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Integro-differential equations

On the existence of weakly continuous solutions of the nonlinear functional equations and functional differential equations in reflexive and nonreflexive Banach spaces

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Abstract

The existence of solution of the functional equations and functional integral and differential equations, in different spaces of functions, have been studied by some authors. Here, we are concerned with an initial value problem of the nonlinear functional differential equation and present two theorems for the existence of at least one weak solution for this functional differential equation in the reflexive and nonreflexive Banach spaces relative to the weak topology.
For this aim we study, firstly, the existence of weakly continuous solutions for a nonlinear functional equation in the reflexive and nonreflexive Banach spaces.

Keywords: Weak solution, functional equation, fixed point theorem, measure of weak noncompactness.

1 Introduction and preliminaries

Let $L_1(I)$ be the space of Lebesgue integrable functions defined on the interval $I = [0,T]$. Let E be a Banach space with norm $\| \ . \ \|$ and its dual E^* and denote by $C[I,E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm.

The existence of weak solutions of the integral and differential equations has been studied by many authors such as [2], [3], [5], [8-10] and [12].

Let $\alpha \in (0,1)$. In [12], the authors studied, in the reflexive Banach space, the existence of week continuous solution of the initial value problem of the arbitrary (fractional) orders differential equation

$$\frac{d}{dx}x(t) \ = \ f(t, D^{\alpha}x(t)), \quad t \in (0,T]$$

where D^{α} is the fractional order derivative in the Caputo sense.
The case when $\alpha = 0$

$$\frac{d}{dx}x(t) \ = \ f(t, x(t)), \quad t \in (0,T]$$

have been proved, before, by D. O'Regan in [10].
In this paper we study (the case when $\alpha = 1$) the existence of weak continuous solutions for the initial value problem of functional differential equation

$$\frac{dx}{dt} = f(t, \frac{dx}{dt}), \ t \in (0,T], \tag{1}$$

$$x(0) = x_0. \tag{2}$$

in the reflexive and nonreflexive Banach space E.

For this aim we study, firstly, the existence of weak continuous solutions of the nonlinear functional equation

$$x(t) \ = \ f(t, x(t)), \ t \in I = [0,T] \tag{3}$$

in the nonreflexive and also reflexive Banach space E.

Now, we shall present some auxiliary results that will be needed in this work.

(1) A function $h : E \to E$ is said to be weakly Lipschitz if for every $\phi \in E^*$ there exists a positive constant K such that

$$\phi(h(x(.)) - h(y(.))) \leq K\ \phi(x(.) - y(.)).$$

(2) $h(.)$ is said to be weakly continuous if for every $\phi \in E^*$, $\phi(h(.))$ is continuous see ([7] and [9]).

(3) A function $h : E \to E$ is said to be weakly sequentially continuous if h maps weakly convergent sequences in E to weakly convergent sequences in E.

It is clear that (1) implies (2) and (2) implies (3). If h linear, then (2) and (3) are equivalent. The relation between weak and weak sequentially continuous of mapping is studied in details in [1].
Now, we have the following theorem due to Rubin (see[11]) and some propositions which will be used in the sequel (see [5]-[12]).

Theorem 1 *If X is metrizable (i.e., the topology is induced by a metric). Then the weakly sequentially continuous functions are weakly continuous.*

Proposition 1.1 A subset of a reflexive Banach space is weakly closed if and only if it is closed in the weak topology and bounded in the norm topology.
Proposition 1.2 Let E be a normed space with $y \in E$ and $y \neq 0$. Then there exists a $\phi \in E^*$ with $\| \phi \|= 1$ and $\| y \|= \phi(y)$.

Also, we have the following fixed point theorem, due to O'Regan, in reflexive Banach space (see [9]).

Theorem 2 *(O'Regan fixed point theorem)*
Let E be a Banach space and let Q be a nonempty, bounded, closed and convex subset of the space E and let $F : Q \to Q$ be a weakly sequentially continuous and assume that $FQ(t)$ is relatively weakly compact in E for each $t \in I$. Then, F has a fixed point in the set Q.

Here we apply the O'Regan fixed point theorem, in reflexive Banach space, to prove the existence of weak solutions to the functional equation (3).

In nonreflexive Banach space, our main condition that guarantees the existence of weak solutions of (3) will be formulated in terms of measure of weak noncompactness β introduced by De Blasi in [4].

Further on, denote by m_E the family of nonempty and bounded subsets of E. Let us recall that for any subset $A \in m_E$ of a Banach space E,

$\beta(A) = \inf\{r > 0 :$ there exists a weakly compact set C such that $A \subset C + rB_1\}$,

where B_1 is the closed unit ball in E. Recall that β has the following properties :

(1) $\beta(A) = 0$ iff A is weakly relatively compact;

(2) If $A \subset B$, then $\beta(A) \leq \beta(B)$;

(3) $\beta(A) = \beta(\overline{Conv} A)$;

(4) $\beta(\lambda A + (1-\lambda)A) \leq \lambda\beta A + (1-\lambda)\beta A, \quad \lambda \in [0,1]$;

(5) If $X_n \in m_E$, $X_n = \overline{X_n^w}$ and $X_{n+1} \subset X_n$ for $n = 1, 2, ...$
and if $\lim_{n\to\infty} \beta(X_n) = 0$, then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \phi$.

A measure of weak noncompactness β is said to be sublinear whenever it fulfils the two conditions

(6) $\beta(A + B) \leq \beta(A) + \beta(B)$;

(7) $\beta(\lambda A) = \mid \lambda \mid \beta(A)$, for all $\lambda \in R$.
Moreover,

(8) $\beta(x + A) = \beta(A)$, for all $x \in E$.

Theorem 3 *[6] Let E be a metrizable locally convex topological vector space and let Q be a closed convex subset of E , and let K be a weakly sequentially continuous mapping of Q into itself. If for some $y \in Q$ the implication*

$$\overline{V} = \overline{conv}(K(V) \cup \{y\}) \Rightarrow V \text{ is relatively weakly compact}$$

holds, for every subset V of Q, then K has a fixed point.

2 Weak solutions in reflexive Banach space

In this section we discuss the existence of weak solution of the functional equation (3) in the reflexive Banach spaces E.

Now, Let $B_r = \{x \in C[I, E] : \| x \| \leq r\}$, $r > 0$.
The functional equation (1) will be investigated under the assumptions:
$f : I \times B_r \to E$ such that

(i) for each $x \in B_r$, $f(., x(.))$ is weakly continuous on I;

(ii) for each $t \in I$, $f(t, .)$ is weakly Lipschitz in x with Lipschitz constant K.

Definition 1. A function $x \in C[I, E]$ is called a weakly continuous solution of the functional equation (3) if

$$\phi(x(t)) = \phi(f(t, x(t))), \ t \in I$$

is satisfied for all $\phi \in E^*$.
Now, we can prove the following theorem.

Theorem 4 *Under the assumptions (i)-(ii) the functional equation (3) has at least one weak solution $x \in C[I, E]$.*

Proof. Define the operator A by

$$Ax(t) = f(t, x(t)), \ t \in I.$$

Now we shall prove that $A : C[I, E] \to C[I, E]$.
Let $t_1, t_2 \in I$ and (without loss of generality) assume that $Ax(t_2) - A_x(t_1) \neq 0$

$$\begin{aligned} Ax(t_2) - Ax(t_1) &= f(t_2, x(t_2)) - f(t_1, x(t_1)) \\ &= f(t_2, x(t_2)) - f(t_1, x(t_1)) - f(t_2, x(t_1)) + f(t_2, x(t_1)). \end{aligned}$$

Therefore, as a consequence of proposition 1.2, we obtain

$$\begin{aligned} \| Ax(t_2) - Ax(t_1) \| &= \phi(Ax(t_2) - Ax(t_1)) \\ &= \phi(f(t_2, x(t_2)) - f(t_1, x(t_1)) - f(t_2, x(t_1)) + f(t_2, x(t_1))) \\ &= \phi(f(t_2, x(t_2)) - f(t_2, x(t_1))) + \phi(f(t_2, x(t_1)) - f(t_1, x(t_1))) \\ &\leq K \ \phi(x(t_2) - x(t_1)) + \phi(f(t_2, x(t_1)) - f(t_1, x(t_1))) \\ &\leq K \ \| x(t_2) - x(t_1) \| + \| f(t_2, x(t_1)) - f(t_1, x(t_1)) \| . \end{aligned}$$

This prove that $Ax \in C[I, E]$.
Now, for any $x \in B_r$ we have

$$\| Ax(t) \| = \phi(Ax(t)) = \phi(f(t, x(t))) = \| f(t, x(t)) \|$$

$$\leq \mid f(t,0) \mid +K \;\; \| x \|$$
$$\leq \; a + K\, r, \quad r = \frac{a}{1-K}$$

where $a = \sup\{\mid f(t,0) \mid : t \in I\}$

i.e., $\forall x \in B_r \Rightarrow Ax \in B_r \Rightarrow AB_r \subset B_r$. Thus $A : B_r \to B_r$.
Then B_r is nonempty, uniformly bounded subset of $C[I,E]$. Also, it can be shown B_r is convex and closed. As consequence of proposition 1.1, then AB_r is relatively weakly compact.

It remains to prove that A is weakly sequentially continuous.
Let $\{x_n(t)\}$ be a sequence in B_r converges weakly to $x(t), \forall t \in I$. Since $f(t,x(t))$ is weakly Lipschitz $\Rightarrow$ $f(t,x(t))$ is weakly continuous $\Rightarrow$ $f(t,x(t))$ is weakly sequentially continuous in the second argument, then $f(t,x_n(t))$ converges weakly to $f(t,x(t))$ and hence $\phi(f(t,x_n(t)))$ converges strongly to $\phi(f(t,x(t)))$.
Then A is weakly sequentially continuous. Since all conditions of Theorem 1.2 are satisfied, then the operator A has at least one fixed point $x \in B_r$ which completes the proof.

2.1 Uniqueness of the solution

Now we prove the uniqueness solution of the functional equation (3).

Theorem 5

Let the assumptions of Theorem 2.1 are satisfied with $K < 1$, then the solution of the functional equation (3) is unique.

Proof. Let x_1, x_2 be two weak solutions of (3) in the reflexive Banach space E. Then

$$\| x_1 - x_2 \| = \; \phi(x_1 - x_2) = \phi(f(t,x_1(t)) - f(t,x_2(t)))$$
$$\leq \; K\, \phi(x_1(t) - x_2(t))$$
$$\leq \; K \;\; \| x_1 - x_2 \|,$$

then

$$\| x_1 - x_2 \| \;\; (1-K) \; \leq \; 0, \; K < 1$$

implies that

$$\| x_1 - x_2 \| \leq \; 0 \Rightarrow \| x_1 - x_2 \| = \; 0$$

and

$$x_1(t) = x_2(t),\ t \in I$$

which implies that the uniqueness of a weak solution.

3 Weak solutions in nonreflexive Banach space

Here we shall discuss the existence of the weak solutions of the functional equation (3) in the nonreflexive Banach spaces E. Now, Let $B_r = \{x \in C[I, E] : \| x \| \leq r\}$, $r > 0$.
The functional equation (1) will be investigated under the assumptions:
$f : I \times B_r \to E$ such that

(i:) for each $x \in B_r$, $f(., x(.))$ is weakly continuous on I;

(ii:) for each $t \in I$, $f(t, .)$ is weakly Lipschitz in x with Lipschitz constant K;

(iii:) for any bounded set $X \subseteq B_r$ there exists a constant $\alpha \in [0, 1)$ such that

$$\beta(f(I \times X)) \leq \alpha\beta(X).$$

A function $x \in C[I, E]$ is a solution of the functional equation (3) if

$$\phi(x(t)) = \phi(f(t, x(t))),\ t \in I$$

is satisfied for all $\phi \in E^*$.

Theorem 6 *Under the assumptions (i)-(iii) the functional equation (3) has at least one weak solution $x \in C[I, E]$.*

Proof. Define the operator A by

$$Ax(t) = f(t, x(t)),\ t \in I.$$

As done above in Theorem 2.1 we have

1- $A : C[I, E] \to C[I, E]$;

2- B_r is nonempty, uniformly bounded, closed and convex ;

3- $A : B_r \to B_r$ is weakly sequentially continuous.

Now, let $V \subset B_r$, be such that $V = \overline{conv}(A(V) \cup \{x(.)\})$ for some $x(.) \in B_r$. We prove that V is weakly relatively compact in $C[I, E]$.
Using the properties of β we get

$$\beta(V(t)) = \beta(\overline{conv}(A(V(t)) \cup \{x(t)\})) = \beta(A(V)(t)) \; for \; t \in I$$

$$Ax(t) = f(t, x(t)) \in \overline{conv} f(I \times V(I))$$

where $f(I \times V(I)) = \{f(t, x(t)) : t \in I, x \in V\}$

$$AV(t) \subset \overline{conv} f(I \times V(I))$$

and

$$\begin{aligned} \beta(AV(t)) &\leq \beta(\overline{conv} f(I \times V(I))) \\ &\leq \beta(f(I \times V(I))) \leq \alpha\beta(V(I)). \end{aligned}$$

Thus

$$\beta(V(t)) \leq \alpha\beta(V(t)), \; for \; t \in I, \; \alpha < 1.$$

Then $\beta(V(t)) = 0 \quad for \; each \; t \in I$. Hence $V(t)$ is relatively weakly compact in E for $t \in I$, and by Ascoli's theorem, V is relatively weakly compact in $C[I, E]$.
Since all conditions of Theorem 1.3 are satisfied then A has at least one fixed point $x \in B_r$ which complement the proof.
Also, as in Theorem 2.2, we can prove the following Theorem.

Theorem 7

Let the assumptions of Theorem 3.1 with $K < 1$ are satisfied, then the solution of the functional equation (3) is unique.

4 Application

Consider now the initial value problem (1)-(2)

Theorem 8 *Under the assumptions (i)-(ii) of Theorem 2.1 the initial value problem (1)-(2) has at least one weakly differentiable solution in* $C[I, E]$.

Proof. Let $\frac{dx}{dt} = y \in C[I, E]$ weakly, then

$$x(t) = x_0 + \int_0^t y(s)ds \in C[I, E]$$

and weakly differentiable in $C[I,E]$.
And, then y will be the solution of the functional equation

$$y(t) = f(t, y(t)), \ t \in [0, T].$$

Also, we can prove the following Theorem

Theorem 9 *Let the assumptions of Theorem 3.1 are satisfied then the initial value problem (1)-(2) has at least one weakly differentiable solution in* $C[I,E]$.

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