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A New Approach to Solving SIEs and System of PFDEs Using The $\mathcal{L}_2$-Transform

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Abstract

In this article,we will show how we can apply complex inversion formula for the inversion of the $\mathcal{L}_2$-transform and also express some applications of the $\mathcal{L}_2$-transform for solving of singular integral equation (SIEs) with trigonometric kernel and system of partial fractional differential equations(PFDEs).

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1 Introduction

The Laplace-type integral transform called the $\mathcal{L}_2$-transform was introduced by Yurekli and Sadek [9] and denoted as follows

$$\mathcal{L}_2\{f(t);s\} = \int_0^\infty te^{-s^2t^2}f(t)dt. \tag{1-1}$$

Many problems of mathematical interest lead to the $\mathcal{L}_2$-transform whose inverses are not readily expressed in terms of tabulated functions. In the absence of methods for inversion of the $\mathcal{L}_2$-transform, recently the authors [1-4], established a simple formula to invert the $\mathcal{L}_2$-transform for a given function. In this article,we present some new inversion techniques for the $\mathcal{L}_2$-transform and an application of generalized product theorem (Efros Theorem) for solving some singular integral equation with trigonometric kernel is given. At the end, we express how we may choose the $\mathcal{L}_2$-transform in the scope of integral transform methods as a powerful tool for solving of a system of partial fractional differential equation in the Riemann-Liouville sense.

2 Elementary Properties of the $\mathcal{L}_2$-Transform

In this section, we recall some properties of the $\mathcal{L}_2$-transform that will be used to solve partial differential equations.First,we state a lemma about the $\mathcal{L}_2$-transform of the δ-derivatives.

Lemma 2.1 *If $f, f', \cdots, f^{(n-1)}$ are all continuous functions with a piecewise continuous derivative $f^{(n)}$ on the interval $t \geq 0$ and if all functions are of exponential order* $\exp(c^2t^2)$ *as $t \to \infty$ for some constant c then for $n = 1, 2, \cdots$*

$$\mathcal{L}_2\{\delta_t^n f(t); s\} = 2^n s^{2n} \mathcal{L}_2\{f(t); s\} - 2^{n-1} s^{2(n-1)} f(0^+) \\ -2^{n-2} s^{2(n-2)} (\delta_t f)(0^+) - \cdots - (\delta_t^{n-1} f)(0^+) \,. \tag{2-1}$$

where the differential operator δ is defined as

$$\delta_t = \frac{1}{t}\frac{d}{dt}, \quad \delta_t^2 = \delta_t \delta_t = \frac{1}{t^2}\frac{d^2}{dt^2} - \frac{1}{t^3}\frac{d}{dt},$$

and so on.

Proof: See [9].

3 Complex Inversion Formula for the $\mathcal{L}_2$-Transform and Efros Theorem

Lemma 3.1 *Let $F(\sqrt{s})$ be analytic function of s (assuming that $s = 0$ is not a branch point) except at finite number of poles each of which lies to the left of the vertical line Re $s = c$ and if $F(\sqrt{s}) \to 0$ as $s \to \infty$ through the left plane Re $s \leq c$, and*

$$\mathcal{L}_2\{f(t); s\} = \int_0^\infty t \exp(-s^2t^2) f(t) dt = F(s),$$

then

$$\mathcal{L}_2^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2F(\sqrt{s}) e^{st^2} ds = \sum_{k=1}^{m} [Res\{2F(\sqrt{s})e^{st^2}\}, s = s_k] \,. \tag{3-1}$$

Proof: See [2].

Lemma 3.2 ***(The Convolution Theorem for The $\mathcal{L}_2$-Transform)***

If $F(s), G(s)$ be the $\mathcal{L}_2$-transform of the functions $f(t), g(t)$ respectively, then

$$F(s)G(s) = \mathcal{L}_2\{f *_t g\} = \mathcal{L}_2\{\int_0^t x g(x) f(\sqrt{t^2 - x^2}) dx\} \tag{3-2}$$

Proof: Using the definition of the $\mathcal{L}_2$-transform for $F(s), G(s)$, we have

$$
\begin{aligned}
F(s)G(s) &= (\int_0^\infty ye^{-s^2y^2}f(y)dy)(\int_0^\infty xe^{-s^2x^2}g(x)dx) \\
&= \int_0^\infty \int_0^\infty yxe^{-s^2(x^2+y^2)}f(y)g(x)dydx
\end{aligned}
$$

now, by substitution $x^2+y^2=t^2$ and change the order of integration in the double integral for the first case ,we yield

$$
\begin{aligned}
F(s)G(s) &= \int_0^\infty te^{-s^2t^2}dt\int_0^t xg(x)f(\sqrt{t^2-x^2})dx \\
&= \mathcal{L}_2\{\int_0^t xg(x)f(\sqrt{t^2-x^2})dx\}.
\end{aligned}
$$

Lemma 3.3 ***[Generalized product Theorem (Efros Theorem)]***
Let $\mathcal{L}_2(f(t)) = F(s)$ and assuming $\Phi(s)$, $q(s)$ are analytic and such that, $\mathcal{L}_2(\Phi(t,\tau)) = \Phi(s)\tau e^{-\tau^2q^2(s)}$, then one has,

$$
\mathcal{L}_2\left\{\int_0^\infty f(\tau)\Phi(t,\tau)d\tau\right\} = F(q(s))\Phi(s). \tag{3-3}
$$

Proof: By definition of the $\mathcal{L}_2$ -transform and changing the order of integration we get

$$
\begin{aligned}
\mathcal{L}_2\{\int_0^\infty f(\tau)\phi(t,\tau)d\tau\} &= \int_0^\infty te^{-s^2t^2}(\int_0^\infty f(\tau)\phi(t,\tau)d\tau)dt \\
&= \int_0^\infty f(\tau)dt(\int_0^\infty te^{-s^2t^2}\phi(t,\tau)d\tau)d\tau,
\end{aligned}
$$

or equivalently

$$
\mathcal{L}_2\{\int_0^\infty f(\tau)\phi(t,\tau)d\tau\} = \Phi(s)\int_0^\infty f(\tau)\tau e^{-\tau^2q^2(s)}d\tau = \Phi(s)F(q(s))\ .
$$

Example 3.1: Solve the singular integral equation

$$
\frac{2}{\pi}\int_0^\infty f(\tau)\sin(t\tau)d\tau = \operatorname{Erf}(\frac{t}{2a}) \qquad a\in\mathbb{R}
$$

Solution: By applying the $\mathcal{L}_2$-transform followed by generalized product theorem and using the fact that $\mathcal{L}_2\{\sin(t\tau)\} = \frac{\pi}{4s^3}te^{-\frac{t^2}{4s^2}}$,we obtain

$$
\frac{2}{\pi}F(\frac{1}{2s})\frac{\sqrt{\pi}}{4s^3} = \frac{1}{2s^2\sqrt{1+a^2s^2}}
$$

or,

$$
F(s) = \frac{1}{2}\sqrt{\frac{\pi}{a^2+s^2}},
$$

and finally by inversion of the $\mathcal{L}_2$-transform we find

$$
f(t) = \frac{e^{-a^2t^2}}{t}.
$$

In the next section, we give some illustrative examples and lemmas related to the $\mathcal{L}_2$-transform and complex inversion formula for the $\mathcal{L}_2$-transform.

4 Illustrative Lemmas and Examples

Lemma 4.1 *By using generalized product theorem for the* $\mathcal{L}_2$*-transform,show that*

$$\int_0^\infty \operatorname{Erfc}(\frac{\tau^2}{2x})d\tau = \frac{\sqrt{\pi}}{2\Gamma(\frac{5}{4})}x^{\frac{1}{4}} \tag{4-1}$$

where Erfc *is complementary error function.*

Proof: By applying the $\mathcal{L}_2$-transform on the right hand side of above relation and using the fact that $\mathcal{L}_2\{\operatorname{Erfc}(\frac{\tau^2}{2x})\} = \frac{e^{-\tau^2 s}}{2s^2}$, we get

$$\mathcal{L}_2\{\int_0^\infty \operatorname{Erfc}(\frac{\tau^2}{2x})d\tau\} = \mathcal{L}_2\{\int_0^\infty \tau\operatorname{Erfc}(\frac{\tau^2}{2x})\frac{1}{\tau}d\tau\} = \frac{1}{2s^2}\left(\mathcal{L}_2\{\frac{1}{\tau}\}\right)_{s\to\sqrt{s}} = \frac{\sqrt{\pi}}{4s^{\frac{5}{4}}}$$

which, by using complex inversion formula for $\frac{\sqrt{\pi}}{4s^{\frac{5}{4}}}$ we obtain

$$\mathcal{L}_2^{-1}\{\frac{\sqrt{\pi}}{4s^{\frac{5}{4}}}\} = \frac{\sqrt{\pi}}{2\Gamma(\frac{5}{4})}x^{\frac{1}{4}}$$

Lemma 4.2 *The following relationship holds true.*

$$\mathcal{P}\{\frac{\ln x}{x}; y\} = \frac{\pi \ln y}{y}, \tag{4-2}$$

where $\mathcal{P}$ *is Widder potential transform. It is well-known that second iterate of the* $\mathcal{L}_2$*-transform is Widder potential transform [9].*

$$\mathcal{L}_2\{\mathcal{L}_2\{f(x); u\}; y\} = \frac{1}{2}\mathcal{P}\{f(x); y\} = \frac{1}{2}\int_0^\infty \frac{x}{x^2+y^2}f(x)dx \tag{4-3}$$

Proof: We take the function $f(z) = \frac{\log z}{z^2+a^2}$ where $z = re^{i\theta}$, $\theta \in (-\frac{\pi}{2}, \frac{3\pi}{2}]$ and consider the closed path Γ of integration consists of, two segments $[-R, -\epsilon]$, $[\epsilon, R]$,of the x-axis together with the upper semi-circles $C_\epsilon : z = \epsilon e^{i\theta}$ and $C_R : z = re^{i\theta}$ with $0 < \theta < \pi$.Also, we consider the branch of $\log z$ which is analytic on Γ and its interior, hence, so is $f(z)$. By residue theorem one has,

$$\oint_\Gamma f(z)dz = 2\pi i Res\{f(z), yi\} = \frac{\pi \ln y}{y} + \frac{i\pi^2}{2y},$$

or,

$$\int_{-R}^{-\epsilon} f(z)dz + \int_{-C_\epsilon} f(z)dz + \int_\epsilon^R f(z)dz = \frac{\pi \ln y}{y} + \frac{i\pi^2}{2y}.$$

If we take the limit as $\epsilon \to 0$,$R \to +\infty$, integrals along two semi-circles tend to zero, therefore we get

$$P.V.\{\int_{-\infty}^0 \frac{\ln|x| + i\pi}{x^2+y^2}dx + \int_0^{+\infty} \frac{\ln x}{x^2+y^2}dx\} = \frac{\pi \ln y}{y} + \frac{i\pi^2}{2y}.$$

Taking the real part of the two sides, yields

$$\int_0^{+\infty} \frac{\ln x}{x^2+y^2}dx = \frac{\pi \ln y}{y} = \mathcal{P}\{\frac{\ln x}{x}; y\}$$

Example 4.1: By using the above lemma solve the following integral equation with trigonometric kernel.

$$\int_0^\infty \phi(x)\sin\lambda x dx + \gamma = -\ln\lambda.$$

where γ is Euler's constant.
Solution: Taking the Laplace transform of both sides of integral equation with respect to λ ,we get

$$\mathcal{L}\{\int_0^\infty \phi(x)\sin\lambda x dx + \gamma\} = -\mathcal{L}\{\ln\lambda\},$$

or,equivalently

$$\int_0^\infty \frac{x}{x^2+s^2}\phi(x)dx + \frac{\gamma}{s} = \frac{\ln s+\gamma}{s},$$

the left hand side of the above relation is Widder potential transform of $\phi(x)$ [5].Comparing the above integral with lemma 4.2 ,we obtain

$$\mathcal{P}\{\phi(x);s\} = \int_0^\infty \frac{x}{x^2+s^2}\phi(x)dx = \frac{\ln s}{s},$$

or,

$$\phi(x) = \frac{\ln x}{\pi x}.$$

Lemma 4.3 *The following relation holds true*

$$\mathcal{L}_2^{-1}\{\frac{e^{-x\sqrt{s^2+\lambda^2}}}{2s^2(s^2-\mu)}\} = \frac{1}{\mu}\int_{\frac{1}{2}\frac{x}{\sqrt{t+t^2}}}^\infty e^{-\eta^2-\frac{(\lambda^2+1)x^2}{4\eta^2}+\mu(t+t^2)}d\eta - \frac{1}{\mu}\int_{\frac{1}{2}\frac{x}{\sqrt{t+t^2}}}^\infty e^{-\eta^2-\frac{\lambda^2x^2}{4\eta^2}}d\eta. \qquad (4\text{-}4)$$

Proof: By defining $F(s) = \frac{e^{-x\sqrt{s^2+\lambda^2}}}{2s^2(s^2-\mu)}$ we have $2F(\sqrt{s}) = \frac{e^{-x\sqrt{s+\lambda^2}}}{s(s-\mu)}$. In order to avoid complex integration along complicated key-hole contour we use integral representation for e^{-r} as follows

$$e^{-r} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\eta^2-\frac{r^2}{4\eta^2}}d\eta,$$

if we substitute $r = x\sqrt{s+\lambda^2}$ in the above integral then

$$\begin{aligned} f(x,t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-x\sqrt{s+\lambda^2}}}{s(s-\mu)}e^{st^2}ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{s(s-\mu)}\left(\int_0^\infty e^{-\eta^2-\frac{x^2(s+\lambda^2)}{4\eta^2}}d\eta\right)e^{st^2}ds \\ &= \int_0^\infty e^{-\eta^2-\frac{\lambda^2x^2}{4\eta^2}}d\eta\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-(\frac{x^2}{4\eta^2}-t^2)s}}{s(s-\mu)}ds\right). \end{aligned}$$

By using the fact that, $\mathcal{L}^{-1}\{\frac{e^{-as}}{s}\} = H(t-a), \mathcal{L}^{-1}\{\frac{e^{-as}}{s-\mu}\} = e^{\mu(t-a)}H(t-a)$ and setting $a = \frac{x^2}{4\eta^2} - t^2$ for inner integral ,we finally get

$$\begin{aligned} f(x,t) &= \frac{1}{\mu}\int_0^\infty e^{-\eta^2-\frac{\lambda^2x^2}{4\eta^2}}[e^{\mu(t-\frac{x^2}{4\eta^2}+t^2)}-1]H(t-\frac{x^2}{4\eta^2}+t^2)d\eta \\ &= \frac{1}{\mu}\int_{\frac{1}{2}\frac{x}{\sqrt{t+t^2}}}^\infty e^{-\eta^2-\frac{(\lambda^2+1)x^2}{4\eta^2}+\mu(t+t^2)}d\eta - \frac{1}{\mu}\int_{\frac{1}{2}\frac{x}{\sqrt{t+t^2}}}^\infty e^{-\eta^2-\frac{\lambda^2x^2}{4\eta^2}}d\eta. \end{aligned}$$

In the next section,we use the $\mathcal{L}_2$-transform for solving a system of partial fractional differential equation in the Riemann-Liouville sense which introduced by Giona and Roman [6, 8]on initial-value problems in fractals.

5 System of Partial Fractional Differential Equation

Example 5.1: Using the $\mathcal{L}_2$-transform solve the following system of partial fractional differential equation in the Riemann-Liouville sense

$$\begin{cases} {}_tD^{\alpha}_{0^+}u_1(x,t) + Cx^{-1}\frac{\partial u_1(x,t)}{\partial x} + ku_2(x,t) = c_1 \\ {}_tD^{\alpha}_{0^+}u_2(x,t) + Cx^{-1}\frac{\partial u_2(x,t)}{\partial x} - ku_1(x,t) = c_2 \end{cases}, \qquad k, c_1, c_2 \in \mathbb{R}, C > 0,\ 0 < \alpha \leq \frac{1}{2} \tag{5-1}$$

with Cauchy type initial and boundary conditions as

$${}_tD^{\alpha-1}_{0^+}u_1(x,0^+) = f(x), {}_tD^{\alpha-1}_{0^+}u_2(x,0^+) = 0 \quad u_1(0,t) = u_2(0,t) = 0, \quad x,t \in \mathbb{R}^+. \tag{5-2}$$

Solution: By introducing new variable $w = u_1 + iu_2, \eta = c_2 + c_1 i$, we can rewrite the system of partial fractional differential equation (5-1) and (5-2) in the form

$${}_tD^{\alpha}_{0^+}w(x,t) = -Cx^{-1}\frac{\partial w(x,t)}{\partial x} + kiw(x,t) + \eta \tag{5-3}$$

with initial and boundary conditions

$${}_tD^{\alpha-1}_{0^+}w(x,0^+) = f(x), \quad w(0,t) = 0 \tag{5-4}$$

At this point, by applying the $\mathcal{L}_2$-transform in x and the Laplace transform in t

$$\begin{array}{l} \mathcal{L}\{w(x,t);s\} = \tilde{w}(x,s) = \int_0^{\infty} e^{-st}w(x,t)dt, \quad \Re s > 0 \\ \mathcal{L}_2\{w(x,t);p\} = \hat{w}(p,t) = \int_0^{\infty} xe^{-p^2x^2}w(x,t)dx,\ \Re p^2 > 0. \end{array}$$

and utilizing the Cauchy type initial conditions (5-4), we get the transformed equation of (5-3) in the form

$$\hat{\tilde{w}}(p,s) = \frac{\eta}{2sp^2(s^{\alpha} + Cp^2 - ki)} + \frac{1}{s^{\alpha} + Cp^2 - ki}F(p) \tag{5-5}$$

where $F(p)$ is the $\mathcal{L}_2$-transform of the initial condition $f(x)$. To obtain the relation (5-5) we used the Laplace transform of the Riemann-Liouville derivative by the following relation [7]

$$\mathcal{L}\{{}_tD^{\alpha}_{0^+}w(x,t);s\} = s^{\alpha}\tilde{w}(x,s) - {}_tD^{\alpha-1}_{0^+}w(x,0^+),$$

and the $\mathcal{L}_2$-transform of the δ-derivatives by setting $n = 1$ in (2-1).

$$\mathcal{L}_2\{\frac{1}{x}\frac{\partial w(x,t)}{\partial x};p\} = 2p^2\hat{w}(p,t) - w(0,t).$$

Now, by considering the complex inversion formula for the $\mathcal{L}_2$-transform (3-1) and the convolution theorem (3-2), we attain

$$\begin{aligned}\tilde{w}(x,s) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{\tilde{w}}(\sqrt{p},s)e^{px^2}dp = \frac{\eta}{C}\mathcal{L}_2^{-1}\{\frac{1}{sp(\frac{s^\alpha}{C}+p-\frac{ki}{C})}\} + \frac{1}{C}\mathcal{L}_2^{-1}\{\frac{1}{\frac{s^\alpha}{C}+p-\frac{ki}{C}}\}*_x f(x)\\ &= \frac{\eta}{C}\int_0^x \frac{u}{s}e^{(ki-s^\alpha)\frac{u^2}{C}}du + \frac{1}{C}e^{(ki-s^\alpha)\frac{x^2}{C}}*_x f(x)\end{aligned}$$

where $*_x$ is the convolution of the $\mathcal{L}_2$-transform expressed by (3-2) .

At this point, in regard to the inverse Laplace transform of the functions $e^{-s^\alpha\frac{x^2}{C}}$ via the Wright function [7]

$$\mathcal{L}^{-1}\{e^{-s^\alpha\frac{x^2}{C}}\} = \frac{1}{t}W(-\alpha,0;-\frac{x^2}{C}t^{-\alpha}),$$

we obtain the explicit solution of the Cauchy type problem (5-3)-(5-4) in the following form

$$w(x,t) = \frac{\eta}{C}\int_0^x\int_0^t \frac{u}{\tau}W(-\alpha,0;-\frac{u^2}{C}\tau^{-\alpha})e^{\frac{kiu^2}{C}}d\tau du + \int_0^x \tau G^\alpha(x^2-\tau^2,t)f(\tau)d\tau, \tag{5-6}$$

where the Green function G^α is given by

$$G^\alpha(x,t) = \frac{1}{Ct}W(-\alpha,0;-\frac{x}{C}t^{-\alpha})e^{\frac{kxi}{C}}, \tag{5-7}$$

and the Wright function is presented by

$$W(\alpha,\beta;z) = \sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\alpha k+\beta)}, \quad \alpha > -1,\ \beta\in\mathbb{C},\ z\in\mathbb{C}.$$

Finally , by taking the real and imaginary part of the relation (5-6) the solution of the system (5-1) can be determined as

$$u_1(x,t) = \frac{1}{C}\int_0^x\int_0^t \frac{u}{\tau}W(-\alpha,0;-\frac{u^2}{C}\tau^{-\alpha})(c_2\cos(\frac{ku^2}{C})-c_1\sin(\frac{ku^2}{C}))d\tau du + \int_0^x \tau G_1^\alpha(x^2-\tau^2,t)f(\tau)d\tau \tag{5-8}$$

$$u_2(x,t) = \frac{1}{C}\int_0^x\int_0^t \frac{u}{\tau}W(-\alpha,0;-\frac{u^2}{C}\tau^{-\alpha})(c_1\cos(\frac{ku^2}{C})+c_2\sin(\frac{ku^2}{C}))d\tau du + \int_0^x \tau G_2^\alpha(x^2-\tau^2,t)f(\tau)d\tau \tag{5-9}$$

where

$$G_1^\alpha(x,t) = \frac{1}{Ct}W(-\alpha,0;-\frac{x}{C}t^{-\alpha})\cos\frac{kx}{C}, \tag{5-10}$$

$$G_2^\alpha(x,t) = \frac{1}{Ct}W(-\alpha,0;-\frac{x}{C}t^{-\alpha})\sin\frac{kx}{C}. \tag{5-11}$$

6 Conclusion

As concluding remarks,we have presented a general theoretical scheme for some singular integral equations and partial fractional differential equations using the $\mathcal{L}_2$-transform and it is hoped that these results and others derived from this are useful to researchers in the various branches of the integral transforms methods and applied mathematics.

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