DIFFERENTIAL EQUATIONS
AND
CONTROL PROCESSES
N 3, 2006
Electronic Journal,
reg. N P23275 at 07.03.97

http://www.neva.ru/journal
e-mail: diff@osipenko.stu.neva.ru

Control problems in nonlinear systems

Local parametric identifiability of parabolic equations by various discretizations [1]

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Abstract

The problem of local parametric identifiability for a semilinear parabolic equation with a scalar parameter is considered. Sufficient conditions of local parametric identifiability are given for the following two approaches: (i) we observe a discretization of an exact solution, and (ii) we observe an approximate solution generated by a discretization of the exact equation.

The discretization of exact solution is observed at growing time moments with increasing accuracy in the phase space. It is shown that given sufficient conditions of local parametric identifiability can be checked for the Chaffee – Infante problem. In the case of a discretization of the exact equation we do not have to refine the observations as time grows.

For both cases it is shown that local parametric identifiability holds for a solution with initial values from an open and dense subset of the phase space.

[1] *Mathematics Subject Classifications:* 35B30, 35B40, 65N22.
Key words: local parametric identifiability, parabolic equations, discretizations, observations.

1 Introduction

The general identifiability problem for dynamical systems is formulated as follows: given a dynamical system $S(\lambda, t)$ depending on a parameter λ, is it possible to determine the value of λ from observation of a solution (or a function of a solution)?

This problem has attracted a lot of attention. Let us mention, for example, the work [8], in which general conditions of identifiability were obtained. In a recent work [16], the identifiability problem was considered for an analytic finite-dimensional system of differential equations; it was shown in [16] that if such a system depends on r parameters, then the parameters are identifiable by any randomly chosen set of $2r + 1$ experiments.

We are interested in this paper in the problem of local parametric identifiability near a given value λ_0 of the parameter. In our case local parametric identifiability at λ_0 means that the observed values for $\lambda = \lambda_0$ differ from those for λ with small positive values of $|\lambda - \lambda_0|$.

We study evolutionary systems generated by semilinear parabolic equations. From the practical point of view, it is impossible to observe exact solutions of such equations; it is only possible to observe their values at discrete points or to observe approximate solutions given, for example, by discretization schemes.

Our goal is to give sufficient conditions under which the problem is locally identifiable by observations of a solution with initial values from an open and dense subset of the phase space.

In Sec. 2, we observe discretizations of exact solutions at growing time moments Tn (where T is fixed and n grows) with increasing accuracy as $n \to \infty$. We give sufficient conditions of local parametric identifiability (Theorem 2.1) and show that these conditions can be checked for the Chaffee – Infante problem (Theorem 2.2).

In Sec. 3, we observe approximate solutions given by a semi-implicit discretization of a parabolic equation. Sufficient conditions of local parametric identifiability are given (Theorem 3.1); it is worth noting that in this case we do not have to increase the accuracy of observations as time grows.

In Sec. 4,we discuss some applications.

2 Discretization of a solution

Consider a semilinear parabolic equation

$$u_t = u_{xx} + f(\lambda, u), \tag{2.1}$$

where $x \in (0, \pi)$, $t > 0$, with Dirichlet boundary conditions

$$u(0, t) = u(\pi, t) = 0. \tag{2.2}$$

In equation (2.1), $\lambda \in R$ is a parameter. Let $u(\lambda, x, t, u_0)$ be a classical solution of equation (2.1), i. e., $u \in C^{2,1}_{x,t}$, with the initial value

$$u(\lambda, x, 0, u_0) = u_0(x). \tag{2.3}$$

We study the problem of local parametric identifiability for problem (2.1)–(2.3) in the following form.

Fix a number $T > 0$. For any natural number n, we fix a natural number $m(n)$ so that

$$m(n) \to \infty, \quad n \to \infty. \tag{2.4}$$

Consider the finite arrays

$$V(\lambda, n, u_0) = \{u(\lambda, kh_n, Tn, u_0): \quad 0 < k \le m(n) - 1\}$$

defined for $n > 0$, where

$$h_n = \frac{\pi}{m(n)}.$$

The array $V(\lambda, n, u_0)$ is the set of values of a solution $u(\lambda, x, t, u_0)$ on a finite subset

$$\{(kh_n, Tn): \quad 0 < k \le m(n) - 1\}$$

of the set $(0, \pi) \times \{Tn\}$.

Condition (2.4) means that we observe the solution at time moments Tn with increasing accuracy as $n \to \infty$.

First we fix the spaces we work with.

We consider the standard Sobolev space $H^1_{[0,\pi]}$ with the norm defined by

$$\|u\|^2 = \int_0^\pi |u|^2 dx + \int_0^\pi \left|\frac{\partial u}{\partial x}\right|^2 dx.$$

Let H^1_0 be the following subspace of $H^1_{[0,\pi]}$:

$$H^1_0 = \left\{u \in H^1_{[0,\pi]}: \ u(0) = u(\pi) = 0\right\}.$$

We use below the norm

$$\|u\| = \left(\int_0^\pi \left|\frac{\partial u}{\partial x}\right|^2 dx\right)^{1/2},$$

which is equivalent to the above-mentioned norm. We also consider the Sobolev space $H^2_0 \subset H^1_0$ with the norm

$$\|u\|_2 = \left(\int_0^\pi \left|\frac{\partial^2 u}{\partial x^2}\right|^2 dx\right)^{1/2}$$

We assume that $f(\lambda, \cdot) \in C^2(R)$.

Our basic structural assumption on the nonlinearity f is as follows: there exists a function $C(\lambda)$, $\lambda \in R$, such that

$$uf(\lambda, u) \le C(\lambda). \tag{2.5}$$

It is well known [9] that under condition (2.5) problem (2.1)–(2.3) generates an evolutionary system $S(\lambda, t)$, $t > 0$, in the space H^1_0, so that for any $u_0 \in H^1_0$, the solution

$$u(\lambda, x, t, u_0) = S(\lambda, t)u_0(x)$$

is defined for all $t > 0$. In addition, condition (2.5) implies that $S(\lambda, t)$ has a global attractor $A(\lambda)$ in H^1_0 [9, 15].

Let us give the main definition.

We say that problem (2.1)–(2.3) is locally identifiable at $\lambda = \lambda_0$ via refining observations of a solution $u(\lambda_0, x, t, u_0)$ if there exists a number $\varepsilon > 0$ such that for any λ, $0 < |\lambda - \lambda_0| < \varepsilon$, and for any $v_0 \in H^1_0$ there exists $n_0 > 0$ such that

$$V(\lambda_0, n, u_0) \ne V(\lambda, n, v_0)$$

for $n \geq n_0$.

This definition corresponds to the general definition of local identifiability of nonlinear systems of ordinary differential equations via observations of their solutions at discrete time moments [4].

Denote by $F(\lambda)$, $\lambda \in R$, the set of fixed points of the system $S(\lambda, t)$. Obviously, a function $u(x)$ is a fixed point of $S(\lambda, t)$ if and only if $u(x)$ is a solution of the following boundary-value problem:

$$\frac{d^2 u}{dx^2} + f(\lambda, u) = 0, \quad u(0) = u(\pi) = 0. \tag{2.6}$$

It is known that any system $S(\lambda, t)$ has a global Lyapunov function

$$V(u) = \int\limits_0^\pi \left(\frac{1}{2} |\nabla u|^2 - F(u) \right) dx,$$

where F is the antiderivative of f with respect to u; this function decreases along nonconstant solutions [9,15]. Thus, for any $u_0 \in H_0^1$, the solution $S(\lambda, t)u_0$ tends to the set $F(\lambda)$. If we denote by $\omega(\lambda, u_0)$ the ω-limit set of the solution $S(\lambda, t)u_0$ [15], then the following inclusions obviously hold:

$$\omega(\lambda, u_0) \subset F(\lambda) \subset A(\lambda), \quad u_0 \in H_0^1, \quad \lambda \in R. \tag{2.7}$$

Now let us impose the following condition on $f(\lambda, u)$ at λ_0.

Condition I at λ_0. For any $\delta > 0$, there exist numbers $v^+ \in (0, \delta)$, $v^- \in (-\delta, 0)$, and $\mu > 0$ such that

$$f(\lambda, v^+) \neq f(\lambda_0, v^+) \quad \text{and} \quad f(\lambda, v^-) \neq f(\lambda_0, v^-)$$

for $0 < |\lambda - \lambda_0| < \mu$.

Obviously, *Condition I at* λ_0 is a corollary of the following condition.

Condition II at λ_0. The function $f(\lambda, u)$ is continuously differentiable in λ on the set $R \times \{0\}$ and

$$\frac{\partial f}{\partial \lambda}(\lambda, 0)\bigg|_{\lambda=\lambda_0} \neq 0.$$

The main result of Sec. 2 is the following statement.

Theorem 2.1. *Assume that*

(a) *all fixed points of the system* $S(\lambda_0, t)$ *are hyperbolic;*

(b) *if* $f(\lambda_0, 0) = 0$, *then the fixed point* $u \equiv 0$ *of* $S(\lambda_0, t)$ *is unstable;*

(c) *Condition I at* λ_0 *is satisfied.*

Then there exists an open and dense subset $\mathcal{H}$ *of the space* H_0^1 *such that, for* $u_0 \in \mathcal{H}$, *problem (2.1)–(2.2) is locally identifiable at* λ_0 *via refining observations of the solution* $u(\lambda_0, x, t, u_0)$.

If $f(\lambda_0, 0) \neq 0$, *then one may take* $\mathcal{H} = H_0^1$.

We begin the proof of Theorem 2.1 with an auxiliary statement.

Lemma 2.1. *Let condition I at* λ_0 *be satisfied. If* $u(\lambda_0, x)$ *is a nonzero solution of the boundary-value problem* (2.6), *then there exists* $\Delta > 0$ *such that, for any solution* $u(\lambda, x)$ *of* (2.6) *with* $0 < |\lambda - \lambda_0| < \Delta$,

$$u(\lambda, x) \not\equiv u(\lambda_0, x). \tag{2.8}$$

Proof. Since $u(\lambda_0, x) \not\equiv 0$,

$$u^* = \max_{x \in [0,\pi]} |u(\lambda_0, x)| > 0.$$

Apply *Condition I at* λ_0 to find numbers $v^+ \in (0, u^*)$, $v^- \in (-u^*, 0)$, and the corresponding $\mu > 0$. We claim that we may take $\Delta = \mu$.

Fix $x_0 \in (0, \pi)$ such that

$$u^* = |u(\lambda_0, x_0)|.$$

Assume that $u(\lambda_0, x_0) > 0$ (the case $u(\lambda_0, x_0) < 0$ is considered similarly). Then there exists $x_1 \in (0, \pi)$ such that

$$u(\lambda_0, x_1) = v^+$$

(since $u(\lambda_0, 0) = 0$).

Take $\lambda \in R$ with $0 < |\lambda - \lambda_0| < \Delta$ and any solution $u(\lambda, x)$ of the corresponding problem (2.6). If $u(\lambda, x_1) \neq u(\lambda_0, x_1)$, then (2.8) holds. Otherwise, $u(\lambda, x_1) = u(\lambda_0, x_1) = v^+$, and it follows from *Condition I at* λ_0 that

$$\left.\frac{\partial^2 u(\lambda_0, x)}{\partial x^2}\right|_{x=x_1} = -f(\lambda_0, v^+) \neq -f(\lambda, v^+) = \left.\frac{\partial^2 u(\lambda, x)}{\partial x^2}\right|_{x=x_1}.$$

This proves (2.8).

Corollary. *The sets of nonzero fixed paints of the systems* $S(\lambda_0, t)$ *and* $S(\lambda, t)$ *are disjoint if* $0 < |\lambda - \lambda_0| < \Delta$.

Let us proceed with the proof of Theorem 2.1.

Take an array $V(\lambda, n, u_0)$ and construct a continuous piecewise-linear function $v^*(\lambda, x, n, u_0)$ on $[0, \pi]$ as follows:

$$v^*(\lambda, kh_n, n, u_0) = u(\lambda, kh_n, Tn, u_0), \quad k = 0, ..., m(n),$$

and $v^*(\lambda, \cdot, n, u_0)$ is linear on any segment

$$[kh_n, (k+1)h_n], \quad k = 0, ..., m(n) - 1.$$

It is known (see [1, Chapter 1]) that, for any $\lambda \in R$, the operator $S(\lambda, t)$ is a bounded operator from H_0^1 into H_0^2, i. e., for any $v_0 \in H_0^1$ there exists a constant $K = K(\lambda, v_0)$ such that

$$\|S(\lambda, t)v_0\|_2 \leq K, \quad t > 0.$$

Elementary estimates (see [2] for details) show that, for any $v_0 \in H_0^1$ and any $\lambda \in R$,

$$\|v^*(\lambda, x, n, v_0) - u(\lambda, x, Tn, v_0)\| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.9}$$

Assume that $f(\lambda_0, 0) = 0$ (the case $f(\lambda_0, 0) \neq 0$ is more simple since in this case $u \equiv 0$ is not a fixed point of the system $S(\lambda_0, t)$).

Assumptions (a) and (b) of our theorem imply that $u \equiv 0$ is a hyperbolic unstable fixed point of the system $S(\lambda_0, t)$, hence its stable manifold, $W^S(0)$, has positive codimension in the space H_0^1.

Obviously, in this case the set

$$\mathcal{H} = H_0^1 \setminus W^S(0) \tag{2.10}$$

is an open and dense subset of H_0^1.

Take $u_0 \in \mathcal{H}$ and the corresponding solution $u(\lambda_0, x, t, u_0)$. It was mentioned above that

$$u(\lambda_0, x, t, u_0) \to F(\lambda_0) \quad \text{as} \quad t \to \infty.$$

Since all fixed points of $S(\lambda_0, t)$ are hyperbolic, they are isolated. The global attractor $A(\lambda_0)$ is compact and contains all fixed points of $S(\lambda_0, t)$, hence the set $F(\lambda_0)$ is finite. Since the set $\omega(\lambda_0, u_0)$ is connected [15], it coincides with a single fixed point; denote this point by $w(x)$. It follows from (2.10) that $w(x) \not\equiv 0$.

Apply Lemma 2.1 to find, for the solution $w(x)$, a number $\varepsilon > 0$ such that if $0 < |\lambda - \lambda_0| < \varepsilon$, then any solution $u(\lambda, x)$ of (2.6) satisfies

$$u(\lambda, x) \not\equiv w(x). \tag{2.11}$$

We claim that this number $\varepsilon > 0$ has the property described in the definition of local identifiability via refining observations.

Indeed, take λ such that $0 < |\lambda - \lambda_0| < \varepsilon$ and an arbitrary initial function $v_0 \in H_0^1$.

The set $\omega(\lambda, v_0)$ is a closed subset of the compact set $A(\lambda)$ (see inclusion (2.7)). This set consists of solutions $u(\lambda, x)$ of the boundary-value problem (2.6). Hence, inequality (2.11) implies that there exists a positive number $a > 0$ such that

$$\operatorname{dist}(w(x), \omega(\lambda, v_0)) = a, \tag{2.12}$$

where dist is the distance generated by the norm of the space H_0^1.

To obtain a contradiction, assume that there exists a sequence n_m of natural numbers such that $n_m \to \infty$ as $m \to \infty$ and

$$V(\lambda_0, n_m, u_0) = V(\lambda, n_m, v_0) \tag{2.13}$$

for all m.

Construct the corresponding continuous piecewise-linear functions $v^*(\lambda_0, x, n_m, u_0)$ and $v^*(\lambda, x, n_m, v_0)$; for brevity, we denote these functions by $v^*_{0,m}(x)$ and $v^*_m(x)$, respectively. By (2.13),

$$v^*_{0,m}(x) \equiv v^*_m(x) \tag{2.14}$$

for all m.

Since

$$\|u(\lambda_0, x, n_m, u_0) - w(x)\| \to 0,$$

it follows from (2.9) that

$$\|v^*_{0,m}(x) - w(x)\| \to 0 \tag{2.15}$$

as $m \to \infty$.

Similarly, since

$$\operatorname{dist}\,(u(\lambda, x, n_m, v_0), \omega(\lambda, v_0)) \to 0,$$

it follows from (2.9) that

$$\operatorname{dist}\,(v^*_m(x), \omega(\lambda, v_0)) \to 0 \tag{2.16}$$

as $m \to \infty$.

Combining relations (2.12), (2.14), (2.15) and (2.16), we get the desired contradiction. Theorem 2.1 is proved.

Remark 2.1. I. Kukavica and J. C. Robinson in [12] studied the problem of distinguishability of global attractors for PDEs (in particular, for reaction-diffusion equations of the type of Eq. (2.1)). They show that it is possible to distinguish between different elements of the attractor by measurement of the solutions at almost every set of k points of the domain (where k is estimated by the dimension of the attractor).

Our approach is based on a quite different idea. Though we refer to the existence of the global attractor (and to some of its properties), we do not have to find its elements; we just start with an almost arbitrary initial point u_0 of the phase space and compare the solution $S(\lambda_0, t)u_0$ with **any** solution $S(\lambda, t)v$.

The proof of the main result in [12] in the nonanalytic case is based on a theorem by C. C.Poon [14] stating that differences of solutions on the global attractor have "finite order of vanishing". Since no such result is available for differences of arbitrary solutions $S(\lambda_0, t)u$ and $S(\lambda, t)v$ with $\lambda \neq \lambda_0$, we cannot replace our condition of refining the space resolution by measurements at a finite set of points whose cardinality does not depend on the unknown value of λ (but our results in Sec. 3 show that the latter is possible if we work with a discretization of Eq. (2.1)).

It follows from Lemma 2.1 that if *Condition I at* λ_0 is satisfied, then, for any λ with $0 < |\lambda - \lambda_0| < \Delta$, there exists $\delta(\lambda) > 0$ such that

$$\|u - v\| \geq \delta(\lambda)$$

for any nonzero fixed points u of $S(\lambda_0, t)$ and v of $S(\lambda, t)$, respectively. Let us note that $\delta(\lambda) \to 0$ as $\lambda \to \lambda_0$ in our case, since for hyperbolic fixed points u of $S(\lambda_0, t)$ there exist families of fixed points $v(\lambda)$ of $S(\lambda, t)$ such that $\|u - v(\lambda)\| \to 0$ as $\lambda \to \lambda_0$ (see [9]).

If we fix λ with $0 < |\lambda - \lambda_0| < \Delta$, $u_0 \notin W^S(0)$, and an arbitrary v_0, then there exist $w(x) \in F(\lambda_0)$, $w(x) \not\equiv 0$, $u(\lambda, x) \in F(\lambda)$, and $T(\lambda) > 0$ such that

$$\|u(\lambda_0, x, t, u_0) - w(x)\| < \delta(\lambda)/4$$

and

$$\|u(\lambda, x, t, v_0) - u(\lambda, x)\| < \delta(\lambda)/4$$

for $t \geq T(\lambda)$.

At the same time, the proof of estimate (2.9) in [2] shows that there exists $N_0 = N_0(\lambda)$ such that if the discretization space step is

$$h = \frac{\pi}{N_0},$$

then the corresponding piecewise-linear functions $v^*(\lambda_0, x, n, u_0)$ and $v^*(\lambda, x, n, v_0)$ (constructed by the mesh with fixed step h) satisfy the inequalities

$$\|v^*(\lambda_0, x, n, u_0) - u(\lambda_0, x, Tn, u_0)\| < \delta(\lambda)/4$$

and

$$\|v^*(\lambda, x, n, v_0) - u(\lambda, x, Tn, v_0)\| < \delta(\lambda)/4,$$

respectively, for $n \geq 1$.

It follows that the solutions $u(\lambda_0, x, t, u_0)$ and $u(\lambda, x, t, v_0)$ are distinguishable by observations at a fixed (not refined) mesh.

But our reasoning shows that the step h of this mesh will depend on λ, and we have no hope to get reasonable estimates of this dependence (since our *Condition I at* λ_0 does not exclude cases of very degenerate behavior of $f(\lambda, u)$ near λ_0).

For this reason, we prefer to formulate Theorem 2.1 for measurements with refining space resolution.

Remark 2.2. Let us discuss the assumptions of Theorem 2.1. While it is relatively easy to check conditions (b) and (c), condition (a) does not seem very natural.

Nevertheless, let us recall that, by the main result of [5], if we consider, instead of equation (2.1), an equation without a parameter,

$$u_t = u_{xx} + f(u),$$

then any fixed point of the corresponding evolutionary system is hyperbolic for nonlinearities $f(u)$ belonging to a residual subset of the function space $C^k(R, R)$, $k \geq 2$ (i. e., for a countable intersection of dense and open subsets of this function space).

Conditions of Theorem 2.1 are easily checked for a Chaffee−Infante problem [9, 6] with linear dependence on the parameter.

Assume that $f(\lambda, u) = \lambda g(u)$ in (2.1), where the function $g \in C^2(R, R)$ satisfies the following conditions:

$$\text{(CI1)} \quad g(0) = 0, \quad g'(0) = 1;$$

$$\text{(CI2)} \quad \limsup_{|u|\to\infty} \frac{g(u)}{u} \leq 0;$$

$$\text{(CI3)} \quad ug''(u) < 0 \text{ for } u \neq 0.$$

A typical example of a function $g(u)$ satisfying conditions (CI1)–(CI3) is $g(u) = u - u^3$.

It is known [9] that, under these conditions, the corresponding semigroup $S(\lambda, t)$ has a global attractor for any $\lambda > 0$, and its trajectories tend to the union of fixed points as $t \to \infty$.

In addition, it is known that if

$$\lambda > 1, \ \lambda \neq m^2, \ m \in Z, \tag{2.17}$$

then any fixed point of $S(\lambda, t)$ is hyperbolic, while the zero fixed point is unstable.

Finally, it is easy to see that if a function $g(u)$ satisfies condition (CI3), then it has not more than 2 nonzero roots. Hence, *Condition I* is obviously satisfied by $f(\lambda, u) = \lambda g(u)$ at any $\lambda_0 \neq 0$.

Thus, the following statement is a corollary of Theorem 2.1.

Theorem 2.2. *Assume that $f(\lambda, u) = \lambda g(u)$ in equation* (2.1)*, where the function $g(u)$ satisfies conditions* (CI1)–(CI3)*. Then, for any λ_0 satisfying inequalities* (2.17)*, there exists an open and dense subset $\mathcal{H}$ of the space H_0^1 such that, for $u_0 \in \mathcal{H}$, problem* (2.1) – (2.3) *is locally identifiable at λ_0 via refining observations of the solution $u(\lambda_0, x, t, u_0)$.*

This theorem generalizes the main result of [2].

3 Discretization of the equation

Let us consider the following usual semi-implicit discretization of equation (2.1). We fix a natural number N, set $d = \dfrac{\pi}{N+1}$, and let $h > 0$ to be the time step of the discretization. We approximate the values $u(\lambda, md, nh, u_0)$ by values v_m^n, $n \geq 0$, $m = 0, ..., N+1$, given by the following discretization scheme:

$$\Delta v^{n+1} = Av^{n+1} + \underline{f}(\lambda, v^n), \ n \geq 0, \tag{3.1}$$

where

$$v^n = (v_1^n, ..., v_N^n) \in R^N, \ \underline{f}(\lambda, v) = (f(\lambda, v_1), ..., f(\lambda, v_N)),$$

$$\Delta v^{n+1} = \frac{1}{h}\left(v^{n+1} - v^n\right), \ (Av)_m = \frac{1}{d^2}\left(v_{m+1} - 2v_m + v_{m-1}\right),$$

and $v_0 = v_{N+1} = 0$. Scheme (3.1) generates a mapping

$$\varphi(\lambda, \cdot): \ R^N \to R^N \tag{3.2}$$

such that $v^{n+1} = \varphi(\lambda, v^n)$.

We assume everywhere that

$$h\|A\| < 1, \tag{3.3}$$

where $\|A\|$ is the operator norm of the matrix A. Under condition (3.3), the mapping (3.2) is given by

$$\varphi(\lambda, v) = J^{-1}\left(v + h\underline{f}(\lambda, v)\right), \tag{3.4}$$

where $J = E_N - hA$.

It is shown in [13] that if, for a fixed λ,

$$f(\lambda, \cdot) \in C^1, \quad \left|\frac{\partial}{\partial u} f(\lambda, u)\right| \le M, \quad \text{and} \quad hM < 1, \tag{3.5}$$

then $\varphi(\lambda, v)$ is a diffeomorphism of the space R^N. We also assume that inequalities (2.5) hold for $\lambda \in R$.

In this section, we pay the main attention to the case $f(\lambda, u) = \lambda g(u)$, since in this case the sufficient conditions of local identifiability are quite simple (we discuss the general case later, in Remark 3.2).

Thus, let $f(\lambda, u) = \lambda g(u)$.

We fix $\lambda_0 \in R$ and assume that conditions (3.5) are satisfied for $(\lambda, u) \in \Lambda \times R$, where Λ is a neighborhood of λ_0.

Theorem 3.1. *Assume that $f(\lambda, u) = \lambda g(u)$ and*

(a) *all fixed points of the diffeomorphism $\varphi(\lambda_0, \cdot)$ are hyperbolic;*

(b) *if $g(0) = 0$, then the fixed point $u = 0$ of $\varphi(\lambda_0, \cdot)$ is unstable.*

Then there exists an open and dense subset $\mathcal{H} \subset R^N$ such that, for any $u_0 \in \mathcal{H}$, scheme (3.1) is locally identifiable at λ_0 via observations of the trajectory $\{\varphi^n(\lambda_0, u_0): \ n \ge 0\}$ in the following sense: there exists a number $\varepsilon > 0$ such that for any λ, $0 < |\lambda - \lambda_0| < \varepsilon$, and any $v_0 \in R^N$, there exists $n_0 > 0$ such that

$$\varphi^n(\lambda_0, u_0) \ne \varphi^n(\lambda, v_0)$$

for $n \ge n_0$.

If $g(0) \ne 0$ and $\lambda_0 \ne 0$, then one may take $\mathcal{H} = R^N$.

Proof. It is shown in [13] that, for any λ, the diffeomorphism $\varphi(\lambda, \cdot)$ has a global Lyapunov function. Hence, any its trajectory tends to the set $F(\lambda)$ of fixed points of $\varphi(\lambda, \cdot)$.

Since fixed points of $\varphi(\lambda_0, \cdot)$ are hyperbolic, each of them is isolated in $F(\lambda_0)$. It follows that, for any $u_0 \in R^N$, its trajectory $\varphi^n(\lambda_0, u_0)$ tends to a single fixed point.

Assume that $g(0) = 0$ (the case $g(0) \ne 0$ is more simple since in this case $u = 0$ is not a fixed point of $\varphi(\lambda_0, \cdot)$).

Similarly to Theorem 2.1 (formula (2.10)), we define $\mathcal{H}$ by the formula $\mathcal{H} = R^N \setminus W^S(0)$, where $W^S(0)$ is the stable manifold of the hyperbolic unstable fixed point $u = 0$ of $\varphi(\lambda_0, \cdot)$. The set $\mathcal{H}$ is an open and dense subset of R^N.

Fix $u_0 \in \mathcal{H}$, and let

$$\varphi^n(\lambda_0, u_0) \underset{n\to\infty}{\longrightarrow} w_0.$$

It follows from formula (3.4) that $w \in R^N$ is a fixed point of $\varphi(\lambda, \cdot)$ if and only if

$$Aw + \lambda \underline{g}(w) = 0. \tag{3.6}$$

Consider the function

$$\Phi(\lambda, w) = \varphi(\lambda, w) - w.$$

This function is continuously differentiable in λ and w. In addition,

$$\Phi(\lambda_0, w_0) = 0$$

since w_0 is a fixed point of $\varphi(\lambda_0, \cdot)$.

The point w_0 is a hyperbolic fixed point of $\varphi(\lambda_0, \cdot)$, hence the eigenvalues λ_j of the Jacobi matrix

$$\left.\frac{\partial \varphi(\lambda_0, w)}{\partial w}\right|_{w=w_0}$$

satisfy the inequalities $|\lambda_j| \neq 1$. It follows that $\lambda_j \neq 1$, and

$$\det\left(\left.\frac{\partial \Phi(\lambda_0, w)}{\partial w}\right|_{w=w_0}\right) = \det\left(\left.\frac{\partial \varphi(\lambda_0, w)}{\partial w}\right|_{w=w_0} - E_N\right) \neq 0.$$

By the implicit function theorem, the equation

$$\Phi(\lambda, w) = 0$$

has a unique (and continuously differentiable) solution $w(\lambda)$ in a neighborhood of λ_0, and $w(\lambda_0) = w_0$.

Thus, we may differentiate equation (3.6) at λ_0; we obtain the following equality:

$$A\left.\frac{\partial w}{\partial \lambda}\right|_{\lambda=\lambda_0} + \underline{g}\left(w(\lambda_0)\right) + \lambda_0 D \left.\frac{\partial w}{\partial \lambda}\right|_{\lambda=\lambda_0} = 0, \tag{3.7}$$

where

$$D = \operatorname{diag}\left(g'\left(w_{0,1}(\lambda_0)\right), ..., g'\left(w_{0,N}(\lambda_0)\right)\right)$$

and

$$w_0 = \left(w_{0,1}, ..., w_{0,N}\right).$$

Since $w_0 \neq 0$ and the matrix A is nondegenerate, $Aw_0 \neq 0$. It follows from equality (3.6) (with $\lambda = \lambda_0$) that $\underline{g}(w_0) \neq 0$.

Now equality (3.7) implies that

$$\left.\frac{\partial w}{\partial \lambda}\right|_{\lambda=\lambda_0} \neq 0. \tag{3.8}$$

It follows that there exists $\varepsilon > 0$ such that, for $0 < |\lambda - \lambda_0| < \varepsilon$, the diffeomorphism $\varphi(\lambda, \cdot)$ does not have fixed points coinciding with w_0.

Fix λ such that $0 < |\lambda - \lambda_0| < \varepsilon$ and let $A(\lambda)$ be the global attractor of $\varphi(\lambda, \cdot)$ (such an attractor exists under condition (2.5), see [7]). Take arbitrary $v_0 \in R^N$. For the ω-limit set $\omega(\lambda, v_0)$ of the trajectory $\varphi^n(\lambda, v_0)$, the following inclusions hold:

$$\omega(\lambda, v_0) \subset F(\lambda) \subset A(\lambda).$$

$F(\lambda)$ is a closed subset of the compact set $A(\lambda)$, hence $F(\lambda)$ is compact. It was shown above that

$$w_0 \notin F(\lambda),$$

hence

$$\text{dist}\,(w_0, \omega(\lambda, v_0)) > 0. \tag{3.9}$$

Relation (3.9) and the relations

$$\text{dist}\,(\varphi^n(\lambda, v_0), \omega(\lambda, v_0)) \to 0, \quad n \to \infty,$$

and

$$\varphi^n(\lambda_0, u_0) \to w_0, \quad n \to \infty,$$

imply the statement of Theorem 3.1.

Remark 3.1. Note that, in contrast to Sec. 2, in the case of scheme (3.1) we do not have to refine the observations of trajectories $\varphi^n(\lambda_0, u_0)$ and $\varphi^n(\lambda, v_0)$.

Remark 3.2. It is possible to apply a similar reasoning in the general case where the nonlinearity in (2.1) has the form $f(\lambda, u)$. Obviously, in this case the condition

$$\frac{\partial f}{\partial \lambda} \in C, \quad \frac{\partial}{\partial \lambda} f(\lambda, u)\Big|_{\lambda=\lambda_0} \neq 0 \quad \text{for all} \quad u \tag{3.10}$$

is sufficient.

Indeed, in this case equality (3.7) is replaced by the equality

$$A\,\frac{\partial w}{\partial \lambda}\Big|_{\lambda=\lambda_0} + \left(\frac{\partial f}{\partial \lambda}\,(\lambda_0, w_{0,1}), ..., \frac{\partial f}{\partial \lambda}\,(\lambda_0, w_{0,N})\right) +$$

$$+\text{diag}\left(\frac{\partial f}{\partial u}\,(\lambda_0, w_{0,1}), ..., \frac{\partial f}{\partial u}\,(\lambda_0, w_{0,N})\right)\,\frac{\partial w}{\partial \lambda}\Big|_{\lambda=\lambda_0} = 0.$$

This equality and condition (3.10) imply that inequality (3.8) holds. The rest of the proof is the same as in Theorem 3.1.

Remark 3.3. For a fixed λ_0, the genericity of condition (a) in Theorem 3.1 is established in [7].

Remark 3.4. Theorem 3.1 generalizes the main result of [3], established for the Chaffee – Infante problem.

4 Applications

Equations of the type (2.1) are widely used in applications (it is enough to mention the famous Kolmogorov – Petrovski – Piskunov equation [10]).

For new applications, let us mention the work [11] in which the Landau – Khalatnikov equation is reduced to the Chaffee – Infante problem

$$\eta_t = \eta_{xx} + \lambda\eta - \eta^3, \quad \eta(0) = \eta(\pi) = 0. \tag{4.1}$$

Equation (4.1) models the evolution of a binary alloy; the parameter λ is directly related to the temperature of the alloy.

Of course, the results of this paper are of theoretical character; they only give us conditions of parametric identifiability (i. e., the possibility of parameter identification) and not an algorithm of identification itself.

Nevertheless, it is interesting to apply the method of Sec. 3 to check the “rate of divergence” of solutions of scheme (3.1) for different values of λ.

Below, we represent results of numerical simulation of Eq. (4.1) for some values of λ_0 and λ.

We fix time step $h = 0.01$, space step $d = \frac{\pi}{21}$ (so that $N = 20$), and the initial value $v^0 = (v_1^0, ..., v_{20}^0)$ with $v_{2i+1}^0 = 0$, $i = 0, ..., 9$; $v_{2i}^0 = 1$, $i = 1, ..., 10$.

The tables below show the number of steps at which the norm of the difference between the solutions $\varphi^n(\lambda_0, v^0)$ and $\varphi^n(\lambda, v^0)$ reaches the prescribed value Δ. The calculations are stopped if

$$|\varphi^n(\lambda_0, v^0) - \varphi^n(\lambda, v^0)| < \Delta \quad \text{for} \quad n \leq 100\,000.$$

In Table 1, $\lambda_0 = 1.4$, $\lambda = 1.5$; in Table 2, $\lambda_0 = 1.4$, $\lambda = 1.6$.

Table 1

Δ	0.1	0.2	0.25	0.3
number of steps	54	133	206	stopped

Table 2

Δ	0.1	0.2	0.3	0.4	0.5	0.55	0.6
number of steps	26	53	86	130	208	383	stopped

5 Aknowledgements

The author is deeply grateful to anonymous referees for valuable comments which helped to improve the presentation.

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