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Delay differential equations

New Conditions on the Solutions of a Certain Third Order Delay Differential Equations with Multiple Deviating Arguments

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Abstract

In this work a class of third order delay differential equations (where the nonlinear functions, especially the first two restoring terms, are sum of multiple deviating arguments and the forcing term simultaneously depend explicitly on the independent variable t for all i, the last restoring term has variable coefficient and deviating arguments $\tau_i(t)$ vary for all i) is considered.

By employing the direct technique of Lyapunov, where a complete Lyapunov functional is constructed and used, we obtain sufficient conditions that guarantee the existence of solutions which are periodic, uniformly asymptotically stable, uniformly ultimately bounded. The behaviour of solutions as t tends to infinity is studied. The obtained results are new and include many recent results in the literature. Finally, two examples are given to show the feasibility of our results.

Keywords: Third order, nonlinear differential equation, uniform stability, uniform ultimate boundedness, periodic solutions

1 Introduction

Functional or delay differential equation is one of the major branches of differential equations because it generalizes most of the classical equations such as ordinary differential equations, difference differential equations, integral equations and integro-differential equations. Accordingly, much attention has been devoted to the study of delay differential equations. See for instance Burton [16, 17], Driver [18], Hale [20], Lakshmikantham *et al.* [21], Yoshizawa [41, 42, 43] which contain basic background to the study. In a wide variety of problems involving non linear delay differential equations of higher order, it is essential to establish conditions on the solutions that are periodic, stable and bounded. Third order nonlinear differential equations with and without delay have attracted and still attracting the attention of many researchers because of their importance in real world situations. We can mention for instance the papers of Ademola *et al.* [1]-[13], Adesina [14], Afuwape and Omeike [15], Olutimo and Adams [22], Graef and Tunç [19],Omeike [23, 24], Remili *et al.* [25, 26], Tunç [28]-[36], Tunç and Ergören [37], Tunç and Gözen [38], Tunç and Mohammed [39], Yao and Wang [40], Zhu [44]. In this regard, these cited papers and their references contain outstanding results on the qualitative behaviour of solutions of considered differential equations.

In an interesting contribution, Tunç [33] discussed the asymptotic stability and boundedness of solutions to a kind of nonlinear third order differential equations with retarded arguments:

$$\begin{aligned}&x'''(t) + h(x(t), x'(t), x''(t), x(t-\tau(t)), x'(t-\tau(t)), x''(t-\tau(t)))x''\\&+ g(x(t-\tau(t)), x'(t-\tau(t))) + f(x(t-\tau(t)))\\&= p(t, x(t), x'(t), x(t-\tau(t)), x'(t-\tau(t)), x''(t))).\end{aligned}$$

Tunç and Ergören [37] studied the third order delay equations with multiple deviating arguments

$$x'''(t) + f_1(t, x(t))x''(t) + f_2(t, x(t))x' + g_0(t, x(t)) + \sum_{i=1}^{n} g_i(t, x(t-\tau_i(t))) = p(t)$$

and obtained conditions that guarantee the existence of solutions that are uniformly bounded.
Another contribution worthy of mentioning is the work of Remili and Oudjedi [25] which dwells on the stability and boundedness of the solutions of non autonomous third order delay differential equation of form

$$[g(x(t))x'(t)]'' + a(t)x''(t) + b(t)x' + c(t)f(x(t-\tau)) = p(t),$$

where τ is a constant. Recently, Ademola *et al.* [12] established criteria for stability, boundedness and existence of periodic solutions to third order delay differential equations with multiple deviating arguments:

$$\begin{aligned}
&x'''(t) + \sum_{i=1}^{n} f_i(t, x(t), x(t-\tau_i(t)), x'(t), x'(t-\tau_i(t)), x''(t), x''(t-\tau_i(t))) \\
&+ \sum_{i=1}^{n} g_i(x'(t-\tau_i(t))) + \sum_{i=1}^{n} h_i(x(t-\tau_i(t))) \\
&= \sum_{i=1}^{n} p_i(t, x(t), x(t-\tau_i(t)), x'(t), x'(t-\tau_i(t)), x''(t), x''(t-\tau_i(t))).
\end{aligned}$$

However, the problem of uniform asymptotic stability, uniform ultimate boundedness and existence of a unique periodic solution for the non autonomous third order delay differential equation (1.1) (where the nonlinear functions are sum of multiple deviating arguments, the functions f_i, g_i, p_i simultaneously depend explicitly on the independent variable t for all i, h_i has variable coefficient and $\tau_i(t)$ varies for all i) is yet to be explored. Regardless of the difficulties associated with the construction of a suitable complete Lyapunov functional, the aim of this paper therefore is to fill these vacuums. In effect,we will discuss the equation

$$\begin{aligned}
&x'''(t) + \sum_{i=1}^{n} f_i(\cdot) + \sum_{i=1}^{n} g_i(t, x(t), x(t-\tau_i(t)), x'(t), x'(t-\tau_i(t))) \\
&+ \phi(t) \sum_{i=1}^{n} h_i(x(t-\tau_i(t))) = \sum_{i=1}^{n} p_i(\cdot),
\end{aligned} \tag{1.1}$$

where

$$f_i(\cdot) = f_i(t, x(t), x(t-\tau_i(t)), x'(t), x'(t-\tau_i(t)), x''(t), x''(t-\tau_i(t)))$$

and

$$p_i(\cdot) = p_i(t, x(t), x(t-\tau_i(t)), x'(t), x'(t-\tau_i(t)), x''(t), x''(t-\tau_i(t)))$$

with the equivalent system

$$
\begin{aligned}
x'(t) &= y(t),\\
y'(t) &= z(t)\\
z'(t) &= \sum_{i=1}^{n} p_i(t, x(t), x(t-\tau_i(t)), y(t), y(t-\tau_i(t)), z(t), z(t-\tau_i(t)))\\
&\quad - \sum_{i=1}^{n} f_i(t, x(t), x(t-\tau_i(t)), y(t), y(t-\tau_i(t)), z(t), z(t-\tau_i(t)))\\
&\quad - \sum_{i=1}^{n} g_i(t, x(t), x(t-\tau_i(t)), y(t), y(t-\tau_i(t))) - \phi(t)\sum_{i=1}^{n} h_i(x(t))\\
&\quad + \phi(t)\int_{t-\tau_i(t)}^{t}\sum_{i=1}^{n} h_i'(x(s))y(s)ds,
\end{aligned}
\tag{1.2}
$$

where the functions f_i, g_i, h_i, ϕ and p_i in their respective arguments on $\mathbb{R}^+ \times \mathbb{R}^{3n+3}$, $\mathbb{R}^+ \times \mathbb{R}^{2n+2}$, $\mathbb{R}^n$, $\mathbb{R}^+$ and $\mathbb{R}^+ \times \mathbb{R}^{3n+3}$ respectively with $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$, $n \in \mathbb{N}$ finite and $0 \leq \tau_i(t) \leq \gamma$, $\gamma > 0$ being a constant whose value will be established later. Continuity of these functions is sufficient for the existence of the solutions of equation (1.1). Furthermore, it is assumed that the functions f, g and p in the equation (1.1) satisfy a Lipschitz condition in their respective arguments. The primes stand for differentiation with respect to the independent variable t and the derivatives h_i' and ϕ' exist and continuous for all $t \geq 0, x$ with $h_i(0) = 0$.

This work is motivated by the recent works in [7, 12, 13, 22, 25, 26] and [38]. The obtained results are new and include many existing results in the literature. Despite the attention that has been given to the third order nonlinear differential equation with deviating arguments, we are yet to come across a situation where the unknown functions f_i, g_i and p_i depend explicitly on the independent variable t, the function h_i has variable coefficient and $\tau_i(t)$ varies for all i, $(i = 1, 2, \cdots, n)$. This is our principal contribution. In the next section, we give basic preliminary results used in this work. The main results of the paper are stated and proved in Section 3. Examples and discussions are given in Section 4 while Section 5 is dedicated to conclusion of this work.

2 Preliminary Results

Consider the following general nonlinear non-autonomous delay differential equation

$$\dot{X} = \frac{dX}{dt} = F(t, X_t), \quad X_t = X(t+\theta), \quad -r \leq \theta < 0, \quad t \geq 0, \tag{2.1}$$

where $F : \mathbb{R}^+ \times C_H \to \mathbb{R}^n$ is a continuous mapping, $F(t+\omega, \phi) = F(t, \phi)$ for all $\phi \in C$ and for some positive constant ω. We assume that F takes closed bounded sets into bounded sets in $\mathbb{R}^n$. $(C, \|\cdot\|)$ is the Banach space of continuous function $\varphi : [-r, 0] \to \mathbb{R}^n$ with supremum norm, $r > 0$; for $H > 0$, we define $C_H \subset C$ by $C_H = \{\varphi \in C : \|\varphi\| < H, \}$ C_H is the open $H-$ball in C, $C = C([-r, 0], \mathbb{R}^n)$.

Definition 2.1. (See [20]). Suppose that $F(t, 0) = 0$ for all $t \in \mathbb{R}$. The solution $X_t = 0$ is said to be *stable* for any $t_0 \in \mathbb{R}$, $\epsilon > 0$, there is a $\delta = \delta(\epsilon, t_0)$ such that $||\phi||_r < \delta$ implies $||X_t(t; t_0, \phi)|| < \epsilon$ for $t \geq t_0 - r$. The solution $X_t = 0$ of (2.1) is *uniformly stable* on (t_0, ∞) if it is stable at each t_0 and the number δ is independent of t_0 (i.e $\delta(\epsilon)$ depends only on ϵ).

Definition 2.2. (See [20]). The solution $X_t = 0$ of (2.1) is said to be *asymptotically stable* at t_0 if it is stable at t_0 then there exists number $\delta_1 = \delta_1(t_0) > 0$ such that whenever $||\phi||_r < \delta_1$,

$$\lim_{t \to \infty} X_t(t; t_0, \phi) = 0.$$

The solution $X_t = 0$ of (2.1) is *uniformly asymptotically stable* if it is uniformly stable and furthermore, there exits $\delta_1 > 0$ (independent of t_0) such that for each t_0 and $||\phi||_r < \delta_1$,

$$X_t(t; t_0, \phi) \to 0$$

as $t \to \infty$.

Definition 2.3. (See [21]). A solution of (2.1) is *bounded* if there is an $M = M(t_0, \phi) > 0$ such that

$$\|X_t(t; t_0, \phi)\| < M$$

for all $t \geq t_0 - r$.

Definition 2.4. (See [21]). The solutions of (2.1) are *uniformly bounded* if the M in Definition 2.3 is independent of t_0.

Definition 2.5. (See [21]). The solutions of (2.1) are *ultimately bounded for bound M*, if there exist an $M > 0$ and a $T = T(t_0, \phi) > 0$ such that for every solution $X_t(t_0, \phi)$ of (2.1)

$$\|X_t(t; t_0, \phi)\| < M$$

for all $t \geq t_0 + T$.

Definition 2.6. (See [21]). The solutions of (2.1) are *uniformly ultimately bounded for bound M*, if the T in Definition 2.5 is independent of t_0.

Definition 2.7. (See [17]). A continuous function $W : \mathbb{R}^+ \to \mathbb{R}^+$ with $W(0) = 0$, $W(s) > 0$ if $s \neq 0$, and W strictly increasing is a *wedge.* (We denote wedges by W or W_i, where i is an integer).

Lemma 2.1. (See [42]). Suppose that $F(t, \phi) \in \overline{C}_0(\phi)$ and $F(t, \phi)$ is periodic in t of period ω, $\omega \geq r$ and consequently for any $\alpha > 0$ there exists an $L(\alpha) > 0$ such that $\phi \in C_\alpha$ implies $|F(t, \phi)| \leq L(\alpha)$. Suppose that a continuous Lyapunov functional $V(t, \phi)$ exists, defined on $t \in \mathbb{R}^+$, $\phi \in S^*$, S^* is the set of $\phi \in C$ such that with $|\phi(0)| \geq H$ (H may be large) and that $V(t, \phi)$ satisfies the following conditions:

(i) $a(|\phi(0)|) \leq V(t, \phi) \leq b(\|\phi\|)$, where $a(r)$ and $b(r)$ are continuous, increasing and positive for $r \geq H$ and $a(r) \to \infty$ as $r \to \infty$;

(ii) $\dot{V}_{(2.1)}(t, \phi) \leq -c(|\phi(0)|)$, where $c(r)$ is continuous and positive for $r \geq H$.

Suppose that there exists an $H_1 > 0$, $H_1 > H$, such that

$$hL(\gamma^*) < H_1 - H, \tag{2.2}$$

where $\gamma^* > 0$ is a constant which is determined in the following way: By the condition on $V(t, \phi)$ there exist $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ such that $b(H_1) \leq a(\alpha)$, $b(\alpha) \leq a(\beta)$ and $b(\beta) \leq a(\gamma)$. γ^* is defined by $b(\gamma) \leq a(\gamma^*)$. Under the above conditions, there exists a periodic solution of (2.1) of period ω. In particular, the relation (2.2) can always be satisfied if h is sufficiently small.

Lemma 2.2. (See [42]). Suppose that $F(t, \phi)$ is defined and continuous on $0 \leq t \leq c$, $\phi \in C_H$ and that there exists a continuous Lyapunov functional $V(t, \phi, \varphi)$ defined on $0 \leq t \leq c$, $\phi, \varphi \in C_H$ which satisfy the following conditions:

(i) $V(t, \phi, \varphi) = 0$ if $\phi = \varphi$;

(ii) $V(t, \phi, \varphi) > 0$ if $\phi \neq \varphi$;

(iii) for the associated system

$$\dot{x}(t) = F(t, x_t), \qquad \dot{y}(t) = F(t, y_t) \tag{2.3}$$

we have $V'_{(2.3)}(t, \phi, \varphi) \le 0$, where for $\|\phi\| = H$ or $\|\varphi\| = H$, we understand that the condition $V'_{(2.3)}(t, \phi, \varphi) \le 0$ is satisfied in the case V' can be defined.

Then, for given initial value $\phi \in C_{H_1}$, $H_1 < H$, there exists a unique solution of (2.1).

Lemma 2.3. (See [42]). Suppose that a continuous Lyapunov functional $V(t, \phi)$ exists, defined on $t \in \mathbb{R}^+$, $\|\phi\| < H$, $0 < H_1 < H$ which satisfies the following conditions:

(i) $a(\|\phi\|) \le V(t, \phi) \le b(\|\phi\|)$, where $a(r)$ and $b(r)$ are continuous, increasing and positive,

(ii) $\dot{V}_{(2.1)}(t, \phi) \le -c(\|\phi\|)$, where $c(r)$ is continuous and positive for $r \ge 0$,

then the zero solution of (2.1) is uniformly asymptotically stable.

Lemma 2.4. (See [16]). Let $V : \mathbb{R}^+ \times C \to \mathbb{R}$ be continuous and locally Lipschitz in ϕ. If

(i) $W_0(|X_t|) \le V(t, X_t) \le W_1(|X_t|) + W_2\left(\int\limits_{t-r(t)}^{t} W_3(X_t(s))ds\right)$ and

(ii) $\dot{V}_{(2.1)}(t, X_t) \le -W_4(|X_t|) + N$, for some $N > 0$ where W_i $(i = 0, 1, 2, 3, 4)$ are wedges.

Then X_t of (2.1) is uniformly bounded and uniformly ultimately bounded for bound M.

3 Main Results

We shall give the following notations before we state our main results. Let $x(t) = x$, $y(t) = y$, $z(t) = z$, $\sum\limits_{i=1}^{n} f_i(t, x(t), x(t-\tau_i(t)), y(t), y(t-\tau_i(t)), z(t), z(t-\tau_i(t))) = \sum\limits_{i=1}^{n} f_i(\cdot)$, $\sum\limits_{i=1}^{n} g_i(t, x(t), x(t-\tau_i(t)), x'(t), x'(t-\tau_i(t))) = \sum\limits_{i=1}^{n} g_i(\cdot)$ and $\sum\limits_{i=1}^{n} p_i(t, x(t), x(t-\tau_i(t)), y(t), y(t-\tau_i(t)), z(t), z(t-\tau_i(t))) = \sum\limits_{i=1}^{n} p_i(\cdot)$. Let

(x_t, y_t, z_t) be any solution of system (1.2), the continuously differentiable functional employed in the proof of our results is $V = V(t, x_t, y_t, z_t)$ defined as follows

$$
\begin{aligned}
2V = {} & 2\phi(t)\Big(\alpha + \sum_{i=1}^{n} a_i\Big)\int_0^x \sum_{i=1}^{n} h_i(\xi)d\xi + 4\phi(t)y\sum_{i=1}^{n} h_i(x) + 2\Big(\alpha + \sum_{i=1}^{n} a_i\Big)yz \\
& + 2z^2 + \Big(\alpha^2 + \beta + \sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n} b_i\Big)y^2 + \beta\sum_{i=1}^{n} b_i x^2 + 2\beta\sum_{i=1}^{n} a_i xy + 2\beta xz \\
& + \int_{-\tau_i(t)}^{0}\int_{t+s}^{t}\Big[\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)\Big]d\theta ds,
\end{aligned}
\tag{3.1}
$$

where α and β are positive constants satisfying the following inequalities

$$
\phi_1\Big(\sum_{i=1}^{n} b_i\Big)^{-1}\sum_{i=1}^{n} c_i \le \alpha \le \sum_{i=1}^{n} a_i; \tag{3.2}
$$

$$
0 < \beta < \min\Big\{\sum_{i=1}^{n} b_i,\ \Big(\alpha\sum_{i=1}^{n} b_i - \phi_1\sum_{i=1}^{n} c_i\Big)A_1^{-1},\ \Big(\sum_{i=1}^{n} a_i - \alpha\Big)A_2^{-1}\Big\}; \tag{3.3}
$$

with

$$
A_1 := 2\Big[1 + \sum_{i=1}^{n} a_i + \phi_0^{-1}\Big(\sum_{i=1}^{n}\delta_i\Big)^{-1}\Big(\sum_{i=1}^{n}\frac{g_i(\cdot)}{y} - \sum_{i=1}^{n} b_i\Big)^2\Big];
$$

$$
A_2 := 4\Big[1 + \phi_0^{-1}\Big(\sum_{i=1}^{n}\delta_i\Big)^{-1}\Big(\sum_{i=1}^{n}\frac{f_i(\cdot)}{z} - \sum_{i=1}^{n} a_i\Big)^2\Big]
$$

and λ_1 and λ_2 are nonnegative constants which will be determined later.

Remark 3.1. *The following remarks hold for the continuously differentiable functional* (3.1)*:*

(i) if $\phi(t)h_i(x) = h_i(x)$ and $g_i(\cdot) = g_i(y)$ then the functional (3.1) *reduces to that used in [12];*

(ii) whenever $\alpha^2 = \alpha$, $\sum_{i=1}^{n} a_i = a$, the functional V in equation (3.1) *specializes to the functional V defined in [7]; and*

(iii) when $\alpha^2 = \alpha$, $\sum_{i=1}^{n} a_i = a$, $\sum_{i=1}^{n} b_i = b$, $\sum_{i=1}^{n} c_i = c$ and $\sum_{i=1}^{n} \delta_i = \delta$, the functional V in equation (3.1) *specializes to that used in [2] and [3].*

Thus the functional V defined in equation (3.1) is an improvement on the one used in [2, 3, 7] and [12].

In what follows we state the main results and give their proofs.

Theorem 3.1. *In addition to the basic assumptions on the functions f_i, g_i, h_i, ϕ and p_i*
$i = 1, 2, \cdots, n$, suppose that $a_i, b_i, c_i, \delta_i, \epsilon, \beta_0, M_i, \phi_0, \phi_1$ and γ are positive constants such that

(i) $\phi_0 \le \phi(t) \le \phi_1$ *and* $|\phi'(t)| < \epsilon$ *for all* $t \ge 0$;

(ii) $a_i \le f_i(\cdot)/z$ *for all* $t \ge 0, x, y, z \ne 0$, $x(t-\tau_i(t)), y(t-\tau_i(t))$ *and* $z(t-\tau_i(t))$;

(iii) $b_i \le g_i(\cdot)/y$ *for all* $t \ge 0, x, y \ne 0$, $x(t-\tau_i(t))$ *and* $y(t-\tau_i(t))$;

(iv) $h_i(0) = 0$, $\delta_i \le h_i(x)/x \le c_i$ *for all* $t \ge 0$ *and* $x \ne 0$, *and* $\phi_1 c_i \le a_i b_i$;

(v) $\tau_i(t) \le \gamma$, $\tau_i'(t) \le \beta_0$ $\beta_0 \in (0,1)$ *where*

$$\gamma < \min\left\{ \left(\phi_1 \sum_{i=1}^{n} c_i \right)^{-1} A_3,\ A_4 \cdot A_5,\ A_6 \right\} \tag{3.4}$$

where

$$A_3 = \beta\phi_0 \sum_{i=1}^{n} \delta_i - 2\epsilon \sum_{i=1}^{n} c_i \left(1 + \alpha + \sum_{i=1}^{n} a_i \right)$$

$$A_4 = \left(\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i - \phi_1 \sum_{i=1}^{n} c_i - 2\epsilon \sum_{i=1}^{n} c_i (1 + \alpha + \sum_{i=1}^{n} a_i) \right)$$

$$A_5 = \left\{ \left[\alpha + \sum_{i=1}^{n} a_i + (1-\beta_0)^{-1} (2 + \alpha + \beta + \sum_{i=1}^{n} a_i) \right] \phi_1 \sum_{i=1}^{n} c_i \right\}^{-1},$$

$$A_6 = \left[\sum_{i=1}^{n} a_i - \alpha \right] \left[4\phi_1 \sum_{i=1}^{n} c_i \right]^{-1}; \ \textit{and}$$

(vi) $|p_i(\cdot)| \le M_i$, *for all* $t \ge 0, x, y, z$, $x(t-\tau_i(t)), y(t-\tau_i(t))$ *and* $z(t-\tau_i(t))$.

Then the solution (x_t, y_t, z_t) of system (1.2) is uniformly bounded and uniformly ultimately bounded.

Remark 3.2. *We observed the following:*

(i) If $f_i(\cdot) = a\ddot{x}$, $g_i(\cdot) = b\dot{x}$, $\phi(t)h_i(x(t-\tau_i(t))) = cx$ *and* $p_i(\cdot) = 0$, *then the equation* (1.1) *reduces to a linear constant coefficients third order differential equation*

$$x''' + ax'' + bx' + cx = 0,$$

and hypotheses (i) to (vi) of Theorem 3.1 reduce to Routh Hurwitz criteria $a > 0, b > 0, ab > c$ *and* $c > 0$ *for asymptotic stability for third order differential equation.*

(ii) When $i = 1$, *we have* $f_1(\cdot) = h(\cdot)$, $g_1(\cdot) = g(\cdot)$ *and* $\phi(t)h_1(x(t-\tau_1(t))) = f(x(t-\tau(t)))$ *where* $h(\cdot)$, $g(\cdot)$ *and* $h(\cdot)$ *are independent of* t, *then the equation* (1.1) *becomes the case discussed in [31].*

(iii) Whenever $i = 1$, *we have* $f_1(\cdot) = \varphi(x, x')x''$, $g_1(\cdot) = g(x'(t-\tau(t)))$, $\phi(t)h_1(x(t-\tau_1(t))) = f(x(t-\tau(t)))$ *and* $p_i(\cdot) = p(t, x, x', x'')$ *with* $0 \le \tau_1 \le \gamma$, *then* (1.1) *is a special case discussed in [24].*

(iv) If $i = 1$, *we have* $f_1(\cdot) = a(t)x''$, $g_1(\cdot) = b(t)g_1(x'(t-\tau)) + g_2(x')$, $\phi(t)h_1(x(t-\tau_1(t))) = h(x(t-\tau))$ *and* $p_i(\cdot) = p(t, x, x((t-\tau)), x', x'((t-\tau)), x'')$ *then the equation* (1.1) *is a special case considered in [33].*

(v) whenever $i = 1$, *so that* $\tau_1(t) = \tau(t)$, $f_1(\cdot) = a(t)h(x(t), x'(t))x''$, $g_1(\cdot) = g(x'(t-\tau(t)))$, $\phi(t)h_1(x(t-\tau_1(t))) = c(t)f(x(t-\tau(t)))$ *and* $p_i(\cdot) = p(t, x, x((t-\tau)), x', x'((t-\tau)), x'')$ *then the equation* (1.1) *is a special case of the recent work discussed in [22].*

(vi) When the nonlinear functions g *and* h *are independent of the independent variable* t, *then the equation* (1.1) *is a case discussed in [12].*

Hence, Theorem 3.1 includes and extends the results in [2]-[15], [22]-[40] and the references cited therein.

Next, we will state and prove a result that would be useful in the proof of Theorem 3.1 and the succeeding ones.

Lemma 3.1. *Under the assumptions of Theorem 3.1, there exist constants* $D_0 = D_0(\alpha, \beta, a_i, b_i, c_i,$
$\delta_i, \phi_0, \phi_1) > 0$, $D_1 = D_1(\alpha, \beta, \phi_1, a_i, b_i, c_i) > 0$ *and* $D_2 = D_2(\lambda_1, \lambda_2) > 0$ *such that*

$$\begin{aligned} D_0(x^2(t) + y^2(t) + z^2(t)) \le V(t, x_t, y_t, z_t) &\le D_1(x^2(t) + y^2(t) + z^2(t)) \\ &+ D_2 \int_{\tau_i(t)}^{0} \int_{t+s}^{t} (y^2(\theta) + z^2(\theta)) d\theta ds \end{aligned} \tag{3.5}$$

for all $t \geq 0$, x, y *and* z. *In addition, there exist constants* $D_3 = D_3(\alpha, \beta, \beta_0, \gamma, \delta_i, \epsilon, \gamma, a_i, b_i, c_i, \phi_0,$
$\phi_1, M_i) > 0$ *and* $D_4 = D_4(2, \alpha, \beta, a_i, M_1) > 0$ *such that*

$$V'_{(1.2)} \leq -D_3(x^2(t) + y^2(t) + z^2(t)) + D_4, \tag{3.6}$$

for all $t \geq 0$, x, y *and* z.

Proof. Let (x_t, y_t, z_t) be any solution of system (1.2). Setting $x_t = y_t = z_t = 0$ in equation (3.1), we find that

$$V(t, 0, 0, 0) = 0$$

for all $t \geq 0$. Also, from hypothesis (iv) of Theorem 3.1, $h_i(0) = 0$, it follows that the functional V defined in the equation (3.1) can be written in the form

$$\begin{aligned} V = {} & \phi(t)\Big(\sum_{i=1}^n b_i\Big)^{-1} \int_0^x \Big[\Big(\alpha \sum_{i=1}^n b_i + \sum_{i=1}^n a_i \sum_{i=1}^n b_i\Big) \\ & - 2\phi(t) \sum_{i=1}^n h_i'(\xi)\Big] \sum_{i=1}^n h_i(\xi) d\xi + \Big(\sum_{i=1}^n b_i\Big)^{-1} \Big(\phi(t) \sum_{i=1}^n h_i(x) + \sum_{i=1}^n b_i y\Big)^2 \\ & + \frac{1}{2}\Big(\beta x + \sum_{i=1}^n a_i y + z\Big)^2 + \frac{1}{2}(\alpha y + z)^2 + \frac{1}{2}\beta y^2 + \frac{1}{2}\beta\Big(\sum_{i=1}^n b_i - \beta\Big) x^2 \\ & + \frac{1}{2} \int_{-\tau_i(t)}^0 \int_{t+s}^t \Big[\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)\Big] d\theta ds. \end{aligned} \tag{3.7}$$

Applying assumptions (i), (iv), the inequalities (3.2), (3.3) and the positivity of the double integrals in equation (3.7), there exists a positive constant $K_0 = K_0(\alpha, \beta, \delta, \phi_0, \phi_1, \delta_i, a_i, b_i, c_i)$ such that

$$V \geq K_0(x^2 + y^2 + z^2) \tag{3.8}$$

for all $t \geq 0$, x, y and z. We have the following conclusions emanating from inequality (3.8):

(i) Inequality (3.8) establishes the lower inequality in (3.5) with K_0 equivalent to D_0;

(ii) It is clear from the inequality (3.8) that

$$V(t, x_t, y_t, z_t) = 0 \ \text{ if and only if } \ x^2 + y^2 + z^2 = 0; \tag{3.9}$$

$$V(t, x_t, y_t, z_t) > 0 \ \text{ if and only if } \ x^2 + y^2 + z^2 \neq 0; \ \text{ and} \tag{3.10}$$

$$V(t, x_t, y_t, z_t) \to +\infty \ \text{ as } \ x^2 + y^2 + z^2 \to \infty. \tag{3.11}$$

Furthermore, $\phi(t) \leq \phi_1$ for all $t \geq 0$, $h_i(x)/x \leq c_i$ for all $x \neq 0$, and from the inequality $2mn \leq m^2 + n^2$ for all $m, n \in \mathbb{R}$, it follows from (3.1), the existence of positive constants K_1 and K_2 such that

$$V \leq K_1(x^2 + y^2 + z^2) + K_2 \int_{-\tau_i(t)}^{0} \int_{t+s}^{t} (y^2(\theta) + z^2(\theta)) d\theta ds \qquad (3.12)$$

for all $t \geq 0, x, y$ and z, where

$$\begin{aligned} K_1 :=& \frac{1}{2} \max \Bigg\{ 2\Big(1 + \alpha + \sum_{i=1}^{n} a_i\Big)\phi_1 \sum_{i=1}^{n} c_i + \beta\Big(1 + \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i\Big), \\ & 2\Big(\sum_{i=1}^{n} b_i + \phi_1 \sum_{i=1}^{n} c_i\Big) + \alpha(1+\alpha) + \Big(1 + \sum_{i=1}^{n} a_i\Big)\Big(\beta + \sum_{i=1}^{n} a_i\Big), \\ & 2 + \alpha + \beta + \sum_{i=1}^{n} a_i \Bigg\} \end{aligned}$$

and

$$K_2 := \max\{\lambda_1, \lambda_2\}.$$

Inequality (3.12) satisfies the upper estimate in (3.5) with K_1 and K_2 equivalent to D_1 and D_2 respectively. Combining estimates (3.8) and (3.12), inequality (3.5) hold.

Next, the derivative of the functional V defined in the equation (3.1), with respect to the independent variable t along the solution path of system (1.2) and simplify to give

$$\begin{aligned} V'_{(1.2)} = & -\sum_{j=1}^{5} W_j - [1 - \tau_i'(t)] \int_{t-\tau_i(t)}^{t} \Big[\lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)\Big] d\theta ds \\ & + (\lambda_1 y^2 + \lambda_2 z^2)\tau_i(t) + \sum_{j=6}^{7} W_j + \Big[\beta x + \Big(\alpha + \sum_{i=1}^{n} a_i\Big) y \\ & + 2z\Big] \sum_{i=1}^{n} p_i(\cdot) + \beta\Big[\sum_{i=1}^{n} a_i y^2 + 2yz\Big], \end{aligned} \qquad (3.13)$$

where

$$W_1 := \frac{1}{2}\beta\phi(t) \sum_{i=1}^{n} \frac{h_i(x)}{x} x^2 + \frac{1}{2}\Big[\Big(\alpha + \sum_{i=1}^{n} a_i\Big) \sum_{i=1}^{n} \frac{g_i(\cdot)}{y} - 2\phi(t) \sum_{i=1}^{n} h_i'(x)\Big] y^2$$

$$+\frac{1}{2}\Big[2\sum_{i=1}^{n}\frac{f_i(\cdot)}{z}-\Big(\alpha+\sum_{i=1}^{n}a_i\Big)\Big]z^2;$$

$$W_2 := \frac{1}{4}\beta\phi(t)\sum_{i=1}^{n}\frac{h_i(x)}{x}x^2+\beta\Big[\sum_{i=1}^{n}\frac{g_i(\cdot)}{y}-\sum_{i=1}^{n}b_i\Big]xy$$

$$W_3 := \frac{1}{4}\beta\phi(t)\sum_{i=1}^{n}\frac{h_i(x)}{x}x^2+\beta\Big[\sum_{i=1}^{n}\frac{f_i(\cdot)}{z}-\sum_{i=1}^{n}a_i\Big]xz;$$

$$W_4 := \frac{1}{4}\Big\{\Big[\Big(\alpha+\sum_{i=1}^{n}a_i\Big)\sum_{i=1}^{n}\frac{g_i(\cdot)}{y}-2\phi(t)\sum_{i=1}^{n}h_i'(x)\Big]y^2$$
$$+8\Big(\sum_{i=1}^{n}\frac{g_i(\cdot)}{y}-\sum_{i=1}^{n}b_i\Big)yz+\Big[2\sum_{i=1}^{n}\frac{f_i(\cdot)}{z}-\Big(\alpha+\sum_{i=1}^{n}a_i\Big)\Big]z^2\Big\}$$

$$W_5 := \frac{1}{4}\Big\{\Big[\Big(\alpha+\sum_{i=1}^{n}a_i\Big)\sum_{i=1}^{n}\frac{g_i(\cdot)}{y}-2\phi(t)\sum_{i=1}^{n}h_i'(x)\Big]y^2+4\Big[\Big(\alpha+\sum_{i=1}^{n}a_i\Big)\times$$
$$\sum_{i=1}^{n}\frac{f_i(\cdot)}{z}-\Big(\alpha^2+\sum_{i=1}^{n}a_i^2\Big)\Big]yz+\Big[2\sum_{i=1}^{n}\frac{f_i(\cdot)}{z}-\Big(\alpha+\sum_{i=1}^{n}a_i\Big)\Big]z^2\Big\}$$

$$W_6 := \Big[\Big(\alpha+\sum_{i=1}^{n}a_i\Big)\int_0^x\sum_{i=1}^{n}h_i(\xi)d\xi+2y\sum_{i=1}^{n}h_i(x)\Big]\phi'(t);\ \text{and}$$

$$W_7 :=\Big[\beta x+\Big(\alpha+\sum_{i=1}^{n}a_i\Big)y+2z\Big]\phi(t)\int_{t-\tau_i(t)}^{t}\sum_{i=1}^{n}h_i'(x(s))y(s)ds.$$

Applying the assumptions (i) to (iv) of the Theorem 3.1 in W_1, we obtain the following estimates

$$W_1 \geq \frac{1}{2}\beta\phi_0\sum_{i=1}^{n}\delta_i x^2+\frac{1}{2}\Big[\Big(\alpha\sum_{i=1}^{n}b_i+\sum_{i=1}^{n}a_i\sum_{i=1}^{n}b_i\Big)-2\phi_1\sum_{i=1}^{n}c_i\Big]y^2+\frac{1}{2}\Big(\sum_{i=1}^{n}a_i-\alpha\Big)z^2,$$

for all $t \geq 0$, x, y and z. In view of assumptions (i) and (iv) of the Theorem 3.1 and noting that β, ϕ_0, δ_i are positive constants, with the view that

$$\Big[x+2\phi_0^{-1}\Big(\sum_{i=1}^{n}\delta_i\Big)^{-1}\Big(\sum_{i=1}^{n}\frac{g_i(\cdot)}{y}-\sum_{i=1}^{n}b_i\Big)y\Big]^2\geq 0$$

and

$$\Big[x+2\phi_0^{-1}\Big(\sum_{i=1}^{n}\delta_i\Big)^{-1}\Big(\sum_{i=1}^{n}\frac{f_i(\cdot)}{z}-\sum_{i=1}^{n}a_i\Big)z\Big]^2\geq 0$$

for all $t \geq 0$, x, y and z, it follows that

$$W_2 \geq -\beta\phi_0^{-1}\left(\sum_{i=1}^{n}\delta_i\right)^{-1}\left(\sum_{i=1}^{n}\frac{g_i(\cdot)}{y}-\sum_{i=1}^{n}b_i\right)^2 y^2$$

and

$$W_3 \geq -\beta\phi_0^{-1}\left(\sum_{i=1}^{n}\delta_i\right)^{-1}\left(\sum_{i=1}^{n}\frac{f_i(\cdot)}{z}-\sum_{i=1}^{n}a_i\right)^2 z^2$$

for all $t \geq 0$, x, y and z. Next, we employ the inequalities

$$64\left(\sum_{i=1}^{n}\frac{g_i(\cdot)}{y}-\sum_{i=1}^{n}b_i\right)^2 < \left[\left(\alpha+\sum_{i=1}^{n}a_i\right)\sum_{i=1}^{n}\frac{g_i(\cdot)}{y}-2\phi(t)\sum_{i=1}^{n}h_i'(x)\right]\times \left[2\sum_{i=1}^{n}\frac{f_i(\cdot)}{z}-\left(\alpha+\sum_{i=1}^{n}a_i\right)\right]$$

and

$$16\left[\left(\alpha+\sum_{i=1}^{n}a_i\right)\sum_{i=1}^{n}\frac{f_i(\cdot)}{z}-\left(\alpha^2+\sum_{i=1}^{n}a_i^2\right)\right]^2 < \left[2\sum_{i=1}^{n}\frac{f_i(\cdot)}{z}-\left(\alpha+\sum_{i=1}^{n}a_i\right)\right]\times \left[\left(\alpha+\sum_{i=1}^{n}a_i\right)\sum_{i=1}^{n}\frac{g_i(\cdot)}{y}-2\phi(t)\sum_{i=1}^{n}h_i'(x)\right]$$

in W_4 and W_5 respectively, to arrive at

$$W_4 = W_5 \geq \frac{1}{4}\left[\sqrt{\left[\left(\alpha+\sum_{i=1}^{n}a_i\right)\sum_{i=1}^{n}\frac{g_i(\cdot)}{y}-2\phi(t)\sum_{i=1}^{n}h_i'(x)\right]}\, y - \sqrt{\left[2\sum_{i=1}^{n}\frac{f_i(\cdot)}{z}-\left(\alpha+\sum_{i=1}^{n}a_i\right)\right]}\, z\right]^2 \geq 0,$$

for all $t \geq 0$, x, y and z. In addition, since $|\phi'(t)| < \epsilon$ for all $t \geq 0$, $\epsilon > 0$ is chosen sufficiently small, $h_i(x)/x \leq c_i$ for all $x \neq 0$, $x \leq |x|$ and the inequality $2x_1x_2 \leq x_1^2 + x_2^2$ for all $x_1, x_2 \in \mathbb{R}$, we find that

$$W_6 \leq \epsilon\left(1+\alpha+\sum_{i=1}^{n}a_i\right)\sum_{i=1}^{n}c_i\left(x^2+y^2\right)$$

for all $t \geq 0$, x, y and z. Finally, since $\phi(t) \leq \phi_1$ for all $t \geq 0$ and $h_i(x)/x \leq c_i$ for all $x \neq 0$, we obtain

$$
\begin{aligned}
W_7 \leq & \frac{1}{2}\phi_1 \sum_{i=1}^{n} c_i \Bigg[\Bigg[\beta x^2 + \Bigg(\alpha + \sum_{i=1}^{n} a_i\Bigg) y^2 + 2z^2\Bigg]\tau_i(t) \\
& + \Bigg(2 + \alpha + \beta + \sum_{i=1}^{n} a_i\Bigg) \int_{t-\tau_i(t)}^{t} y^2(s)ds\Bigg]
\end{aligned}
$$

for all $t \geq 0$, x and y. Substituting estimates W_j $(j = 1, 2, \cdots, 7)$ in the equation (3.13), we obtain

$$
\begin{aligned}
V'_{(1.2)} \leq & -\frac{1}{2}\beta\phi_0 \sum_{i=1}^{n} \delta_i x^2 - \frac{1}{2}\Bigg(\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i - \phi_1 \sum_{i=1}^{n} c_i\Bigg) y^2 \\
& - \frac{1}{4}\Bigg(\sum_{i=1}^{n} a_i - \alpha\Bigg) z^2 - (1 - \tau_i'(t))\lambda_2 \int_{t-\tau_i(t)}^{t} z^2(\theta)d\theta \\
& - \Bigg\{\frac{1}{2}\Bigg(\alpha \sum_{i=1}^{n} b_i - \phi_1 \sum_{i=1}^{n} c_i\Bigg) - \beta\Bigg[1 + \sum_{i=1}^{n} a_i + \phi_0^{-1}\Bigg(\sum_{i=1}^{n} \delta_i\Bigg)^{-1} \times \\
& \Bigg(\sum_{i=1}^{n} \frac{g_i(\cdot)}{y} - \sum_{i=1}^{n} b_i\Bigg)^2\Bigg]\Bigg\} y^2 - \Bigg\{\frac{1}{4}\Bigg(\sum_{i=1}^{n} a_i - \alpha\Bigg) \\
& - \beta\Bigg[1 + \phi_0^{-1}\Bigg(\sum_{i=1}^{n} \delta_i\Bigg)^{-1}\Bigg(\sum_{i=1}^{n} \frac{f_i(\cdot)}{z} - \sum_{i=1}^{n} a_i\Bigg)^2\Bigg]\Bigg\} z^2 \\
& - \Bigg[(1 - \tau_i'(t))\lambda_1 - \frac{1}{2}\Bigg(2 + \alpha + \beta + \sum_{i=1}^{n} a_i\Bigg)\phi_1 \sum_{i=1}^{n} c_i\Bigg] \times \\
& \int_{t-\tau_i(t)}^{t} y^2(\theta)d\theta + \epsilon\Bigg(1 + \alpha + \sum_{i=1}^{n} a_i\Bigg) \sum_{i=1}^{n} c_i\Bigg(x^2 + y^2\Bigg) \\
& + \Bigg[\beta|x| + \Bigg(\alpha + \sum_{i=1}^{n} a_i\Bigg)|y| + 2|z|\Bigg] \sum_{i=1}^{n} |p_i(\cdot)|, + \frac{1}{2}\tau_i(t)\Bigg[\beta x^2 \\
& + \Bigg(\alpha + \sum_{i=1}^{n} a_i\Bigg) y^2 + 2z^2\Bigg]\phi_1 \sum_{i=1}^{n} c_i + \Bigg(\lambda_1 y^2 + \lambda_2 z^2\Bigg)\tau_i(t)
\end{aligned}
\tag{3.14}
$$

for all $t \geq 0$, x, y and z. Noting that $\tau_i(t) \leq \gamma$ and $\tau_i'(t) \leq \beta_0$, $0 < \beta_0 < 1$ for all $t \geq 0$, choose

$$
\lambda_1 = 2^{-1}(1 - \beta_0)\Bigg(2 + \alpha + \beta + \sum_{i=1}^{n} a_i\Bigg)\phi_1 \sum_{i=1}^{n} c_i > 0, \quad \lambda_2 = 0 \tag{3.15}
$$

$\epsilon > 0$ sufficiently small such that

$$\beta\phi_0 \sum_{i=1}^{n} \delta_i > 2\epsilon\left(1 + \alpha + \sum_{i=1}^{n} a_i\right) \sum_{i=1}^{n} c_i,$$

$$\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i - \phi_1 \sum_{i=1}^{n} c_i > 2\epsilon\left(1 + \alpha + \sum_{i=1}^{n} a_i\right) \sum_{i=1}^{n} c_i$$

and use inequalities (3.2) (3.3) and (3.15) in estimate (3.14), we obtain

$$\begin{aligned} V'_{(1.2)} \leq\ & -\frac{1}{2}\left[\beta\phi_0 \sum_{i=1}^{n} \delta_i - 2\epsilon\left(1 + \alpha + \sum_{i=1}^{n} a_i\right) \sum_{i=1}^{n} c_i - \beta\phi_1 \sum_{i=1}^{n} c_i\gamma\right] x^2 \\ & -\frac{1}{2}\left[\alpha \sum_{i=1}^{n} b_i - \phi_1 \sum_{i=1}^{n} c_i - 2\epsilon\left(1 + \alpha + \sum_{i=1}^{n} a_i\right) \sum_{i=1}^{n} c_i \right. \\ & \left. - \left[2\epsilon\left(1 + \alpha + \sum_{i=1}^{n} a_i\right) \sum_{i=1}^{n} c_i\right] \phi_1 \sum_{i=1}^{n} c_i\gamma\right] y^2 \\ & -\frac{1}{4}\left(\sum_{i=1}^{n} a_i - \alpha - 4\phi_1 \sum_{i=1}^{n} c_i\gamma\right) z^2 \\ & + \left[\beta|x| + \left(\alpha + \sum_{i=1}^{n} a_i\right)|y| + 2|z|\right] \sum_{i=1}^{n} |p_i(\cdot)|, \end{aligned} \tag{3.16}$$

for all $t \geq 0$, x, y and z. In view of estimate (3.14) there exist positive constants K_3 and K_4 such that inequality (3.16) becomes

$$V'_{(1.2)} \leq -K_3(x^2 + y^2 + z^2) + K_4(|x| + |y| + |z|) \sum_{i=1}^{n} |p_i(\cdot)| \tag{3.17}$$

for all $t \geq 0$, x, y and z, where

$$\begin{aligned} K_3 := \frac{1}{2} \min \Bigg\{ & \beta\phi_0 \sum_{i=1}^{n} \delta_i - 2\epsilon\left(1 + \alpha + \sum_{i=1}^{n} a_i\right) \sum_{i=1}^{n} c_i - \beta\phi_1 \sum_{i=1}^{n} c_i\gamma, \\ & \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i - \phi_1 \sum_{i=1}^{n} c_i - 2\epsilon\left(1 + \alpha + \sum_{i=1}^{n} a_i\right) \sum_{i=1}^{n} c_i \\ & - \left[2\epsilon\left(1 + \alpha + \sum_{i=1}^{n} a_i\right) \sum_{i=1}^{n} c_i\right] \phi_1 \sum_{i=1}^{n} c_i\gamma, \\ & \sum_{i=1}^{n} a_i - \alpha - 4\phi_1 \sum_{i=1}^{n} c_i\gamma \Bigg\} \end{aligned}$$

and

$$K_4 := \max\left\{\beta, \alpha + \sum_{i=1}^{n} a_i, 2\right\}.$$

In view of the inequality $|x| < 1 + x^2$ and hypothesis (vi) of Theorem 3.1, there exist positive constants K_5 and K_6 such that

$$V'_{(1.2)} \leq -K_5(x^2 + y^2 + z^2) + K_6 \tag{3.18}$$

for all $t \geq 0, x, y$ and z, where

$$K_5 = K_3 - K_4 \sum_{i=1}^{n} M_i \;\text{ and }\; K_6 = 3K_4 \sum_{i=1}^{n} M_i.$$

The inequality (3.18) satisfies estimate (3.6) of Lemma 3.1 with K_5 and K_6 equivalent to D_3 and D_4 respectively. This completes the proof of Lemma 3.1. □

Proof of Theorem 3.1. Let (x_t, y_t, z_t) be any solution of the system (1.2). From the inequalities (3.8), (3.12) and (3.18), assumptions (i) and (ii) of Lemma 2.4 are satisfied. Thus by Lemma 2.4, solution (x_t, y_t, z_t) of (1.2) is uniformly bounded and uniformly ultimately bounded. This completes the proof of the Theorem 3.1. □

Next, we will state and proof existence and uniqueness results for the system (1.2).

Theorem 3.2. *If all conditions of the Theorem 3.1 are satisfied, then there exists a periodic solution of the system* (1.2) *of period* ω.

Proof. Let (x_t, y_t, z_t) be any solution of the system (1.2), the proof of the Theorem 3.2 will depend upon the same scalar valued functional defined in (3.1), as used in the proof of Lemma 3.1. In view of estimates (3.8), (3.11) and (3.12) hypothesis (i) of Lemma 2.1 hold. Furthermore, using condition (vi) of the Theorem 3.1 and the inequality $(|x|+|y|+|z|)^2 \leq 3(x^2+y^2+z^2)$ in estimate (3.17) there exist positive constants K_7 and K_8 such that

$$V'_{(1.2)} \leq -K_7(x^2 + y^2 + z^2) \tag{3.19}$$

for all $t \geq 0, x, y$ and z provided that

$$(x^2 + y^2 + z^2)^{1/2} \geq K_8 \tag{3.20}$$

where $K_7 := \frac{1}{2}K_3$ and $K_8 := 2\times 3^{1/2}K_3^{-1}K_4\sum\limits_{i=1}^{n} M_i$. In view of inequalities (3.19) and (3.20) condition (ii) of Lemma 2.1 follows. Thus by Lemma 2.1 a periodic solution of system (1.2) of period ω is assured. This completes the proof of the Theorem 3.2. □

Theorem 3.3. *If all assumptions of the Theorem 3.1 hold, then for any given initial value in $\mathbb{R}^3$ there exists a unique solution of the system* (1.2).

Proof. Let (x_t, y_t, z_t) be any solution of the system (1.2), the proof of the Theorem 3.3 also depends on the continuously differentiable functional (3.1). From inequalities (3.8), (3.9) and (3.10) conditions (i) and (ii) of Lemma 2.2 hold. Furthermore, from inequalities (3.19) and (3.20), we obtain

$$V'_{(1.2)} \leq 0, \tag{3.21}$$

for all $t \geq 0, x, y$ and z. Inequality (3.21) satisfies condition (iii) of Lemma 2.2, hence by Lemma 2.2, the existing solution of the system (1.2) is unique. This completes the proof of the Theorem 3.3. □

Next, we will establish the bound on the solution (x_t, y_t, z_t) for the system (1.2).

Theorem 3.4. *If conditions (i) to (v) of Theorem 3.1 hold and condition (vi) is replaced by*

$$\int_{t_0}^{t}\sum_{i=1}^{n}|p_i(\cdot)|ds \leq \sum_{i=1}^{n} N_i, \quad 0 < N_i < \infty, \tag{3.22}$$

then for any given finite $x_{t_0}, y_{t_0}, z_{t_0}$, there exists a positive constant $M = M(x_{t_0}, y_{t_0}, z_{t_0}, \alpha, \beta, \beta_0, \delta_i,$
$\epsilon, \phi_0, \phi_1, a_i, b_i, c_i, t_0)$ such that the unique solution (x_t, y_t, z_t) of the system (1.2) *which satisfies the initial condition*

$$x_t(t_0) = x_{t_0}, \quad y_t(t_0) = y_{t_0}, \quad z_t(t_0) = z_{t_0}, \tag{3.23}$$

satisfies

$$|x_t| \leq M, \quad |y_t| \leq M, \quad |z_t| \leq M, \tag{3.24}$$

for all $t \geq t_0$.

Proof. Let (x_t, y_t, z_t) be any solution of the system (1.2), the proof of the Theorem 3.4 is also based on the continuously differentiable functional (3.1).

Since $K_3(x^2+y^2+z^2)\geq 0$ for all x, y and z, inequality (3.17) becomes

$$V'_{(1.2)} \leq 3K_4\sum_{i=1}^{n}|p_i(\cdot)| + K_3(x^2+y^2+z^2)\sum_{i=1}^{n}|p_i(\cdot)| \tag{3.25}$$

for all $t\geq 0, x, y$ and z. Applying inequality (3.8) in (3.25), there exists a positive constant K_9 such that

$$V'_{(1.2)} - K_9\sum_{i=1}^{n}|p_i(\cdot)|V \leq K_9\sum_{i=1}^{n}|p_i(\cdot)| \tag{3.26}$$

for all $t\geq 0, x, y$ and z, where $K_9 := \max\{K_0^{-1}K_3,\quad 3K_4\}$. Solving the first order differential inequality (3.26) using integrating factor

$$\exp\left[-K_9\int_{t_0}^{t}\sum_{i=1}^{n}|p_i(\cdot)|ds\right],$$

we find that

$$\begin{aligned} V(t) \leq V(t_0)\exp\left[K_9\int_{t_0}^{t}\sum_{i=1}^{n}|p_i(\cdot)|ds\right] \\ + k_9\left[\exp\left(K_9\int_{t_0}^{t}\sum_{i=1}^{n}|p_i(\cdot)|ds\right)-1\right], \end{aligned} \tag{3.27}$$

where $V(t) = V(t, x_t, y_t, z_t)$. Using inequality (3.22) and (3.23) in estimate (3.27), there exists a positive constant K_{10} such that

$$V(t) \leq K_{10} \tag{3.28}$$

for all $t\geq 0, x, y$ and z, where

$$K_{10} := V(t_0, x_{t_0}, y_{t_0}, z_{t_0})\exp\left[K_9\sum_{i=1}^{n}N_i\right] + k_9\left[\exp\left(K_9\sum_{i=1}^{n}N_i\right)-1\right].$$

Finally, using inequality (3.8) in (3.28), there exists a positive constant K_{11} such that

$$|x_t|\leq K_{11}, \quad |y_t|\leq K_{11}, \quad |z_t|\leq K_{11}, \tag{3.29}$$

for all $t\geq t_0$, where $K_{11} := (K_0^{-1}K_{10})^{1/2}$. Inequalities (3.29) establish inequalities (3.24) with K_{11} equivalent to M. This completes the proof of the Theorem 3.4. □

Next, we will discuss the behaviour of solution (x_t, y_t, z_t) of the system (1.2) as $t \to \infty$.

Theorem 3.5. *If all conditions of the Theorem 3.1 hold and in addition, for all* i, $(i = 1, 2, \cdots, n)$

(i)′ $f_i(t, 0, x(t-\tau_i(t)), 0, y(t-\tau_i(t)), 0, z(t-\tau_i(t))) = 0$ *for all* $t \geq 0, x(t-\tau_i(t)), y(t-\tau_i(t))$ *and* $z(t-\tau_i(t))$,

(ii)′ $g_i(t, 0, x(t-\tau_i(t)), 0, y(t-\tau_i(t))) = 0$ *for all* $t \geq 0, x(t-\tau_i(t))$ *and* $y(t-\tau_i(t))$.

Then every solution (x_t, y_t, z_t) *of the system* (1.2) *is uniformly bounded and*

$$x_t \to 0, \quad y_t \to 0, \quad z_t \to 0, \tag{3.30}$$

as $t \to \infty$.

Proof. Let (x_t, y_t, z_t) be any solution of the system (1.2). The proof of the Theorem 3.5 depends on the functional V defined in the equation (3.1). Theorem 3.1 established uniform boundedness of solution (x_t, y_t, z_t) of the system (1.2). From the inequality (3.19), we define a positive definite function (or wedge) as

$$W_1(X_t) := K_7(x^2 + y^2 + z^2), \quad X_t = (x_t, y_t, z_t) \in \mathbb{R}^3$$

with respect to a closed set

$$\Omega_1 := \Big\{(x_t, y_t, z_t) : x_t = 0, \quad y_t = 0, \quad z_t = 0\Big\}$$

and

$$V'_{(1.2)} \leq -W_1(X_t).$$

Since $\phi(t) \leq \phi_1$ for all $t \geq 0$, it follows from the system (1.2) that

$$F(t, X_t) = \begin{pmatrix} y \\ z \\ -\phi(t)\sum\limits_{i=1}^{n} h_i(x) - \sum\limits_{i=1}^{n} g_i(\cdot) - \sum\limits_{i=1}^{n} f_i(\cdot) + A_7 \end{pmatrix}$$

where

$$A_7 := \phi(t) \int\limits_{t-\tau_i(t)}^{t} \sum_{i=1}^{n} h_i'(x(s))y(s)ds$$

and

$$G(t, X_t) = \begin{pmatrix} 0 \\ 0 \\ \sum\limits_{i=1}^{n} p_i(\cdot) \end{pmatrix}.$$

Since f_i, g_i and h_i are continuous functions in their respective arguments it follows that $F(t, X_t)$ is bounded for all t when X_t belongs to any compact subset of $\mathbb{R}^3$. From conditions (i)$'$ and (ii)$'$ of the Theorem 3.5 and the fact that $x_t = y_t = z_t = 0$ for all t on Ω_1, we have

$$X' = F(t, X_t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = F(X_t) \tag{3.31}$$

so that every solution (x_t, y_t, z_t) of the system (1.2) approaches the largest semi invariant set of

$$X' = F(X_t) \tag{3.32}$$

contained in Ω_1, as $t \to \infty$. From the systems (3.31) and (3.32), we find that

$$\begin{pmatrix} x_t' \\ y_t' \\ z_t' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which has solution

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix},$$

where $C_i (i = 1, 2, 3)$ is a constant. To remain in Ω_1, we must have $C_1 = C_2 = C_3 = 0$. Therefore the only solution of (3.32) which remains in Ω_1 is $X_t = (0, 0, 0)^T$, that is the largest invariant set of the system (3.32) contained in Ω_1 is the set $\{(0, 0, 0)\}$. Thus the relation (3.30) holds as $t \to \infty$. This completes the proof of Theorem 3.5. □

Next, if $p_i(\cdot) = 0$ in (1.1), we have

$$x''' + \sum_{i=1}^{n} f_i(\cdot) + \sum_{i=1}^{n} g_i(\cdot) + \phi(t) \sum_{i=1}^{n} h_i(x(t - \tau_i(t))) = 0 \tag{3.33}$$

or its equivalent system

$$
\begin{aligned}
x' &= y, \\
y' &= z, \\
z' &= -\phi(t)\sum_{i=1}^{n} h_i(x) - \sum_{i=1}^{n} f_i(\cdot) - \sum_{i=1}^{n} g_i(\cdot) \\
&\quad + \phi(t)\int_{t-\tau_i(t)}^{t}\sum_{i=1}^{n} h_i(x(s))y(s)ds
\end{aligned}
\tag{3.34}
$$

Theorem 3.6. *If conditions (i) to (v) of the Theorem 3.1 hold, then the zero solution of the system* (3.34) *is uniformly asymptotically stable.*

Proof. Let (x_t, y_t, z_t) be any solution of the system (3.34). The main tool in the proof of the Theorem 3.6 is the functional V of the equation (3.1). From the inequalities (3.8) and (3.12), condition (i) of Lemma 2.3 hold. Setting $p_i(\cdot) = 0$ in the inequality (3.17), we have

$$
V'_{(3.34)} \leq -K_3(x^2 + y^2 + z^2) \tag{3.35}
$$

for all $t \geq 0, x, y$ and z. Inequality (3.35) satisfies hypothesis (ii) of Lemma 2.3, hence by Lemma 2.3 the zero solution of the system (3.34) is uniformly asymptotically stable. This completes the proof of Theorem 3.6. $\square$

If the function $p_i(\cdot)$ of the system (1.2) is replaced by $p(t)$ and $p(t, x, y, z)$ defined on $\mathbb{R}^+$ and $\mathbb{R}^+ \times \mathbb{R}^3$ respectively. We have the following equations

$$
\begin{aligned}
x' &= y, \\
y' &= z, \\
z' &= p(t) - \phi(t)\sum_{i=1}^{n} h_i(x) - \sum_{i=1}^{n} f_i(\cdot) - \sum_{i=1}^{n} g_i(\cdot) \\
&\quad + \phi(t)\int_{t-\tau_i(t)}^{t}\sum_{i=1}^{n} h_i(x(s))y(s)ds
\end{aligned}
\tag{3.36}
$$

and

$$\begin{aligned}
&x' = y,\\
&y' = z,\\
&z' = p(t,x,y,z) - \phi(t)\sum_{i=1}^{n} h_i(x) - \sum_{i=1}^{n} f_i(\cdot) - \sum_{i=1}^{n} g_i(\cdot)\\
&\qquad + \phi(t)\int_{t-\tau_i(t)}^{t}\sum_{i=1}^{n} h_i(x(s))y(s)ds,
\end{aligned}\tag{3.37}$$

with the following results:

Corollary 3.1. *If conditions (i) to (v) of the Theorem 3.1 hold, and condition (vi) is replaced by boundedness of the function $p(t)$, then the solution (x_t, y_t, z_t) of the system (3.36) is uniformly bounded and uniformly ultimately bounded.*

Corollary 3.2. *If conditions (i) to (v) of the Theorem 3.1 are satisfied and in addition the function $p(t,x,y,z)$ is bounded, then the solution (x_t, y_t, z_t) of the system (3.37) is uniformly bounded and uniformly ultimately bounded.*

Proof. The Proofs of Corollaries 3.1 and 3.2 are similar to the proof of the Theorem 3.1, hence they are omitted. □

In what follows, we give two examples to justify our obtained results.

4 Examples and Discussions

Example 4.1. *Consider a third order delay differential equation with multiple deviating arguments for all $n \in \mathbb{N}$ finite*

$$\begin{aligned}
&x''' + A_8 x'' + A_9 x' + (2+\sin(0.001t))\times\\
&\left(2 + \sum_{i=1}^{n} x'(t-\tau_i(t))\right)\left(1 + \sum_{i=1}^{n} x'(t-\tau_i(t))\right)^{-1}\\
&= 1 + \sin\left(\pi/2 + txx'x''\sum_{i=1}^{n} x'(t-\tau_i(t))x''(t-\tau_i(t))\right)\Bigg].
\end{aligned}\tag{4.1}$$

where

$$A_8 := \frac{5 + 4[t^2 + |xx'x''| + \sum_{i=1}^{n}[x^2(t-\tau_i(t)) + x'^2(t-\tau_i(t)) + x''^2(t-\tau_i(t))]]}{1 + t^2 + |xx'x''| + \sum_{i=1}^{n}[(t-\tau_i(t)) + x'^2(t-\tau_i(t)) + x''^2(t-\tau_i(t))]},$$

$$A_9 := \Bigg[\frac{4+3(t+|xx'|+\sum\limits_{i=1}^{n}[|x(t-\tau_i(t))|+|x'(t-\tau_i(t))|)]}{1+t+|xx'|+\sum\limits_{i=1}^{n}[|x(t-\tau_i(t))|+|x'(t-\tau_i(t))|]}$$

Its equivalent form is

$$\begin{aligned}
x' &= y \\
y' &= z \\
z' &= 1+\sin\left(\pi/2+txyz\sum_{i=1}^{n}[y(t-\tau_i(t))z(t-\tau_i(t))]\right) \\
&\quad -\left(\frac{2+x^2}{1+x^2}\right)(2+\sin(0.001t))-A_{10}z-A_{11}y \\
&\quad +(2+\sin(0.001t))\int_{t-\tau_i(t)}^{t}\left[\frac{(1+x^2(s))^2+1-x^2(s)}{(1+x^2(s))^2}\right]ds.
\end{aligned} \tag{4.2}$$

where

$$A_{10} := \frac{5+4[t^2+|xyz|+\sum\limits_{i=1}^{n}[x^2(t-\tau_i(t))+y^2(t-\tau_i(t))+z^2(t-\tau_i(t))]]}{1+t^2+|xyz|+\sum\limits_{i=1}^{n}[x^2(t-\tau_i(t))+y^2(t-\tau_i(t))+z^2(t-\tau_i(t))]}$$

and

$$A_{11} := \frac{4+3[t+|xy|+\sum\limits_{i=1}^{n}[|x(t-\tau_i(t))|+|y(t-\tau_i(t))|]]}{1+t+|xy|+\sum\limits_{i=1}^{n}[|x(t-\tau_i(t))|+|y(t-\tau_i(t))|]}$$

From the systems (1.2) *and* (4.2)*, we observed the following.*

(i) Set the function

$$\phi(t) := 2+D(t),$$

where

$$D(t) := \sin(0.001t).$$

Since

$$-1 \le D(t) \le 1$$

for all t*, it follows that*

$$1 = \phi_0 \le \phi(t) \le \phi_1 = 3$$

for all $t \geq 0$. *See Figure 1 for the bounds on the function* $D(t)$ *and* $\phi(t)$. *Moreover,*

$$\phi'(t) := 0.001\cos(0.001t)$$

so that

$$|\phi'(t)| \leq \epsilon = 0.001,$$

for all $t \geq 0$. *See Figure 2 for a bound on* $|\phi'(t)|$.

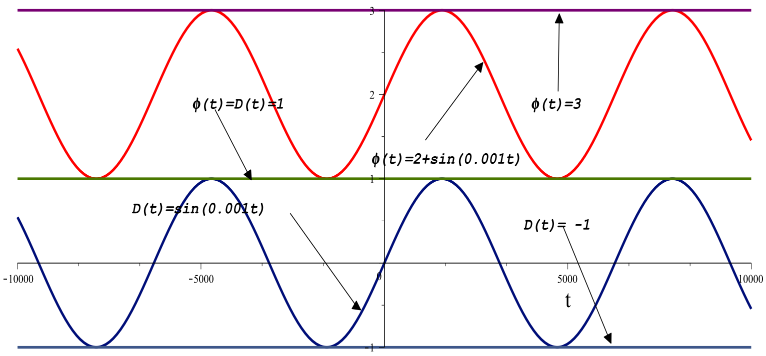

Figure 1: Bounds on the functions $\phi(t)$ and $D(t)$

(ii) Let the function

$$\sum_{i=1}^{n} f_i(\cdot) := \left[\frac{5 + 4[t^2 + |xyz| + \sum_{i=1}^{n}[x^2(t-\tau_i(t)) + y^2(t-\tau_i(t)) + z^2(t-\tau_i(t))]]}{1 + t^2 + |xyz| + \sum_{i=1}^{n}[x^2(t-\tau_i(t)) + y^2(t-\tau_i(t)) + z^2(t-\tau_i(t))]}\right] z$$

or

$$\sum_{i=1}^{n} \frac{f_i(\cdot)}{z} = 4 + \frac{1}{1 + t^2 + |xyz| + \sum_{i=1}^{n}[x^2(t-\tau_i(t)) + y^2(t-\tau_i(t)) + z^2(t-\tau_i(t))]}.$$

Since

$$\frac{1}{1 + t^2 + |xyz| + \sum_{i=1}^{n}[x^2(t-\tau_i(t)) + y^2(t-\tau_i(t)) + z^2(t-\tau_i(t))]} > 0$$

for all $t, x, y, z, x(t-\tau_i(t)), y(t-\tau_i(t))$ *and* $z(t-\tau_i(t))$. *It follows that*

$$\sum_{i=1}^{n} \frac{f_i(\cdot)}{z} \geq \sum_{i=1}^{n} a_i = 4$$

for all $t, x, y, z \neq 0, x(t-\tau_i(t)), y(t-\tau_i(t))$ *and* $z(t-\tau_i(t))$.

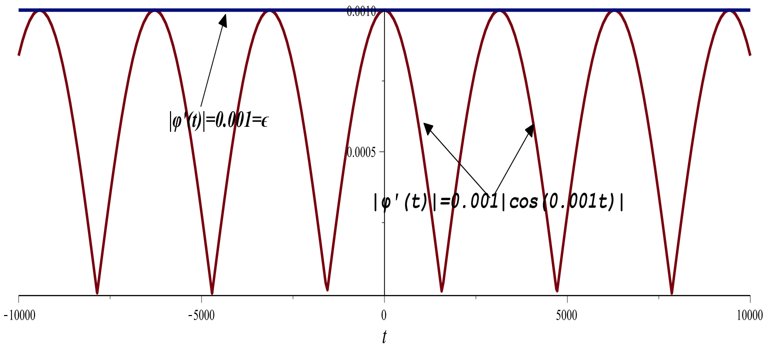

Figure 2: A Bound on $|\phi'(t)|$

(iii) Let the function

$$\sum_{i=1}^{n} g_i(\cdot) := \left[\frac{4+3[t+|xy|+\sum_{i=1}^{n}[|x(t-\tau_i(t))|+|y(t-\tau_i(t))|]]}{1+t+|xy|+\sum_{i=1}^{n}[|x(t-\tau_i(t))|+|y(t-\tau_i(t))|]}\right]y$$

or

$$\sum_{i=1}^{n} \frac{g_i(\cdot)}{y} := 3 + \frac{1}{1+t+|xy|+\sum_{i=1}^{n}[|x(t-\tau_i(t))|+|y(t-\tau_i(t))|]}.$$

It is not difficult to show that

$$\sum_{i=1}^{n} \frac{g_i(\cdot)}{y} \geq \sum_{i=1}^{n} b_i = 3$$

for all $t, x, y \neq 0, x(t-\tau_i(t))$ *and* $y(t-\tau_i(t))$.

(iv) Let the function

$$\sum_{i=1}^{n} h_i(x) := \frac{x(2+x^2)}{1+x^2}$$

or

$$\sum_{i=1}^{n} \frac{h_i(x)}{x} = 1 + H(x),$$

where

$$H(x) := \frac{1}{1+x^2}.$$

Since

$$0 < H(x) \leq 1$$

for all x*, we conclude from the inequality that*

$$1 = \sum_{i=1}^{n} \delta_i \leq \sum_{i=1}^{n} \frac{h_i(x)}{x} \leq \sum_{i=1}^{n} c_i = 2,$$

for all $x \neq 0$*. Moreover,*

$$\sum_{i=1}^{n} h_i(0) = 0$$

and

$$\sum_{i=1}^{n} |h_i'(x)| \leq \sum_{i=1}^{n} c_i = 2$$

for all x*. See the behaviour of* $h(x)/x, H(x)$ *and* $|h'(x)|$ *in Figure 3 for* $n = 1$*.*

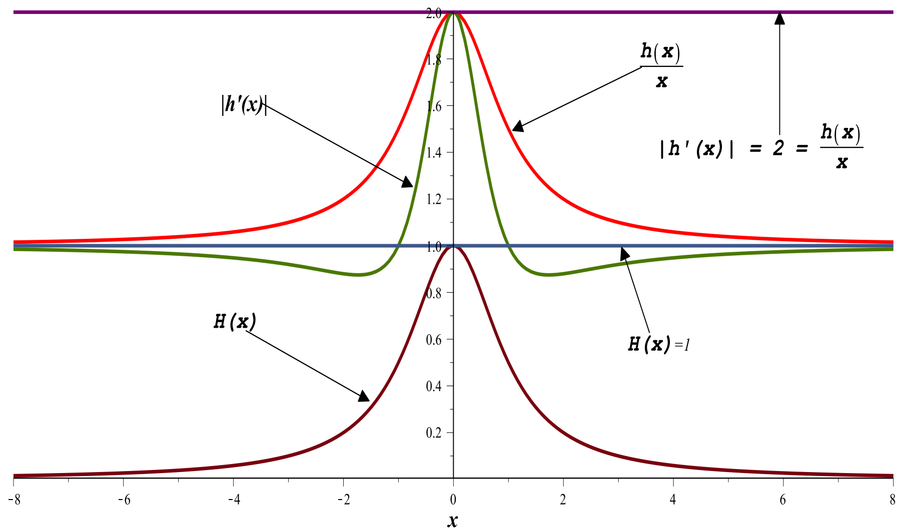

Figure 3: The behaviour of $h(x)/x, H(x)$ and $|h'(x)|$.

(v) *We note, from items (i) to (iv) of Example 4.1, the following relations*

(a) *inequality* (3.2) *becomes*

$$2 < \alpha < 4$$

and we choose $\alpha = 3$*;*

(b) *inequality* (3.3) *yields*

$$\beta < \min\left\{3, \frac{3}{10}, \frac{1}{4}\right\} = \frac{1}{4}$$

and we choose $\beta = \dfrac{1}{5}$*; and* *(c) inequality* (3.4) *yields*

$$\gamma < \min\left\{\frac{28}{1000}, \frac{39}{1000}, \frac{42}{1000}\right\} = \frac{28}{1000}.$$

(vi) Finally, for the function

$$\sum_{i=1}^{n} p_i(\cdot) := 1 + \sin\left(\pi/2 + txyz\sum_{i=1}^{n}[y(t-\tau_i(t))z(t-\tau_i(t))]\right),$$

it is not difficult to show that

$$\sum_{i=1}^{n} p_i(\cdot) \le \sum_{i=1}^{n} M_i = 2$$

for all $t, x, y, z, x(t-\tau_i(t)), y(t-\tau_i(t))$ *and* $z(t-\tau_i(t))$.

From items (i) to (vi) of Example 4.1, all assumptions of the Theorems 3.1 to 3.5 hold. Thus their respective conclusion follows for the system (4.2)*. Also, from items (i) to (v) of Example 4.1, conditions of the Theorem 3.6 hold. Thus by the Theorem 3.6, the trivial solution of the system* (4.2) *is uniformly asymptotically stable.*

Example 4.2. *Consider the third order delay differential equation with multiple deviating arguments*

$$x''' + B_1x'' + B_2x' + (3+\exp(0.01t))(1+\exp(0.01t))^{-1}\times \left[\frac{1+\sum\limits_{i=1}^{n}[x^2(t-\tau_i(t)) + \sin(x(t-\tau_i(t)))]}{1+\sum\limits_{i=1}^{n}x^2(t-\tau_i(t))}\right]\sum_{i=1}^{n} x(t-\tau_i(t)) = B_3, \tag{4.3}$$

$$B_1 := \frac{3+2(|\sin t| + \sum\limits_{i=1}^{n}[|xx'(t-\tau_i(t))| + |x(t-\tau_i(t))x'| + |x''x''(t-\tau_i(t))|])}{1+|\sin t| + \sum\limits_{i=1}^{n}[|xx'(t-\tau_i(t))| + |x(t-\tau_i(t))x'| + |x''x''(t-\tau_i(t))|]},$$

$$B_2 := \frac{4+3(2t+\cos(xx') + 2\sum\limits_{i=1}^{n}|x'(t-\tau_i(t))|)}{1+2t+\cos(xx') + 2\sum\limits_{i=1}^{n}|x'(t-\tau_i(t))|}$$

and

$$B_3 := \frac{2 + 2t^2|xx'x''| + \sum\limits_{i=1}^{n}[|x(t-\tau_i(t))| + x'^2(t-\tau_i(t)) + |x''(t-\tau_i(t))|]}{1 + 2t^2|xx'x''| + \sum\limits_{i=1}^{n}[|x(t-\tau_i(t))| + x'^2(t-\tau_i(t)) + |x''|(t-\tau_i(t))|},$$

Its equivalent system of first order is given by

$$\begin{aligned}
x' &= y \\
y' &= z \\
z' &= -\left(\frac{3+\exp(0.01t)}{1+\exp(0.01t)}\right)\left(\frac{1+x^2+\sin x}{1+x^2}\right)x \\
&\quad - \left[\frac{4+3(2t+\cos(xy)+2\sum\limits_{i=1}^{n}|y(t-\tau_i(t))|)}{1+2t+\cos(xy)+2\sum\limits_{i=1}^{n}|y(t-\tau_i(t))|}\right]y - B_5 z \\
&\quad + \left(\frac{3+\exp(0.01t)}{1+\exp(0.01t)}\right)\int_{t-\tau_i(t)}^{t} B_6(s)ds + B_4,
\end{aligned} \tag{4.4}$$

where

$$B_4 = \frac{2 + 2t^2|xyz| + \sum\limits_{i=1}^{n}[|x(t-\tau_i(t))| + y^2(t-\tau_i(t)) + |z(t-\tau_i(t))|]}{1 + 2t^2|xyz| + \sum\limits_{i=1}^{n}[|x(t-\tau_i(t))| + y^2(t-\tau_i(t)) + |z(t-\tau_i(t))|]},$$

$$B_5 := \frac{3 + 2(|\sin t| + \sum\limits_{i=1}^{n}[|xy(t-\tau_i(t))| + |x(t-\tau_i(t))y| + |zz(t-\tau_i(t))|])}{1 + |\sin t| + \sum\limits_{i=1}^{n}[|xy(t-\tau_i(t))| + |x(t-\tau_i(t))y| + |zz(t-\tau_i(t))|]}$$

and

$$B_6 := \frac{1+x^2+(1-x^2)\sin x + x(1+x^2)\cos x}{(1+x^2)^2}.$$

Now, let us compare the system (1.2) *with the system* (4.4).
(*i*) *Set the function*

$$\phi(t) := \frac{3+\exp(0.01t)}{1+\exp(0.01t)} = \frac{1}{2} + D(t),$$

where

$$D(t) := \frac{1}{1+\exp(0.01t)}.$$

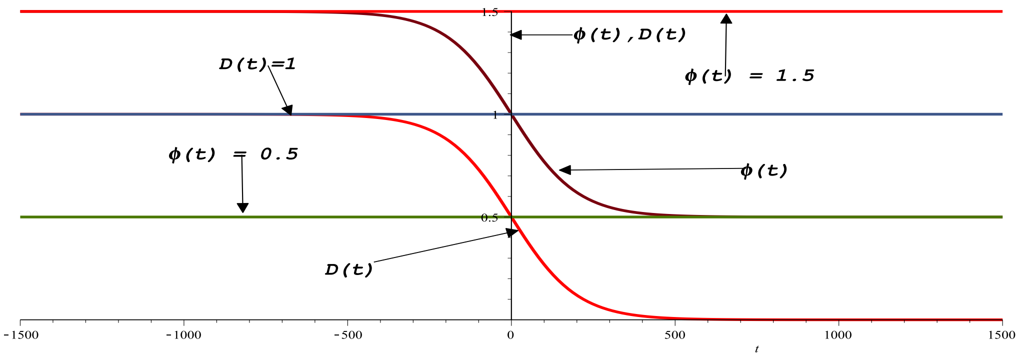

Figure 4: Bounds on $\phi(t)$ and $D(t)$ for Example 4.2

Since

$$0 < D(t) \leq 1$$

for all t*, it follows that*

$$\frac{1}{2} = \phi_0 \leq \phi(t) \leq \phi_1 = \frac{3}{2}$$

for all $t \geq 0$*. Furthermore,*

$$\phi'(t) = -\frac{\exp(0.01t)}{100(1+\exp(0.01t))^2}.$$

It is not difficult to show that

$$|\phi'(t)| < \epsilon = 0.0025$$

for all $t \geq 0$*. (See Figure 5).*

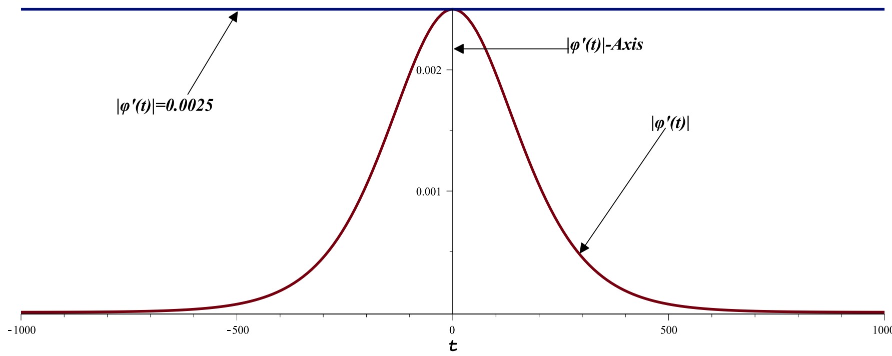

Figure 5: Bound on $|\phi'(t)|$ for Example 4.2

(ii) Let the function

$$\sum_{i=1}^{n} f_i(\cdot) := \left[\frac{3 + 2(|\sin t| + \sum\limits_{i=1}^{n}[|xy(t-\tau_i(t))| + |x(t-\tau_i(t))y| + |zz(t-\tau_i(t))|])}{1 + |\sin t| + \sum\limits_{i=1}^{n}[|xy(t-\tau_i(t))| + |x(t-\tau_i(t))y| + |zz(t-\tau_i(t))|]}\right] z$$

or

$$\sum_{i=1}^{n}\frac{f_i(\cdot)}{z}:=2+\frac{1}{1+|\sin t|+\sum\limits_{i=1}^{n}[|xy(t-\tau_i(t))|+|x(t-\tau_i(t))y|+|zz(t-\tau_i(t))|]}.$$

It is clear from the above equation that

$$\sum_{i=1}^{n}\frac{f_i(\cdot)}{z}\geq\sum_{i=1}^{n}a_i=2$$

for all $t\geq 0, x, y, z\neq 0, x(t-\tau_i(t))$ *and* $z(t-\tau_i(t))$.
(iii) Let the function

$$\sum_{i=1}^{n}g_i(\cdot):=\left[\frac{4+3(2t+\cos(xy)+2\sum\limits_{i=1}^{n}|y(t-\tau_i(t))|)}{1+2t+\cos(xy)+2\sum\limits_{i=1}^{n}|y(t-\tau_i(t))|}\right]y$$

or

$$\sum_{i=1}^{n}\frac{g_i(\cdot)}{y}=3+\frac{1}{1+2t+\cos(xy)+2\sum\limits_{i=1}^{n}|y(t-\tau_i(t))|}.$$

Obviously,

$$\sum_{i=1}^{n}\frac{g_i(\cdot)}{y}\geq\sum_{i=1}^{n}b_i=3$$

for all $t\geq 0, x, y\neq 0, x(t-\tau_i(t))$ *and* $y(t-\tau_i(t))$.
(iv) Let the function

$$\sum_{i=1}^{n}h_i(x):=\left(\frac{1+x^2+\sin x}{1+x^2}\right)x$$

or

$$\sum_{i=1}^{n}\frac{h_i(x)}{x}:=1+H(x)$$

where

$$H(x):=\frac{\sin x}{1+x^2}.$$

Since

$$-0.44\leq H(x)\leq 0.44$$

for all x, *it follows that*

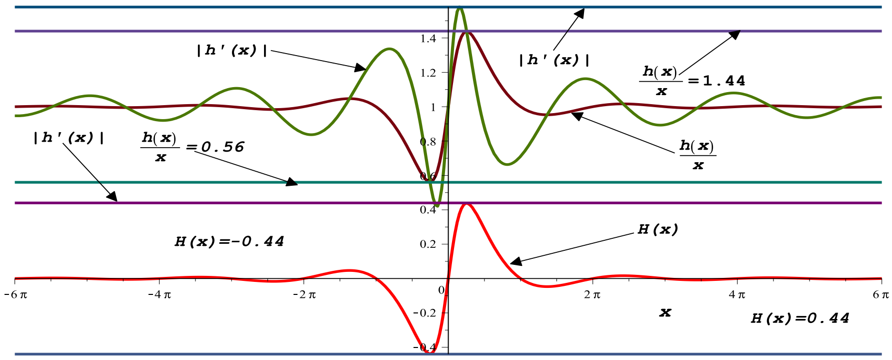

Figure 6: The Behaviour of $H(x)$, $h(x)/x$ and $|h'(x)|$ for Example 4.2

$$0.6 \leq \sum_{i=1}^{n} \delta_i \leq \sum_{i=1}^{n} \frac{h_i(x)}{x} \leq \sum_{i=1}^{n} c_i = 1.44$$

for all $x \neq 0$ *and*

$$\sum_{i=1}^{n} h_i(0) = 0.$$

Furthermore,

$$\sum_{i=1}^{n} h_i'(x) := \frac{1 + x^2 + (1 - x^2)\sin x + x(1 + x^2)\cos x}{(1 + x^2)^2}$$

and the estimate

$$\sum_{i=1}^{n} |h_i'(x)| \leq \sum_{i=1}^{n} c_i = 1.58$$

is obtained for all x. *Thus we choose*

$$\sum_{i=1}^{n} c_i = \max\{1.44, 1.58\} = 1.58.$$

See the behaviour of functions $H(x)$, $h(x)/x$ *and* $|h'(x)|$ *in Figure 6;*
(v) From items (i) to (iv) of the Example 4.2, we have the following relations

(a) *inequality* (3.2) *yields*

$$0.79 < \alpha < 2$$

and we choose $\alpha = 1$;

(b) *inequality* (3.3) *becomes*

$$\beta < \min\{3, 0.11, 0.25\} = 0.11$$

we choose $\beta = 0.1$

(c) inequality (3.4) *yields*

$$\gamma < \min\{0.003, 0.115, 0.105\} = 0.003.$$

and finally
(vi) the function

$$\sum_{i=1}^{n} |p_i(\cdot)| := \frac{2 + 2t^2 + |xyz| + \sum\limits_{i=1}^{n} [|x(t-\tau_i(t))| + y^2(t-\tau_i(t)) + |z(t-\tau_i(t))|]}{1 + 2t^2 + |xyz| + \sum\limits_{i=1}^{n} [|x(t-\tau_i(t))| + y^2(t-\tau_i(t)) + |z(t-\tau_i(t))|]}$$

or

$$\sum_{i=1}^{n} |p_i(\cdot)| = 1 + \frac{1}{1 + 2t^2 + |xyz| + \sum\limits_{i=1}^{n} [|x(t-\tau_i(t))| + y^2(t-\tau_i(t)) + |z(t-\tau_i(t))|]}.$$

It is not very difficult to show that

$$\sum_{i=1}^{n} |p_i(\cdot)| \leq \sum_{i=1}^{n} M_i = 2.$$

From items (i) to (vi) of the Example 4.2 all assumptions of the Theorems 3.1 to 3.5 hold. Thus their respective conclusion follows for the system (4.4)*. Also, from items (i) to (v) of the Example 4.2, conditions of the Theorem 3.6 hold. Thus by Theorem 3.6 the trivial solution of the system* (4.4) *is uniformly asymptotically stable.*

5 Conclusion

In this paper, we study the behaviour of solutions for a certain third order nonlinear differential equations with multiple deviating arguments. Sufficient conditions for the existence of a unique solution that is periodic, uniformly asymptotically stable, uniformly ultimately bounded are established by virtue of a complete Lyapunov functional. As applications, two examples are presented to illustrate the main results.

References

[1] Ademola, A. T., Arawomo, P.O.; *Asymptotic behaviour of solutions of third order nonlinear differential equations.* Acta Univ. Sapientiae, Mathematica, 3(2), 197 - 211 (2011)

[2] Ademola, A. T., Arawomo, P.O.; *Uniform stability and boundedness of solutions of nonlinear delay differential equations of the third order.* Math. J. Okayama Univ. 55, 157 - 166 (2013)

[3] Ademola, A. T., Arawomo, P.O., Ogunlaran, M.O., Oyekan, E.A.; *Uniform stability, boundedness and asymptotic behaviour of solutions of some third order nonlinear delay differential equations. Differential Equations and Control Processes.* (4), 43 - 66 (2013)

[4] Ademola, A. T., Arawomo, P.O.; *Boundedness and asymptotic behaviour of solutions of a nonlinear differential equation of the third order.* Afr. Mat. 23:DOI 10.1007/s13370-011-0034-x 261 271 (2012)

[5] Ademola, A. T., Arawomo, P.O.; *On the asymptotic behaviour of solutions of certain differential equations of the third order.* Proyecciones Journal of Mathematics. 33(1), 111 - 132 (2014)

[6] Ademola, A. T., Arawomo, P.O.; *Stability, boundedness and asymptotic behaviour of solutions of certain nonlinear differential equations of the third order.* Kragujevac Journal of Mathematics. 35(3), 431 - 445 (2011)

[7] Ademola, A. T.; *Existence and uniqueness of a periodic solution to certain third order nonlinear delay differential equation with multiple deviating arguments.* Acta Univ. Sapientiae Mathematica. 5(2), 113 - 131 (2013)

[8] Ademola, A. T.; *Stability, boundedness and uniqueness of solutions to certain third order stochastic delay differential equations Differential Equations and Control Processes.* (2), 25 - 50 (2017)

[9] Ademola, A. T., Mahmoud, A. M.; *On some qualitative properties of solutions to certain third order vector differential equations with multiple constant deviating arguments.* International Journal of Nonlinear Science, 26(3), 180 - 192 (2018)

[10] Ademola, T. A., Ogundiran, M. O.; *On the existence and uniqueness of solutions of a generalized Lipschitz ordinary differential equations.* Ife Journal of Science, 9(2), 241 - 246 (2007)

[11] Ademola, T. A., Ogundiran, M. O., Arawomo, P.O., Adesina, O.A.; *Boundedness results for a certain third order nonlinear differential equation.* Applied Mathematics and Computation 216 3044 - 3049 (2010)

[12] Ademola, A. T., Ogundare, B. S., Ogundiran, M. O., Adesina, O.A.; *Stability, boundedness and existence of periodic solutions to certain third order delay differential equations with multiple deviating arguments.* International Journal of Differential Equations. Volume 2015, Article ID 213935, 12 pages (2015)

[13] Ademola, A. T., Ogundiran, M. O., Arawomo, P. O.; *Stability, boundedness and existence of periodic solutions to certain third order nonlinear differential equations.* Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica. 54(1) 5 - 18, (2015)

[14] Adesina, O. A.; *Results on the qualitative behaviour of solutions for a class of third order nonlinear differential equations.* AIP Conference Proceedings. 1637(1), 5 - 12 (2014)

[15] Afuwape, A. U., Omeike, M. O.; *Stability and boundedness of solutions of a kind of third order delay differential equations.* Comp. Appl. Math. 29(3), 329 - 342 (2010)

[16] Burton, T. A.; *Stability and periodic solutions of ordinary and functional differential equations.* Mathematics in Science and Engineering. 178 Academic Press. Inc., Orlando, FL, (1985)

[17] Burton, T. A.; *Volterra integral and differential equations.* New York: Academic Press. (1983)

[18] Driver, R. D.; *Ordinary and delay differential equations.* Springer-Verlag New York Heidelberg Berlin. (1977)

[19] Graef, J. R. Tunç, C.; *Global asymptotic stability and boundedness of certain multi-delay functional differential equations of third order.* Math. Methods Appl. Sci. 38(17), 3747 - 3752 (2015)

[20] Hale, J. K.; *Theory of functional differential equations.* Springer-Verlag New York. (1977)

[21] Lakshmikantham, V., Wen, L., Zhang, B.; *Theory of differential equations with unbounded delay.* Springer Science+Business Media Dordrecht. (1994)

[22] Olutimo, A. L., Adams, D. O.; *On the stability and boundedness of solutions of certain non-autonomous delay differential equation of third order.* Applied Mathematics. 7, 457 - 467 (2016)

[23] Omeike, M. O.; *New results on the stability of solution of some non-autonomous delay differential equations of the third order.* Differential Equations and Control Processes. (1), 18 - 29 (2010)

[24] Omeike, M. O.; *Uniform ultimate boundedness of solutions of third order nonlinear delay differential equations.* An. Stiint. Univ. Al. I. Cuza Iasi, Ser. Noua, Mat., Tomul LVI. f.2 DOI: 10.2478/v10157-010-0026-4. 363 - 372 (2010)

[25] Remili, M., Beldjerd, D.; *A boundedness and stability results for a kind of third order delay differential equations.* Appl. Appl. Math. 10(2), 772 - 782 (2015)

[26] Remili, M., Oudjedi L. D.; *Stability and boundedness of the solutions of non autonomous third order differential equations with delay.* Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica. 53(2), 139 - 147 (2014)

[27] Tunç, C.; *Global stability and boundedness of solutions to differential equations of third order with multiple delays.* Dynam. Systems Appl. 24(4), 467 - 478 (2015)

[28] Tunç; *On the qualitative behaviors of solutions of some differential equations of higher order with multiple deviating arguments.* J. Franklin Inst. 351(2), 643 - 655 (2014)

[29] Tunç, C.; *Stability and boundedness for a kind of non-autonomous differential equations with constant delay.* Appl. Math. Inf. Sci. 7(1), 355-361 (2013)

[30] Tunç, C.; *Stability and boundedness in differential systems of third order with variable delay Stability and boundedness in differential systems of third order with variable delay.* Proyecciones 35(3), 317-338 (2016)

[31] Tunç, C.; *Qualitative behaviors of functional differential equations of third order with multiple deviating arguments.* Abstract and Applied Analysis. Volume 2012 Article ID 392386, http://dx.doi.org/10.1155/2012/392386, 12 pages (2012)

[32] Tunç, C; *Existence of periodic solutions to nonlinear differential equations of third order with multiple deviating arguments.* International Journal of Differential Equations. Volume 2012, Article ID 406835, doi:10.1155/2012/406835, 13 pages (2012)

[33] Tunç, C.; *On the qualitative behaviors of solutions to a kind of nonlinear third order differential equations with retarded argument.* Italian Journal of Pure and Applied Mathematics. 28, 273 - 284 (2011)

[34] Tunç, C.; *On the stability and boundedness of solutions of nonlinear third order differential equations with delay.* Filomat. 24(3), 1 - 10 (2010)

[35] Tunç, C.; *On the qualitative behaviors of nonlinear functional differential systems of third order.* Advances in nonlinear analysis via the concept of measure of noncompactness, 421 - 439, Springer, Singapore, (2017)

[36] Tunç, C.; *Some stability and boundedness conditions for non-autonomous differential equations with deviating arguments.* Electronic Journal of Qualitative Theory of Differential Equations. 2010(1), 1 - 12 (2010)

[37] Tunç, C., Ergören, H.; *Uniformly boundedness of a class of non linear differential equations of third order with multiple deviating arguments.* Cubo 14(3) , 63 - 69 (2012)

[38] Tunç, C ., Gözen, M.; *Stability and uniform boundedness in multi delay functional differential equations of third order.* Abstract and Applied Analysis. Volume 2013, Article ID 248717, 7, (2013)

[39] Tunç, C., Mohammed, S. A.; *On the qualitative properties of differential equations of third order with retarded argument.* Proyecciones. 33(3), 325 - 347 (2014)

[40] Yao, H., Wang, J.; *Globally asymptotic stability of a kind of third order delay differential system.* International Journal of Nonlinear Science. 10(1), 82 - 87 (2010)

[41] Yoshizawa, T.; *Liapunov's function and boundedness of solutions.* Funkcialaj Ekvacioj. 2, 71 - 103 (1958)

[42] Yoshizawa, T.; *Stability theory and existence of periodic solutions and almost periodic solutions.* Springer-Verlag, New York. Heidelberg. Berlin. (1975)

[43] Yoshizawa, T.; *Stability theory by Liapunov's second method.* The Mathematical Society of Japan. (1966)

[44] Zhu, Y.F.; *On stability, boundedness and existence of periodic solution of a kind of third order nonlinear delay differential system.* Annals of Diff. Equations. 8(2), 249 - 259 (1992)