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Functional differential equations

Existence of positive periodic solutions for two types of third-order nonlinear neutral differential equations with variable coefficients

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Abstract

In this work, we study the existence of positive periodic solutions for two types of third-order nonlinear neutral differential equations with variable coefficients. The results are established by using the Krasnoselskii's fixed point theorem. The results obtained here extend the work of Ren, Siegmund and Chen [30]. Two examples are given to illustrate this work.

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1 Introduction

Third order differential equations arise from in a variety of different areas of applied mathematics and physics, as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves or gravity driven flows and so on [2, 16, 23, 28, 30].

Delay differential equations have received increasing attention during recent years since these equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering, see the monograph [10, 24] and the papers [1]-[6], [9], [11]-[22], [26]-[28], [30], [32], [33], [35], [36] and the references therein.

Ren, Siegmund and Chen [30] discussed the existence of positive ω-periodic solutions for the following neutral functional differential equation

$$(x(t) - cx(t - \tau(t)))''' = -a(t)x(t) + f(t, x(t - \tau(t))),$$

where $|c| < 1$. By employing Krasnoselskii's fixed point theorem, the authors obtained existence results for positive ω-periodic solutions.

In the present article, we study the existence of positive ω-periodic solutions for the following two types of third-order nonlinear neutral differential equations

$$(x(t) - c(t)x(t - \tau))''' = a(t)x(t) - f(t, x(t - \tau)), \tag{1}$$

and

$$(x(t) - c(t)x(t - \tau))''' = -a(t)x(t) + f(t, x(t - \tau)), \tag{2}$$

where $c \in C(\mathbb{R}, \mathbb{R})$, $a \in C(\mathbb{R}, (0, \infty))$, $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\tau, \omega > 0$, and c, a are ω-periodic functions, f is ω-periodic with respect to first variable. To show the existence of positive ω-periodic solutions, we transform (1) and (2) into integral equations and then use Krasnoselskii's fixed point theorem. The obtained integral equations split in the sum of two mappings, one is a contraction and the other is compact.

In this paper, we have two main contributions comparing with the existing results. First, instead of constant c we take variable coefficient $c(t)$. Second, in addition to $|c(t)| < 1$, we consider the range $|c(t)| > 1$ for $c(t)$, which is new in the literature. Also, the results obtained here extend the work of Ren, Siegmund and Chen [30].

The organization of this paper is as follows. In section 2, we introduce some notations and lemmas, and state some preliminary results needed in later sections. Then we give the Green's function of (1) and (2) which plays an important role in this paper. Also, we present the inversions of (1) and (2), and Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [7, 8, 25, 29, 31, 34]. In section 3, we present our main results on existence of positive ω-periodic solutions of (1) and (2). Two examples are also given to illustrate this work.

2 Preliminaries

For $\omega > 0$, let C_ω be the set of all continuous scalar functions x, periodic in t of period ω. Then $(C_\omega, \|.\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,\omega]} |x(t)|.$$

Define

$$C_\omega^+ = \{u \in C_\omega : u > 0\}, \ C_\omega^- = \{u \in C_\omega : u < 0\}.$$

Denote

$$M = \sup\{a(t) : t \in [0,\omega]\}, \ m = \inf\{a(t) : t \in [0,\omega]\}, \ \rho = \sqrt[3]{M}.$$

Lemma 1 ([30]) *The equation*

$$u'''(t) - Mu(t) = h(t), \quad h \in C_\omega^-,$$
$$u(0) = u(\omega), \quad u'(0) = u'(\omega), \quad u''(0) = u''(\omega),$$

has a unique ω-periodic solution

$$u(t) = \int_0^\omega G_1(t,s)(-h(s))ds,$$

where

$$G_1(t,s) = \begin{cases} \frac{2\exp(\frac{1}{2}\rho(s-t))[\sin(\frac{\sqrt{3}}{2}\rho(t-s)+\frac{\pi}{6})-\exp(-\frac{1}{2}\rho\omega)\sin(\frac{\sqrt{3}}{2}\rho(t-s-\omega)+\frac{\pi}{6})]}{3\rho^2(1+\exp(-\rho\omega)-2\exp(-\frac{\rho\omega}{2})\cos(\frac{\sqrt{3}}{2}\rho\omega))} \\ +\frac{\exp(\rho(t-s))}{3\rho^2(\exp(\rho\omega)-1)}, \quad \text{if } 0 \le s \le t \le \omega, \\ \frac{2\exp(\frac{1}{2}\rho(s-t-\omega))[\sin(\frac{\sqrt{3}}{2}\rho(t-s+\omega)+\frac{\pi}{6})-\exp(-\frac{1}{2}\rho\omega)\sin(\frac{\sqrt{3}}{2}\rho(t-s)+\frac{\pi}{6})]}{3\rho^2(1+\exp(-\rho\omega)-2\exp(-\frac{\rho\omega}{2})\cos(\frac{\sqrt{3}}{2}\rho\omega))} \\ +\frac{\exp(\rho(t+\omega-s))}{3\rho^2(\exp(\rho\omega)-1)}, \quad \text{if } 0 \le t \le s \le \omega. \end{cases}$$

Corollary 1 *Green's function G_1 satisfies the following properties*

$$\int_0^\omega G_1(t,s)ds = \frac{1}{M},$$

and if $\sqrt{3}\rho\omega < 4\pi/3$ holds, then

$$0 < A < G_1(t,s) \le B, \quad 0 < A < G_1(t+\tau,s) \le B,$$

where

$$A = \frac{1}{3\rho^2(\exp(\rho\omega) - 1)}, \quad B = \frac{3 + 2\exp(-\frac{\rho\omega}{2})}{3\rho^2(1 - \exp(-\frac{\rho\omega}{2}))^2}.$$

Lemma 2 ([30]) *The equation*

$$u'''(t) + Mu(t) = h(t), \quad h \in C_\omega^+,$$
$$u(0) = u(\omega), \quad u'(0) = u'(\omega), \quad u''(0) = u''(\omega),$$

has a unique ω-periodic solution

$$u(t) = \int_0^\omega G_2(t,s)h(s)ds,$$

where

$$G_2(t,s) = \begin{cases} \frac{2\exp(\frac{1}{2}\rho(t-s))[\sin(\frac{\sqrt{3}}{2}\rho(t-s)-\frac{\pi}{6})-\exp(\frac{1}{2}\rho\omega)\sin(\frac{\sqrt{3}}{2}\rho(t-s-\omega)-\frac{\pi}{6})]}{3\rho^2(1+\exp(\rho\omega)-2\exp(\frac{1}{2}\rho\omega)\cos(\frac{\sqrt{3}}{2}\rho\omega))} \\ +\frac{\exp(\rho(s-t))}{3\rho^2(1-\exp(-\rho\omega))}, \quad \text{if } 0 \le s \le t \le \omega, \\ \frac{2\exp(\frac{1}{2}\rho(t+\omega-s))[\sin(\frac{\sqrt{3}}{2}\rho(t+\omega-s)-\frac{\pi}{6})-\exp(\frac{1}{2}\rho\omega)\sin(\frac{\sqrt{3}}{2}\rho(t-s)-\frac{\pi}{6})]}{3\rho^2(1+\exp(\rho\omega)-2\exp(\frac{1}{2}\rho\omega)\cos(\frac{\sqrt{3}}{2}\rho\omega))} \\ +\frac{\exp(\rho(s-t-\omega))}{3\rho^2(1-\exp(-\rho\omega))}, \quad \text{if } 0 \le t \le s \le \omega. \end{cases}$$

Corollary 2 *Green's function G_2 satisfies the following properties*

$$\int_0^\omega G_2(t,s)ds = \frac{1}{M},$$

and if $\sqrt{3}\rho\omega < 4\pi/3$ holds, then

$$0 < A < G_2(t,s) \le B, \quad 0 < A < G_2(t+\tau,s) \le B.$$

Lemma 3 ([30]) *The equation*

$$u'''(t) - a(t)u(t) = h(t), \quad h \in C_\omega^-,$$

has a unique positive ω-periodic solution

$$(P_1h)(t) = (I - T_1B_1)^{-1}(T_1h)(t),$$

where

$$(T_1h)(t) = \int_0^\omega G_1(t,s)(-h(s))ds, \quad (B_1u)(t) = (-M + a(t))u(t).$$

Lemma 4 ([30]) *If $\sqrt{3}\rho\omega < 4\pi/3$ holds, then P_1 is completely continuous and*

$$0 < (T_1h)(t) \le (P_1h)(t) \le \frac{M}{m}\|(T_1h)(t)\|, \quad \forall h \in C_\omega^-.$$

The following theorem is essential for our results on existence of positive periodic solution of (1).

Theorem 1 *If $x \in C_\omega$ then x is a solution of (1) if and only if*

$$x(t) = c(t)x(t-\tau) + P_1(-f(t, x(t-\tau)) + c(t)a(t)x(t-\tau)). \tag{3}$$

Proof. Let $x \in C_\omega$ be a solution of (1). Rewrite (1) as

$$\begin{aligned}
&(x(t) - c(t)x(t-\tau))''' - M(x(t) - c(t)x(t-\tau)) \\
&= (-M + a(t))(x(t) - c(t)x(t-\tau)) - f(t, x(t-\tau)) + c(t)a(t)x(t-\tau) \\
&= B_1(x(t) - c(t)x(t-\tau)) - f(t, x(t-\tau)) + c(t)a(t)x(t-\tau).
\end{aligned}$$

From Lemmas 1 and 3, we have

$$\begin{aligned}
&x(t) - c(t)x(t-\tau) \\
&= T_1B_1(x(t) - c(t)x(t-\tau)) + T_1(-f(t, x(t-\tau)) + c(t)a(t)x(t-\tau)).
\end{aligned}$$

This yields

$$(I - T_1B_1)(x(t) - c(t)x(t-\tau)) = T_1(-f(t, x(t-\tau)) + c(t)a(t)x(t-\tau)).$$

Therefore

$$\begin{aligned}
x(t) - c(t)x(t-\tau) &= (I - T_1B_1)^{-1}T_1(-f(t, x(t-\tau)) + c(t)a(t)x(t-\tau)) \\
&= P_1(-f(t, x(t-\tau)) + c(t)a(t)x(t-\tau)).
\end{aligned}$$

Obviously,

$$x(t) = c(t)x(t-\tau) + P_1(-f(t, x(t-\tau)) + c(t)a(t)x(t-\tau)).$$

■

Corollary 3 *If $x \in C_\omega$ then x is a solution of (1) if and only if*

$$x(t) = \frac{1}{c(t+\tau)}\left[x(t+\tau) + P_1(-c(t+\tau)a(t+\tau)x(t) + f(t+\tau, x(t)))\right]. \tag{4}$$

Lemma 5 ([30]) *The equation*

$$u'''(t) + a(t)u(t) = h(t), \quad h \in C_\omega^+,$$

has a unique positive ω-periodic solution

$$(P_2h)(t) = (I - T_2B_2)^{-1}(T_2h)(t),$$

where

$$(T_2h)(t) = \int_0^\omega G_2(t,s)h(s)ds, \quad (B_2u)(t) = (M - a(t))u(t).$$

Lemma 6 ([30]) *If $\sqrt{3}\rho\omega < 4\pi/3$ holds, then P_2 is completely continuous and*

$$0 < (T_2h)(t) \le (P_2h)(t) \le \frac{M}{m}\|(T_2h)(t)\|, \quad \forall h \in C_\omega^+.$$

The following theorem is essential for our results on existence of positive periodic solution of (2).

Theorem 2 *If $x \in C_\omega$ then x is a solution of (2) if and only if*

$$x(t) = c(t)x(t-\tau) + P_2(f(t, x(t-\tau)) - c(t)a(t)x(t-\tau)). \tag{5}$$

Proof. Let $x \in C_\omega$ be a solution of (2). Rewrite (2) as

$$\begin{aligned}&(x(t) - c(t)x(t-\tau))''' + M(x(t) - c(t)x(t-\tau))\\ &= (M - a(t))(x(t) - c(t)x(t-\tau)) + f(t, x(t-\tau)) - c(t)a(t)x(t-\tau)\\ &= B_2(x(t) - c(t)x(t-\tau)) + f(t, x(t-\tau)) - c(t)a(t)x(t-\tau).\end{aligned}$$

From Lemmas 2 and 5, we have

$$\begin{aligned}&x(t) - c(t)x(t-\tau)\\ &= T_2B_2(x(t) - c(t)x(t-\tau)) + T_2(f(t, x(t-\tau)) - c(t)a(t)x(t-\tau)).\end{aligned}$$

This yields

$$(I - T_2B_2)(x(t) - c(t)x(t-\tau)) = T_2(f(t, x(t-\tau)) - c(t)a(t)x(t-\tau)).$$

Therefore,

$$\begin{aligned}x(t) - c(t)x(t-\tau) &= (I - T_2B_2)^{-1}T_2(f(t, x(t-\tau)) - c(t)a(t)x(t-\tau))\\ &= P_2(f(t, x(t-\tau)) - c(t)a(t)x(t-\tau)).\end{aligned}$$

Obviously,

$$x(t) = c(t)x(t-\tau) + P_2(f(t, x(t-\tau)) - c(t)a(t)x(t-\tau)).$$

■

Corollary 4 *If $x \in C_\omega$ then x is a solution of (2) if and only if*

$$x(t) = \frac{1}{c(t+\tau)}\left[x(t+\tau) + P_2(c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t)))\right]. \qquad (6)$$

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of positive ω-periodic solutions to (1) and (2). For its proof we refer the reader to [25, 31, 34].

Lemma 7 (Krasnoselskii [25, 31, 34]) *Let $\mathbb{D}$ be a closed convex nonempty subset of a Banach space $(\mathcal{B}, \|\cdot\|)$. Suppose that A and B map $\mathbb{D}$ into $\mathcal{B}$ such that*

(i) $Ax + By \in \mathbb{D}, \ \forall x, y \in \mathbb{D}$,

(ii) A is completely continuous,

(iii) B is a contraction mapping.

Then there exists $z \in \mathbb{D}$ with $z = Az + Bz$.

3 Positive periodic solutions

To apply Lemma 7, we need to define a Banach space $\mathcal{B}$, a closed convex subset $\mathbb{D}$ of $\mathcal{B}$ and construct two mappings, one is contraction and the other is a completely continuous. So we let $(\mathcal{B}, \|\cdot\|) = (C_\omega, \|\cdot\|)$ and $\mathbb{D} = \{\varphi \in C_\omega : M_1 \le \varphi \le M_2\}$, where M_1 is non-negative constant and M_2 is positive constant.

3.1 Positive periodic solutions in the case $|c(t)| > 1$

In this subsection, we obtain the existence of positive ω-periodic solution for (1) and (2) by considering the two cases: $1 < c(t) < \infty$ and $-\infty < c(t) < -1$ for all $t \in [0, \omega]$.

Theorem 3 *Suppose that $\sqrt{3}\rho\omega < 4\pi/3$, $1 < c_1 \le c(t) \le c_2 < \infty$ and*

$$m \le c(t)a(t)x - f(t,x) \le c_1 M, \ \forall (t,x) \in [0,\omega] \times \left[\frac{m}{(c_2-1)M}, \frac{c_1 M}{(c_1-1)m}\right]. \qquad (7)$$

Then (1) has at least one positive ω-periodic solution x in the subset $\mathbb{D}_1$ of $\mathcal{B}$ where $\mathbb{D}_1 = \left\{\varphi \in C_\omega : \frac{m}{(c_2-1)M} \le \varphi \le \frac{c_1 M}{(c_1-1)m}\right\}$.

Proof. We express (4) as

$$\varphi(t) = (B_1\varphi)(t) + (A_1\varphi)(t) := (H_1\varphi)(t),$$

where $A_1, B_1 : \mathbb{D}_1 \to \mathcal{B}$ are defined by

$$(A_1\varphi)(t) = \frac{1}{c(t+\tau)} P_1(-c(t+\tau)a(t+\tau)\varphi(t) + f(t+\tau,\varphi(t))),$$

and

$$(B_1\varphi)(t) = \frac{\varphi(t+\tau)}{c(t+\tau)}.$$

It is obvious that $A_1\varphi$ and $B_1\varphi$ are continuous and ω-periodic. Now we prove that $A_1x + B_1y \in \mathbb{D}_1$, $\forall x, y \in \mathbb{D}_1$. By Corollary 1, Lemma 4 and the condition (7) we obtain

$$\begin{aligned}
&(A_1x)(t) + (B_1y)(t) \\
&= \frac{1}{c(t+\tau)} \left[P_1(-c(t+\tau)a(t+\tau)x(t) + f(t+\tau,x(t))) + y(t+\tau)\right] \\
&\le \frac{1}{c_1} \left[\frac{M}{m} T_1(-c(t+\tau)a(t+\tau)x(t) + f(t+\tau,x(t))) + \frac{c_1 M}{(c_1-1)m}\right] \\
&\le \frac{M}{mc_1} \max_{t\in[0,\omega]} \left| \int_0^\omega G_1(t,s)(c(s+\tau)a(s+\tau)x(s) - f(s+\tau,x(s)))ds \right| \\
&+ \frac{M}{(c_1-1)m} \\
&\le \frac{M}{mc_1} \int_0^\omega G_1(t,s)c_1 M ds + \frac{M}{(c_1-1)m} \\
&\le \frac{M}{mc_1} c_1 M \frac{1}{M} + \frac{M}{(c_1-1)m} \\
&= \frac{c_1 M}{(c_1-1)m}.
\end{aligned} \tag{8}$$

On the other hand

$$
\begin{aligned}
&(A_1x)(t) + (B_1y)(t) \\
&= \frac{1}{c(t+\tau)}\left[P_1(-c(t+\tau)a(t+\tau)x(t) + f(t+\tau, x(t))) + y(t+\tau)\right] \\
&\geq \frac{1}{c_2}\left[T_1(-c(t+\tau)a(t+\tau)x(t) + f(t+\tau, x(t))) + \frac{m}{(c_2-1)M}\right] \\
&\geq \frac{1}{c_2}\int_0^{\omega} G_1(t,s)(c(s+\tau)a(s+\tau)x(s) - f(s+\tau, x(s)))ds + \frac{1}{c_2}\frac{m}{(c_2-1)M} \\
&\geq \frac{1}{c_2}\int_0^{\omega} G_1(t,s)m ds + \frac{1}{c_2}\frac{m}{(c_2-1)M} \\
&\geq \frac{1}{c_2}m\frac{1}{M} + \frac{1}{c_2}\frac{m}{(c_2-1)M} = \frac{m}{(c_2-1)M}.
\end{aligned}
\tag{9}
$$

Combining (8) and (9), we obtain $A_1x + B_1y \in \mathbb{D}_1$, $\forall x, y \in \mathbb{D}_1$. For $\varphi, \psi \in \mathbb{D}_1$, we have

$$
\begin{aligned}
|(B_1\varphi)(t) - (B_1\psi)(t)| &= \left|\frac{\varphi(t+\tau)}{c(t+\tau)} - \frac{\psi(t+\tau)}{c(t+\tau)}\right| \\
&\leq \frac{1}{c_1}|\varphi(t+\tau) - \psi(t+\tau)| \\
&\leq \frac{1}{c_1}\|\varphi - \psi\|,
\end{aligned}
$$

which implies that $\|B_1\varphi - B_1\psi\| \leq \frac{1}{c_1}\|\varphi - \psi\|$. Since $0 < \frac{1}{c_1} < 1$, B_1 is a contraction on $\mathbb{D}_1$. From Lemma 4, we know that P_1 is completely continuous, so is A_1. By Lemma 7, we obtain that $A_1 + B_1$ has a fixed point $x \in \mathbb{D}_1$, i.e. (1) has a positive ω-periodic solution $x \in \mathbb{D}_1$. ■

Corollary 5 *Assume that the hypotheses of Theorem 3 hold. Then (2) has at least one positive ω-periodic solution x in the subset $\mathbb{D}_1$ of $\mathcal{B}$.*

Theorem 4 *Suppose that $\sqrt{3}\rho\omega < 4\pi/3$, $-\infty < c_3 \leq c(t) \leq c_4 < -1$ and*

$$
\frac{c_3}{c_4}M < f(t,x) - c(t)a(t)x \leq -c_4 m, \ \forall (t,x) \in [0,\omega]\times[0,1]. \tag{10}
$$

Then (1) has at least one positive ω-periodic solution x in the subset $\widetilde{\mathbb{D}}_2$ of $\mathcal{B}$, where $\widetilde{\mathbb{D}}_2 = \{\varphi \in C_\omega : 0 < \varphi \leq 1\}$.

Proof. Let $\mathbb{D}_2 = \{\varphi \in C_\omega : 0 \leq \varphi \leq 1\}$. We define $A_1, B_1 : \mathbb{D}_2 \to \mathcal{B}$ as follows

$$(A_1\varphi)(t) = \frac{-1}{c(t+\tau)} P_1(c(t+\tau)a(t+\tau)\varphi(t) - f(t+\tau, \varphi(t))),$$

and

$$(B_1\varphi)(t) = \frac{\varphi(t+\tau)}{c(t+\tau)}.$$

Now we prove that $A_1x + B_1y \in \mathbb{D}_2$, $\forall x, y \in \mathbb{D}_2$. By Corollary 1, Lemma 4 and the condition (10) we obtain

$$\begin{aligned}
&(A_1x)(t) + (B_1y)(t) \\
&= \frac{-1}{c(t+\tau)} P_1(c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))) + \frac{y(t+\tau)}{c(t+\tau)} \\
&\leq \frac{-1}{c_4}\frac{M}{m} T_1(c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))) \\
&\leq \frac{-M}{mc_4} \max_{t\in[0,\omega]} \left| \int_0^\omega G_1(t,s)(f(s+\tau, x(s)) - c(s+\tau)a(s+\tau)x(s))ds \right| \\
&\leq \frac{-M}{mc_4} \int_0^\omega G_1(t,s)(-c_4 m)ds \\
&\leq \frac{-M}{mc_4}(-c_4 m)\frac{1}{M} = 1. \qquad (11)
\end{aligned}$$

On the other hand

$$\begin{aligned}
&(A_1x)(t) + (B_1y)(t) \\
&= \frac{-1}{c(t+\tau)} P_1(c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))) + \frac{y(t+\tau)}{c(t+\tau)} \\
&\geq \frac{-1}{c_3} T_1(c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))) + \frac{1}{c_4} \\
&\geq \frac{-1}{c_3} \int_0^\omega G_1(t,s)(f(s+\tau, x(s)) - c(s+\tau)a(s+\tau)x(s))ds + \frac{1}{c_4} \\
&\geq \frac{-1}{c_3} \int_0^\omega G_1(t,s)(\frac{c_3}{c_4}M)ds + \frac{1}{c_4} \\
&\geq \frac{-1}{c_3}(\frac{c_3}{c_4}M)\frac{1}{M} + \frac{1}{c_4} = 0. \qquad (12)
\end{aligned}$$

Combining (11) and (12), we obtain $A_1x + B_1y \in \mathbb{D}_2$, for all $x, y \in \mathbb{D}_2$. For $\varphi, \psi \in \mathbb{D}_2$, we have

$$\begin{aligned} |(B_1\varphi)(t) - (B_1\psi)(t)| &= \left| \frac{\varphi(t+\tau)}{c(t+\tau)} - \frac{\psi(t+\tau)}{c(t+\tau)} \right| \\ &\le \frac{-1}{c_4} \left| \varphi(t+\tau) - \psi(t+\tau) \right| \\ &\le \frac{-1}{c_4} \left\| \varphi - \psi \right\|, \end{aligned}$$

which implies that $\|B_1\varphi - B_1\psi\| \le \frac{-1}{c_4} \|\varphi - \psi\|$. Since $0 < \frac{-1}{c_4} < 1$, B_1 is a contraction on $\mathbb{D}_2$. From Lemma 4, we know that P_1 is completely continuous, so is A_1. By Lemma 7, we obtain that $A_1 + B_1$ has a fixed point $x \in \mathbb{D}_2$, i.e. (1) has a nonnegative ω-periodic solution x with $0 \le x(t) \le 1$. Since $f(t,x) - c(t)a(t)x > \frac{c_3}{c_4}M$, it is easy to see that $x(t) > 0$, i.e. (1) has positive ω-periodic solution $x \in \widetilde{\mathbb{D}}_2$. ■

Corollary 6 *Assume that the hypotheses of Theorem 4 hold. Then (2) has at least one positive ω-periodic solution x in the subset $\widetilde{\mathbb{D}}_2$ of $\mathcal{B}$.*

3.2 Positive periodic solutions in the case $|c(t)| < 1$

In this subsection, we obtain the existence of a positive periodic solution for (1) and (2) by considering the three cases; $0 < c(t) < 1$, $-1 < c(t) \le 0$ and $c(t) = 0$ for all $t \in [0, \omega]$.

Theorem 5 *Suppose that $\sqrt{3}\rho\omega < 4\pi/3$, $0 < c_5 \le c(t) \le c_6 < 1$, and*

$$c_5 m \le f(t,x) - c(t)a(t)x \le M, \ \forall (t,x) \in [0,\omega] \times \left[\frac{c_5 m}{(1-c_5)M}, \frac{M}{(1-c_6)m} \right]. \quad (13)$$

Then (1) has at least one positive ω-periodic solution $x(t)$ in the subset $\mathbb{D}_3$ of $\mathcal{B}$, where $\mathbb{D}_3 = \left\{ \varphi \in C_\omega : \frac{c_5 m}{(1-c_5)M} \le \varphi \le \frac{M}{(1-c_6)m} \right\}$.

Proof. We express (3) as

$$\varphi(t) = (B_2\varphi)(t) + (A_2\varphi)(t) := (H_2\varphi)(t),$$

where $A_2, B_2 : \mathbb{D}_3 \to \mathcal{B}$ are defined by

$$(A_2\varphi)(t) = P_1(c(t)a(t)\varphi(t-\tau) - f(t, \varphi(t-\tau))),$$

and

$$(B_2\varphi)(t) = c(t)\varphi(t-\tau).$$

It is obvious that $A_2\varphi$ and $B_2\varphi$ are continuous and ω-periodic. Now we prove that $A_2x + B_2y \in \mathbb{D}_3$, $\forall x, y \in \mathbb{D}_3$. By Corollary 1, Lemma 4 and the condition (13) we obtain

$$\begin{aligned}
&(A_2x)(t) + (B_2y)(t) \\
&= P_1(c(t)a(t)x(t-\tau) - f(t, x(t-\tau))) + c(t)y(t-\tau) \\
&\le \frac{M}{m} T_1(c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))) + c_6\frac{M}{(1-c_6)m} \\
&\le \frac{M}{m} \max_{t\in[0,\omega]} \left| \int_0^\omega G_1(t,s)(f(s+\tau, x(s)) - c(s+\tau)a(s+\tau)x(s))ds \right| \\
&+ c_6\frac{M}{(1-c_6)m} \\
&\le \frac{M}{m} \int_0^\omega G_1(t,s)M ds + c_6\frac{M}{(1-c_6)m} \\
&\le \frac{M}{m} M\frac{1}{M} + c_6\frac{M}{(1-c_6)m} = \frac{M}{(1-c_6)m}.
\end{aligned} \tag{14}$$

On the other hand

$$\begin{aligned}
&(A_2x)(t) + (B_2y)(t) \\
&= P_1(c(t)a(t)x(t-\tau) - f(t, x(t-\tau))) + c(t)y(t-\tau) \\
&\ge T_1(c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))) + c_5\frac{c_5m}{(1-c_5)M} \\
&\ge \int_0^\omega G_1(t,s)(f(s+\tau, x(s)) - c(s+\tau)a(s+\tau)x(s))ds + c_5\frac{c_5m}{(1-c_5)M} \\
&\ge \int_0^\omega G_1(t,s)c_5m ds + c_5\frac{c_5m}{(1-c_5)M} \\
&\ge c_5m\frac{1}{M} + c_5\frac{c_5m}{(1-c_5)M} = \frac{c_5m}{(1-c_5)M}.
\end{aligned} \tag{15}$$

Combining (14) and (15), we obtain $A_2x + B_2y \in \mathbb{D}_3, \forall x, y \in \mathbb{D}_3$. For $\varphi, \psi \in \mathbb{D}_3$, we have

$$
\begin{aligned}
&|(B_2\varphi)(t) - (B_2\psi)(t)| \\
&= |c(t)\varphi(t-\tau) - c(t)\psi(t-\tau)| \\
&\le c_6\,|\varphi(t-\tau) - \psi(t-\tau)| \\
&\le c_6\,\|\varphi - \psi\|,
\end{aligned}
$$

which implies that $\|B_2\varphi - B_2\psi\| \le c_6\|\varphi - \psi\|$. Since $0 < c_6 < 1$, B_2 is a contraction on $\mathbb{D}_3$. From Lemma 4, we know that P_1 is completely continuous, so is A_2. By Lemma 7 we obtain that $A_2 + B_2$ has a fixed point $x \in \mathbb{D}_3$, i.e. (1) has a positive ω-periodic solution $x \in \mathbb{D}_3$. ■

Corollary 7 *Assume that the hypotheses of Theorem 5 hold. Then (2) has at least one positive ω-periodic solution x in the subset $\mathbb{D}_3$ of $\mathcal{B}$.*

Theorem 6 *Suppose that $\sqrt{3}\rho\omega < 4\pi/3$, $-1 < c_7 \le c(t) \le c_8 < 0$ and*

$$
-c_7 M < f(t,x) - c(t)a(t)x \le m,\ \forall (t,x) \in [0,\omega]\times[0,1]. \tag{16}
$$

Then (1) has at least one positive ω-periodic solution x in the subset $\widetilde{\mathbb{D}}_4$ of $\mathcal{B}$, where $\widetilde{\mathbb{D}}_4 = \{\varphi \in C_\omega : 0 < \varphi \le 1\}$.

Proof. Let $\mathbb{D}_4 = \{\varphi \in C_\omega : 0 \le \varphi \le 1\}$. Now we prove that $A_2x + B_2y \in \mathbb{D}_4$, $\forall x, y \in \mathbb{D}_4$. By Corollary 1, Lemma 4 and the condition (16) we obtain

$$
\begin{aligned}
&(A_2x)(t) + (B_2y)(t) \\
&= P_1(c(t)a(t)x(t-\tau) - f(t, x(t-\tau))) + c(t)y(t-\tau) \\
&\le \frac{M}{m} T_1(c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))) \\
&\le \frac{M}{m}\max_{t\in[0,\omega]}\left|\int_0^\omega G_1(t,s)(f(s+\tau,x(s)) - c(s+\tau)a(s+\tau)x(s))ds\right| \\
&\le \frac{M}{m}\int_0^\omega G_1(t,s)m\,ds \\
&\le \frac{M}{m} m \frac{1}{M} = 1.
\end{aligned} \tag{17}
$$

On the other hand

$$
\begin{aligned}
&(A_2x)(t) + (B_2y)(t) \\
&= P_1(c(t)a(t)x(t-\tau) - f(t, x(t-\tau))) + c(t)y(t-\tau) \\
&\geq T_1(c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))) + c_7 \\
&\geq \int_0^\omega G_1(t,s)(f(s+\tau, x(s)) - c(s+\tau)a(s+\tau)x(s))ds + c_7 \\
&\geq \int_0^\omega G_1(t,s)(-c_7M)ds + c_7 \\
&\geq (-c_7M)\frac{1}{M} + c_7 = 0. \qquad (18)
\end{aligned}
$$

Combining (17) and (18), we obtain $A_2x + B_2y \in \mathbb{D}_4,\ \forall x, y \in \mathbb{D}_4$. Obviously, $B_2\varphi$ is continuous and it is easy to show that $(B_2\varphi)(t+\omega) = (B_2\varphi)(t)$. So, for $\varphi, \psi \in \mathbb{D}_4$, we have

$$
\begin{aligned}
|(B_2\varphi)(t) - (B_2\psi)(t)| &= |c(t)\varphi(t-\tau) - c(t)\psi(t-\tau)| \\
&\leq -c_7 |\varphi(t-\tau) - \psi(t-\tau)| \\
&\leq -c_7 \|\varphi - \psi\|,
\end{aligned}
$$

which implies that $\|B_2\varphi - B_2\psi\| \leq -c_7 \|\varphi - \psi\|$. Since $0 < -c_7 < 1$, B_2 is a contraction on $\mathbb{D}_4$. From Lemma 4, we know that P_1 is completely continuous, so is A_2. By Lemma 7, we obtain that $A_2 + B_2$ has a fixed point $x \in \mathbb{D}_4$, i.e. (1) has a nonnegative ω-periodic solution x with $0 \leq x(t) \leq 1$. Since $f(t,x) - c(t)a(t)x > -c_7M$, it is easy to see that $x(t) > 0$, i.e. (1) has positive ω-periodic solution $x \in \widetilde{\mathbb{D}}_4$. ■

Corollary 8 *Assume that the hypotheses of Theorem 6 hold. Then (2) has at least one positive ω-periodic solution x in the subset $\widetilde{\mathbb{D}}_4$ of $\mathcal{B}$.*

Theorem 7 ([30]) *If $\sqrt{3}\rho\omega < 4\pi/3$ holds, $c(t) = 0$ and*

$$
0 < f(t,x) \leq M,\ \forall (t,x) \in [0,\omega] \times \left[0, \frac{M}{m}\right].
$$

Then (1) has at least one positive ω-periodic solution x with $0 < x(t) \leq \frac{M}{m}$.

Remark 1 *In a similar way of Theorem 7 we can prove that the (2) has at least one positive ω-periodic solution x when $c(t) = 0$.*

Example 1 *Consider the third-order nonlinear neutral differential equation*

$$\left(x(t) - \left(2 + \sin^2 t + \frac{1}{0.9 + 8\sin^2 t}\right) x(t - 6\pi)\right)'''$$
$$= \frac{1}{10^3}(1 - \frac{1}{10^2}\sin^2 t)x(t) - \frac{1}{10^4}(6 + \sin t) - \frac{1}{10^3}\exp(\cos(x(t - 6\pi))). \quad (19)$$

Note that (19) of the form (1) with $\omega = 2\pi$, $c(t) = 2 + \sin^2 t + \frac{1}{0.9+8\sin^2 t}$, $a(t) = \frac{1}{10^3}(1 - \frac{1}{10^2}\sin^2 t)$, $f(t, x(t-6\pi)) = \frac{1}{10^4}(6+\sin t) + \frac{1}{10^3}\exp(\cos(x(t-6\pi)))$, *and* $\tau = 6\pi$. *It is easy to verify that the conditions of Theorem 3 are satisfied with* $m = \frac{99}{10^5}$ *and* $M = \frac{1}{10^3}$. *Thus (19) has at least one positive* ω*-periodic solution.*

Example 2 *Consider the third-order nonlinear neutral differential equation*

$$\left(x(t) + (3 + \frac{\sin t}{10})x(t - 4\pi)\right)'''$$
$$= -\frac{1}{10^3}(1 - \frac{1}{2}\sin^2 t)x(t) + \frac{1}{10^4}(2 + \sin t) + \frac{1}{10^3}\sin(x(t - 4\pi)). \quad (20)$$

Note that (20) of the form (2) with $\omega = 2\pi$, $c(t) = -(3 + \frac{\sin t}{10})$, $a(t) = \frac{1}{10^3}(1 - \frac{1}{2}\sin^2 t)$, $f(t, x(t - 4\pi)) = \frac{1}{10^4}(2 + \sin t) + \frac{1}{10^3}\sin(x(t - 4\pi))$ *and* $\tau = 4\pi$. *It is easy to verify that the conditions of Corollary 6 are satisfied with* $m = \frac{1}{2\times 10^3}$ *and* $M = \frac{1}{10^3}$. *Thus (20) has at least one positive* ω*-periodic solution.*

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