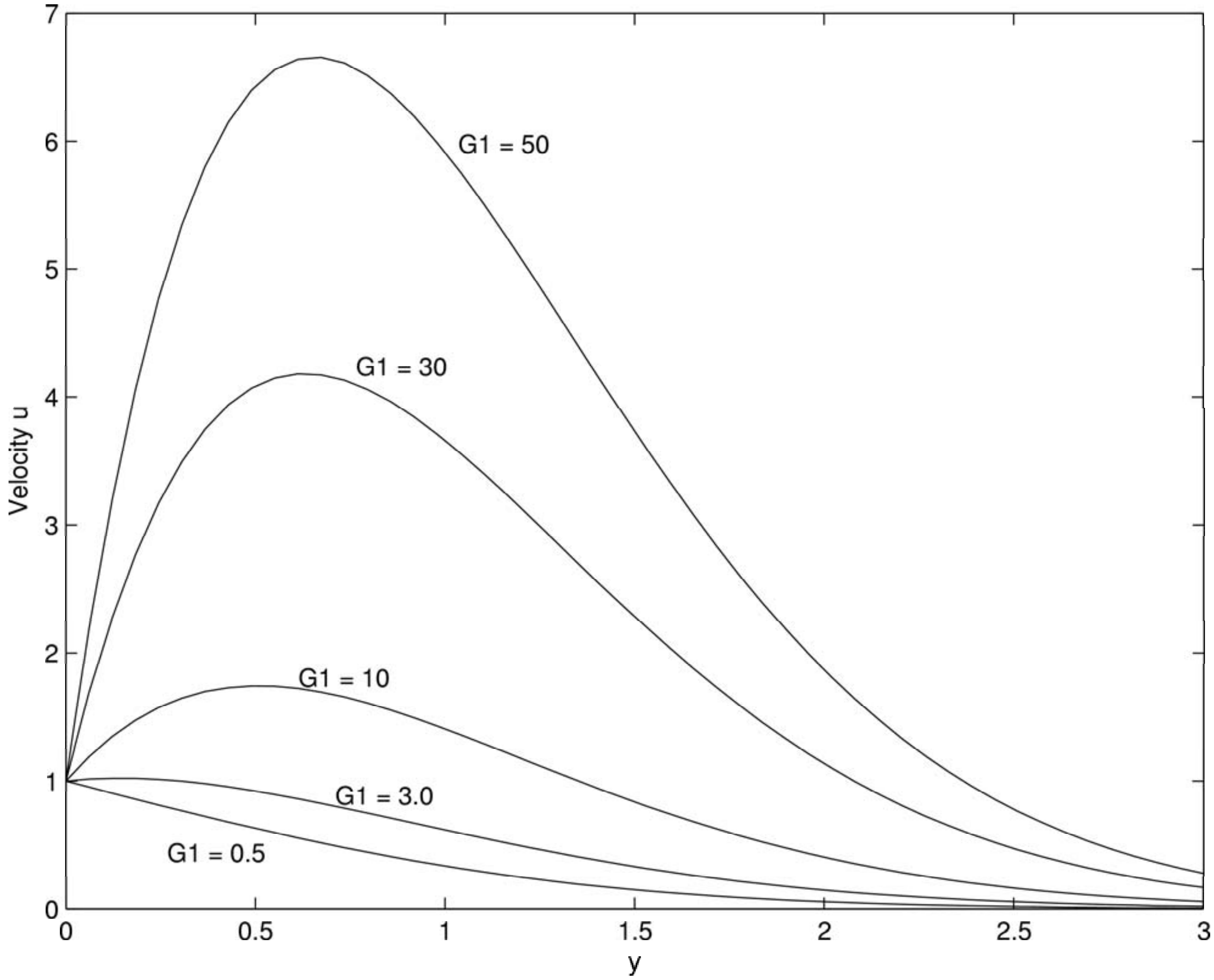

Figure 5: Variation of velocity u [Case (i)]. Effect of G_1.
$(m = 0.5, \mathrm{Pr} = 0.7,\ R = 5,\ t = 0.5)$

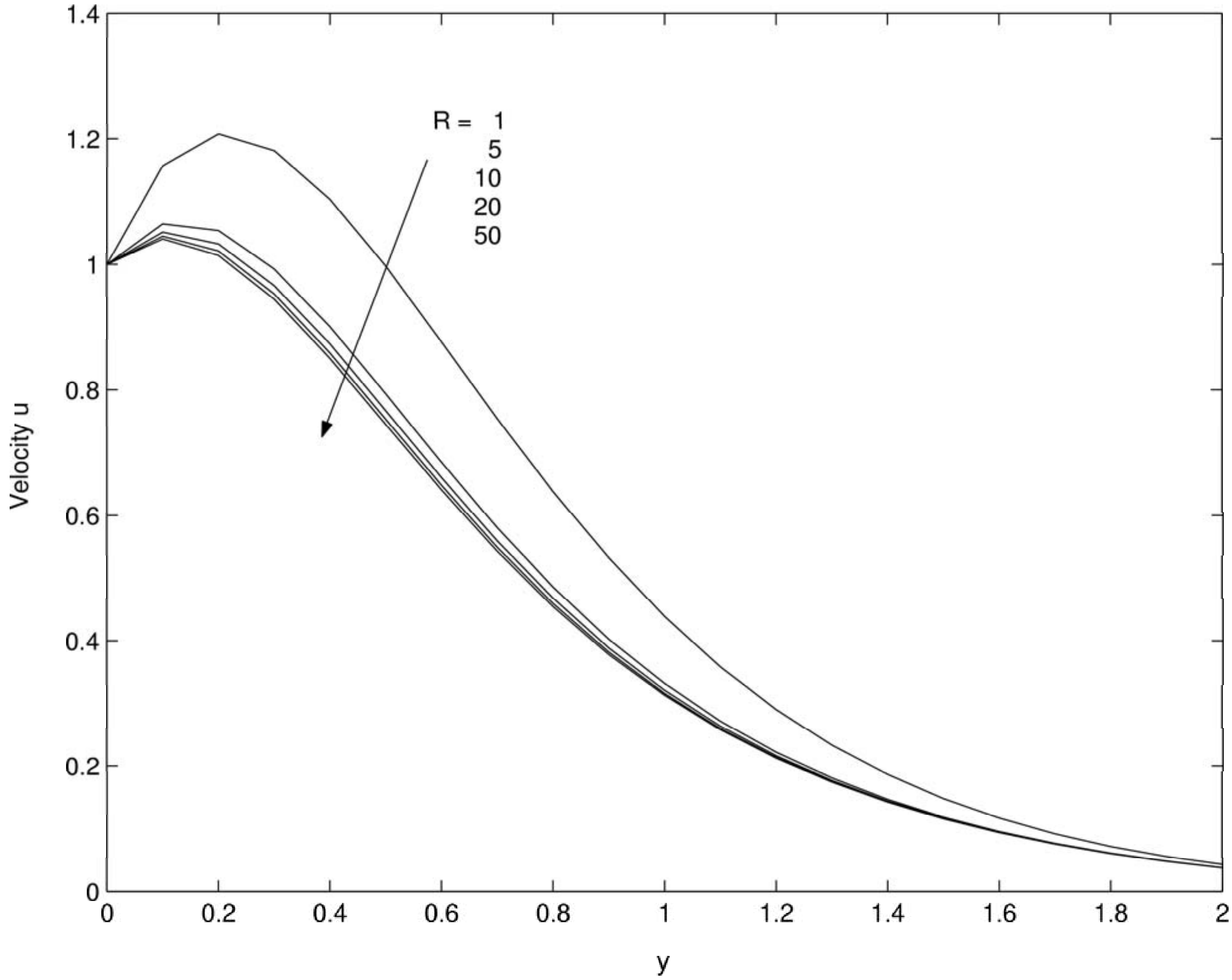

Figure 6: Variation of velocity u [Case (ii)]. Effect of R.
$(G_2 = 10,\ m = 0.5,\ \mathrm{Pr} = 0.7,\ t = 0.5)$

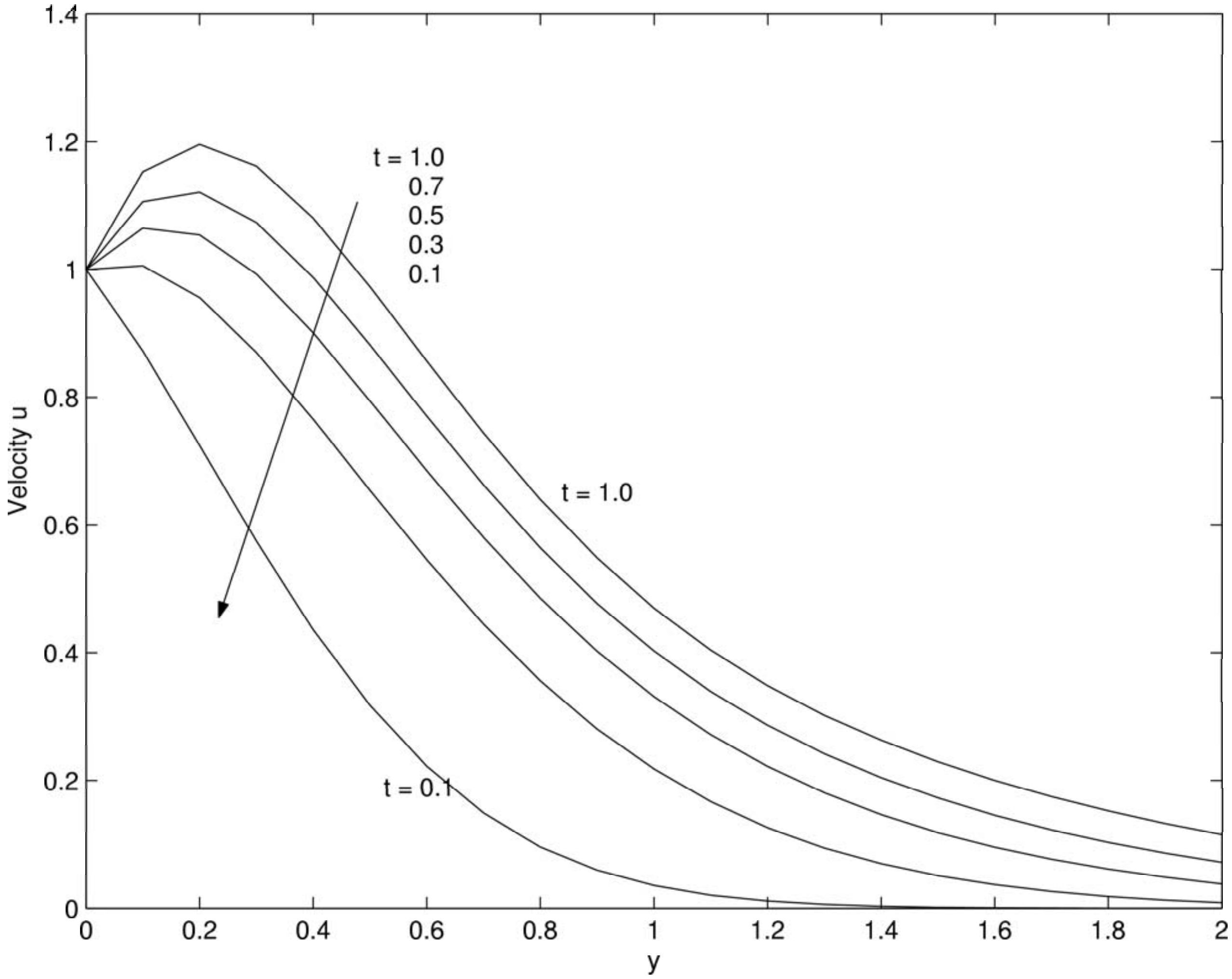

Figure 7: Variation of velocity u [Case (ii)]. Effect of t.
$(G_2 = 10,\ m = 0.5,\ \Pr = 0.7,\ R = 5)$

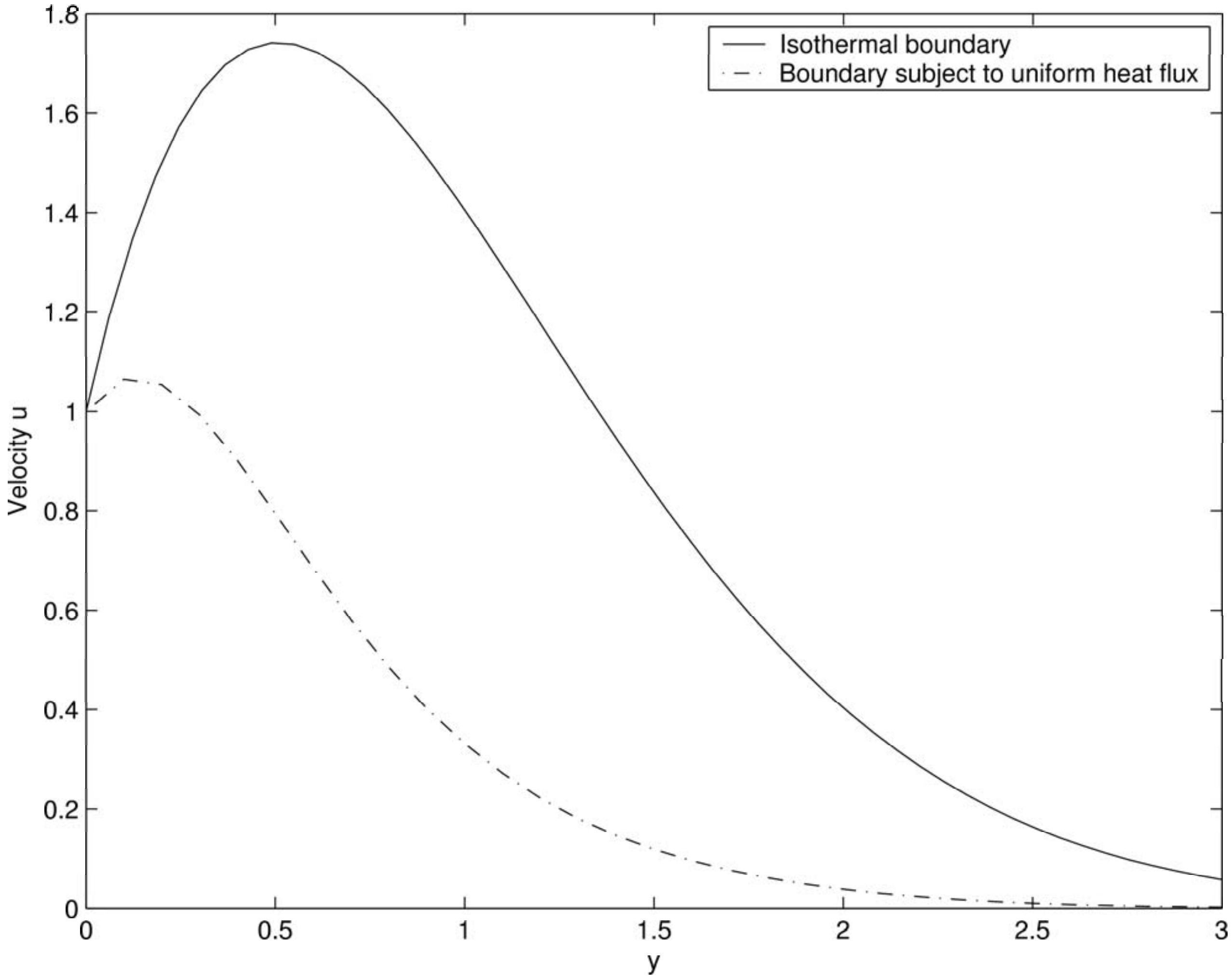

Figure 8: Velocity profiles with different thermal conditions at the boundary.
$(G_1 = G_2 = 10,\ \mathrm{Pr} = 0.7,\ t = 0.5,\ m = 0.5,\ R = 5)$

References

[1] V. M. Soundalgekar: Free convection effects on Stokes problem for a vertical plate, *ASME J. Heat Transf.*, **99**, 499–501, 1977.

[2] G. A. Georgantopoulos: Effects of free convection on the hydromagnetic accelerated flow past a vertical porous limiting surface, *Astrophys. Space Sci.*, **65**, 433–442, 1979.

[3] V. M. Soundalgekar and M. R. Patil: Stokes problem for infinite vertical plate with constant heat flux, *Astrophys. Space Sci.*, **70**, 179–182, 1980.

[4] A. Raptis and A. K. Singh: MHD free convection flow past an accelerated vertical plate, *Int. Comm. Heat Mass Transf.*, **10**, 313–321, 1983.

[5] P. Chandran, N. C. Sacheti and A. K. Singh: On laminar boundary layer flow of electrically conducting liquids near an accelerated vertical plate, *Phys. Chem. Liq.*, **40**, 241–254, 2002.

[6] M. M. Ali, T. S. Chen and B. F. Armaly: Natural convection-radiation interaction in boundary layer flow over horizontal surfaces, *AIAA J.*, **22**, 1797–1803, 1984.

[7] M. A. Hossain, M. A. Alim and D. A. S. Rees: The effect of radiation on free convection from a porous vertical plate, *Int. J. Heat Mass Transf.*, **42**, 181–191, 1999.

[8] A. Raptis and C. Perdikis: Radiation and free convection flow past a moving plate, *Appl. Mech. Engng.*, **4**, 817–821, 1999.

[9] P. Ganesan, P. Loganathan and V. M. Soundalgekar: Radiation effects on flow past an impulsively started infinite vertical plate, *Int. J. Appl. Mech. Engng.*, **6**, 719–730, 2001.

[10] R. Siegel and J. R. Howell: *Thermal Radiation Heat Transfer*. McGraw-Hill, New York, 1972.

[11] M. Q. Brewster: *Thermal Radiative Transfer and Properties*. Wiley, New York, 1992.

[12] P. Chandran and N. C. Sacheti: Transient motion of electrically conducting fluids of different Prandtl numbers: Flow near a vertical boundary, *Diff. Eqns. Control Process.*, **7**, 77–87, 2003.

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