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On the stability criteria of linear non-autonomous time-varying delay system with application to control problem

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Abstract

In this paper we study the stability of linear non-autonomous time-varying system with multiple delays. Using the Lyapunov functionals method we find sufficient conditions for the exponential stability with a given convergence rate, in terms of linear matrix inequalities or the solution of Riccati differential equations. The results are applied to stabilization problem of linear non-autonomous time-varying control system with multiple delays.

***Keywords*:** Non-autonomous systems, Time-delay systems, Exponential stability, Lyapunov functional, Matrix inequalities, Riccati differential equations.
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1 Introduction

Time-delay systems are frequently encountered in various areas, including physical and chemical processes, biology, economics, engineering, control of the growth of global economy, control of epidemics, etc.

The stability problem of linear time-delay systems has attracted a lot of attention in the few past decades, for example, [1, 5, 6, 7, 10, 14], etc.

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Stability criteria for linear time-delay systems can be divided into two categories, delay-independent stability criteria and delay-dependent stability criteria, by whether the criteria include the information about delays or not. In addition linear time-delay systems can be divided into two categories by the number of delays. Single-delay systems which have only a single time-delay constant, and multiple-delay systems which have at least two time-delay constants.

Most of exciting results of stability were derived using different theories. Some of the stability criteria are directly developed from the characteristic equation and are characterized by the location of eigenvalues and determination of measures and norms of matrices of the system. The main technique used in the stability investigation relies on the using of the Lyapunov functionals method, for example, [2, 16, 17, 18].

One of the extended stability properties is the concept of α-stability, which relates to the exponential stability with a convergence rate $\alpha > 0$. The following retarded system

$$\begin{aligned} &\dot{x} = f(t, x(t), x(t-h)), \quad t \geq 0, \\ &x(t) = \phi(t), \quad t \in [-h, 0], \end{aligned}$$

is α-stable with $\alpha > 0$, if there exists a function $\xi(.)$ such that for each $\phi(.)$, the solution $x(t, \phi)$ of the system satisfies

$$\|x(t, \phi)\| \leq \xi(\|\phi\|)\; e^{-\alpha t}, \quad \text{for all } \; t \geq 0,$$

where $\|\phi\| = \max\{\|\phi(t)\| : \; t \in [-h, 0]\}$. This implies that for $\alpha > 0$ the system can be made exponentially stable with the convergence rate α.

For time-varying systems, the investigation of the exponential stability was treated in [3, 5, 8, 9, 16, 17], where stability conditions for time-varying systems are derived in terms of the solution of a certain Riccati differential equation RDE and linear matrix inequalities LMI.

This paper deals also with the exponential stability with convergence rate of linear non-autonomous time-varying system with multiple delays, and then apply the obtained results to control problems. By using Lyapunov functionals method we show that the existence of the solution of certain RDE or of some LMI guarantees the exponential stability with a given convergence rate of linear non-autonomous time-varying delay systems, and how the results can be applied to obtain sufficient conditions for the stabilizability with a given convergence rate of a class of linear non-autonomous time-varying control delay systems.

This paper is organized as follows. In section 2 the problem is stated and the required notation, definitions and auxiliary proposition are formulated. Section 3 presents the main results for the exponential stability problem with a convergence rate of a linear non-autonomous multiple time-varying delays system. Section 4 devotes to apply the obtained results to the stabilization problem. The paper ends with conclusion and cited references.

2 Preliminaries and Statement of the Problem

We start by introducing some notation and definitions, that will be used throughout the paper.
$\mathbb{R}^+$ denotes the set of all real non-negative numbers.
$\mathbb{R}^n$ denotes the n-dimensional space.
A^τ denotes the transpose of the matrix A.
A is symmetric, if $A = A^\tau$.
$< x, y >$ or $x^\tau y$ denotes the scalar product of two vectors x, y.
$\|x\|$ denotes the Euclidean vector norm of x.
$M^{n\times r}$ denotes the space of all $n \times r$- matrices.
I denotes the identity matrix.
$\lambda(A)$ denotes the set of the eigenvalues of A.
$\lambda_{max}(A) = \max\{Re\lambda : \lambda \in \lambda(A)\}$.
$\eta(A)$ denotes the spectral norm of the matrix A, defined by $\eta(A) = \sqrt{\lambda_{max}(A^\tau A)}$.
$\mu(A)$ denotes the measure of the matrix A, defined by $\mu(A) = \frac{1}{2}\lambda_{max}(A + A^\tau)$.
$L_2([0, t], \mathbb{R}^m)$ denotes the Hilbert space of L_2-integrable and $\mathbb{R}^m$-valued functions on $[0, t]$.
Matrix A is called non-negative definite $A \geq 0$, if $< Ax, x > \geq 0$, for all $x \in \mathbb{R}^n$.
A is called positive definite $A > 0$, if $< Ax, x >> 0$ for all $x \neq 0$; or equivalently there exists $c > 0$, such that

$$< Ax, x > \geq c\|x\|^2, \text{ for all } x \in \mathbb{R}^n.$$

Matrix function $A(t)$ is uniformly positive definite, if there exists $c > 0$, such that

$$< A(t)x, x > \geq c\|x\|^2, \text{ for all } t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^n.$$

$C([-h, 0], \mathbb{R}^n)$ denotes the Banach space of all piecewise-continuous vector functions mapping from $[-h, 0]$ into $\mathbb{R}^n$.
In this paper we consider linear non-autonomous time-varying system with mul-

tiple delays

$$\begin{aligned} &\dot{x}(t) = A_0(t)x(t) + \sum_{i=1}^{m} A_i(t)x(t - h_i(t)), \quad t \in \mathbb{R}^+, \\ &x(t) = \phi(t), \quad t \in [-h, 0], \ h \geq 0, \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $A_0(t), A_i(t) \in M^{n\times n}$, $i = 1, 2, ..., m$ are given matrix functions continuous on $\mathbb{R}^+$, $\phi \in C([-h, 0], \mathbb{R}^n)$ is the initial function and $h_i(t), i = 1, 2, ..., m$ are the time-varying delay functions satisfying

$$0 \leq h_i(t) \leq h, \ \ \dot{h}_i(t) \leq \delta_i < 1, \ \text{ for all } t \geq 0.$$

Definition 2.1. Let $\alpha > 0$ be a given number. System (2.1) is said to be α-stable, if there exists a function $\xi(.) : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $\phi(t) \in C([-h, 0], \mathbb{R}^n)$ the solution $x(t, \phi)$ of the system satisfies

$$\|x(t, \phi)\| \leq \xi(\|\phi\|)\, e^{-\alpha t}, \ \text{ for all } \ t \in \mathbb{R}^+.$$

Let us consider the following free-delay linear time-varying control system $[A(t), B(t)]$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \ \ t \in \mathbb{R}^+, \tag{2.2}$$

where $u(t) \in L_2([0, T], \mathbb{R}^r)$, for all $T > 0$ is the control, $A(t) \in M^{n\times n}$, $B(t) \in M^{n\times r}$.

Definition 2.2. Linear control system (2.2) is said to be α-stabilizable, if there exists a feedback control $u(t) = K(t)x(t)$, $K(t) \in M^{r\times n}$, such that the closed-loop system

$$\dot{x}(t) = [A(t) + B(t)K(t)]x(t), \ \ t \geq 0,$$

is α-stable.

Definition 2.3. Linear control system (2.2) is said to be globally null-controllable, if for every $x_0 \in \mathbb{R}^n$, there is a time $T > 0$ and control $u(t) \in L_2([0, T], \mathbb{R}^r)$ such that the solution $x(t)$ of the system satisfies $x(0) = x_0, \quad x(T) = 0$.

Relationship between the global null-controllability and the existence of the solution of Riccati differential equations is given in the following proposition.

Proposition 2.1. [17] **(Kalman condition)** *Assume that the linear control system $[A(t), B(t)]$ is globally null-controllable. Then for every symmetric matrices $Q(t) \geq 0, P(t) \geq 0$ the Riccati differential equation*

$$\dot{P}(t) + A^\tau(t)P(t) + P(t)A(t) + P(t)B(t)B^\tau(t)P(t) + Q(t) = 0, \quad P(0) = P_0,$$

has the solution $P(t) \geq 0, P(t) = P^\tau(t)$.

The following technical result will be used in the proof of the main results.

Proposition 2.2. [1] **(Completing of the square)** *Assume that* $S \in M^{n\times n}$ *is a symmetric positive definite matrix. Then for every matrix* $Q \in M^{n\times n}$ *we have*

$$2 < Qy, x > - < Sy, y > \leq < QS^{-1}Q^{\tau}x, x >, \quad \textit{for all } x, y \in \mathbb{R}^n.$$

3 Main Results

Consider the linear non-autonomous time-varying system with multiple delays (2.1), where the matrix functions $A_0(t), A_i(t)$ are continuous on $\mathbb{R}^+$. Given positive numbers $\alpha, \epsilon, h_i, \epsilon_i,\ i = 1, 2, ..., m$; we set

$$A_{0,\alpha}(t) = A_0(t) + \alpha I, \qquad A_{i,\alpha}(t) = e^{\alpha h_i(t)} A_i(t), \qquad i = 1, 2, ..., m.$$

$$\mu(A_{0,\alpha}) = \sup_{t\in\mathbb{R}^+} \mu(A_{0,\alpha}(t)), \qquad \eta(A_{i,\alpha}) = \sup_{t\in\mathbb{R}^+} \eta(A_{i,\alpha}(t)),$$

$$Q_i(t) = \frac{2}{1-\delta_i} A_{i,\alpha}(t) A^{\tau}_{i,\alpha}(t),$$

$$\sum_{i=1}^{m} \epsilon_i = m + 2\epsilon\mu(A_{0,\alpha}) + \sum_{i=1}^{m} \frac{2\epsilon^2}{1-\delta_i} \eta^2(A_{i,\alpha}).$$

The main results are stated in the following theorems.

Theorem 3.1. *Linear non-autonomous time-varying system with multiple delays (2.1) is* α*-stable, if there exists a symmetric matrix function* $P(t) > 0$ *satisfying the following Riccati differential equation*

$$\dot{P}(t) + A^{\tau}_{0,\alpha}(t)P(t) + P(t)A_{0,\alpha}(t) + \sum_{i=1}^{m} P(t)Q_i(t)P(t) + \sum_{i=1}^{m} \epsilon_i I = 0. \tag{3.1}$$

Proof. Let $P(t) > 0,\ t \in \mathbb{R}^+$ be a solution of Riccati differential equation (3.1). We take the following change of the state variable

$$y(t) = e^{\alpha t} x(t),$$

where $\alpha > 0$ is the convergence rate. Then the linear delay system (2.1) is transformed to the delay system

$$\begin{aligned} \dot{y}(t) &= A_{0,\alpha}(t)y(t) + \sum_{i=1}^{m} A_{i,\alpha}(t)y(t - h_i(t)), \\ y(t) &= e^{\alpha t}\phi(t), \quad t \in [-h, 0]. \end{aligned} \tag{3.2}$$

Consider the following Lyapunov-Krasovskii functional for the system (3.2) as

$$V(t, y(t)) = V_1 + V_2 + V_3, \tag{3.3}$$

where

$$V_1 =< P(t)y(t), y(t) >,$$

$$V_2 = \epsilon\|y(t)\|^2,$$

$$V_3 = \sum_{i=1}^{m} \int_{t-h_i(t)}^{t} \|y(s)\|^2 ds.$$

Taking the derivative of $V(t, y(t))$ in t along the solution $y(t)$ of system (3.2) we have

$$\begin{aligned}
\dot{V}(t, y(t)) &\leq< \dot{P}(t)y(t), y(t) > +2 < P(t)A_{0,\alpha}(t)y(t), y(t) > \\
&+ 2\sum_{i=1}^{m} < P(t)A_{i,\alpha}(t)y(t - h_i(t)), y(t) > +2\epsilon < A_{0,\alpha}(t)y(t), y(t) > \\
&+ 2\epsilon\sum_{i=1}^{m} < A_{i,\alpha}(t)y(t - h_i(t)), y(t) > +m\|y(t)\|^2 - \sum_{i=1}^{m}(1 - \delta_i)\|y(t - h_i(t))\|^2 \\
&=< [\dot{P}(t) + A^{\tau}_{0,\alpha}(t)P(t) + P(t)A_{0,\alpha}(t) + mI]y(t), y(t) > \\
&+ 2\sum_{i=1}^{m} < P(t)A_{i,\alpha}(t)y(t - h_i(t)), y(t) > +2\epsilon < A_{0,\alpha}(t)y(t), y(t) > \\
&+ 2\epsilon\sum_{i=1}^{m} < A_{i,\alpha}(t)y(t - h_i(t)), y(t) > - \sum_{i=1}^{m}(1 - \delta_i)\|y(t - h_i(t))\|^2.
\end{aligned}$$

Therefore we get

$$\begin{aligned}
\dot{V}(t, y(t)) &\leq< [\dot{P}(t) + A^{\tau}_{0,\alpha}(t)P(t) + P(t)A_{0,\alpha}(t) + mI]y(t), y(t) > \\
&+ \epsilon < [A_{0,\alpha}(t) + A^{\tau}_{0,\alpha}(t)]y(t), y(t) > +2\sum_{i=1}^{m} < P(t)A_{i,\alpha}(t)y(t - h_i(t)), y(t) > \\
&- \sum_{i=1}^{m} \frac{1 - \delta_i}{2} < y(t - h_i(t)), y(t - h_i(t)) > \\
&+ 2\epsilon\sum_{i=1}^{m} < A_{i,\alpha}(t)y(t - h_i(t)), y(t) > \\
&- \sum_{i=1}^{m} \frac{1 - \delta_i}{2} < y(t - h_i(t)), y(t - h_i(t)) > .
\end{aligned} \tag{3.4}$$

By using Proposition 2.1 we can obtain

$$\sum_{i=1}^{m}\left\{2 < P(t)A_{i,\alpha}(t)y(t-h_i(t)), y(t) > -\frac{1-\delta_i}{2} < y(t-h_i(t)), y(t-h_i(t)) > \right\}$$
$$\leq \sum_{i=1}^{m}\frac{2}{1-\delta_i} < P(t)A_{i,\alpha}(t)A^{\tau}_{i,\alpha}(t)P(t)y(t), y(t) >,$$

$$\sum_{i=1}^{m}\left\{2\epsilon < A_{i,\alpha}(t)y(t-h_i(t)), y(t) > \right. \tag{3.5}$$
$$\left. -\frac{1-\delta_i}{2} < y(t-h_i(t)), y(t-h_i(t)) > \right\}$$
$$\leq \sum_{i=1}^{m}\frac{2\epsilon^2}{1-\delta_i} < A_{i,\alpha}(t)A^{\tau}_{i,\alpha}(t)y(t), y(t) > .$$

Thus we can write (3.4) in the following form

$$\begin{aligned}\dot{V}(t, y(t)) \leq & < [\dot{P}(t) + A^{\tau}_{0,\alpha}(t)P(t) + P(t)A_{0,\alpha}(t) + mI]y(t), y(t) > \\ & + \epsilon < [A_{0,\alpha}(t) + A^{\tau}_{0,\alpha}(t)]y(t), y(t) > \\ & + \sum_{i=1}^{m}\left\{\frac{2}{1-\delta_i} < P(t)A_{i,\alpha}(t)A^{\tau}_{i,\alpha}(t)P(t)y(t), y(t) > \right. \\ & \left. + \frac{2\epsilon^2}{1-\delta_i} < A_{i,\alpha}(t)A^{\tau}_{i,\alpha}(t)y(t), y(t) > \right\}.\end{aligned}$$

If we let

$$\begin{aligned} & < [A_{0,\alpha}(t) + A^{\tau}_{0,\alpha}(t)]y(t), y(t) > \leq 2\mu(A_{0,\alpha})\|y(t)\|^2, \\ & \sum_{i=1}^{m} < A_{i,\alpha}(t)A^{\tau}_{i,\alpha}(t)y(t), y(t) > \leq \sum_{i=1}^{m}\eta^2(A_{i,\alpha})\|y(t)\|^2. \end{aligned} \tag{3.6}$$

Then we have

$$\begin{aligned}\dot{V}(t, y(t)) \leq & < [\dot{P}(t) + A^{\tau}_{0,\alpha}(t)P(t) + P(t)A_{0,\alpha}(t) + \sum_{i=1}^{m}P(t)Q_i(t)P(t) \\ & + mI]y(t), y(t) > + 2\epsilon\mu(A_{0,\alpha})\|y(t)\|^2 \\ & + \sum_{i=1}^{m}\frac{2\epsilon^2}{1-\delta_i}\eta^2(A_{i,\alpha})\|y(t)\|^2.\end{aligned}$$

Because

$$\sum_{i=1}^{m}\epsilon_i = m + 2\epsilon\mu(A_{0,\alpha}) + \sum_{i=1}^{m}\frac{2\epsilon^2}{1-\delta_i}\eta^2(A_{i,\alpha}).$$

Therefore we find

$$\dot{V}(t,y(t)) \leq < [\dot{P}(t) + A_{0,\alpha}^{\tau}(t)P(t) + P(t)A_{0,\alpha}(t) + \sum_{i=1}^{m} P(t)Q_i(t)P(t)$$
$$+ \sum_{i=1}^{m} \epsilon_i I]y(t), y(t) > .$$

Since $P(t)$ is a solution of (3.1), it follows that

$$\dot{V}(t,y(t)) \leq 0, \qquad \text{for all } t \in \mathbb{R}^+.$$

By integrating both sides of this inequality from 0 to t we find

$$V(t,y(t)) - V(0,y(0)) \leq 0, \qquad \text{for all } t \in \mathbb{R}^+,$$

then we get

$$< P(t)y(t), y(t) > +\epsilon\|y(t)\|^2 + \sum_{i=1}^{m} \int_{t-h_i(t)}^{t} \|y(s)\|^2 ds$$
$$\leq < P(0)y(0), y(0) > +\epsilon\|y(0)\|^2 + \sum_{i=1}^{m} \int_{-h_i(0)}^{0} \|y(s)\|^2 ds,$$

where $P_0 = P(0) > 0$ is any initial condition. Since

$$< P(t)y(t), y(t) > \geq 0, \quad \sum_{i=1}^{m} \int_{t-h_i(t)}^{t} \|y(s)\|^2 ds \geq 0,$$
$$\sum_{i=1}^{m} \int_{-h_i(0)}^{0} \|y(s)\|^2 ds \leq \sum_{i=1}^{m} \int_{-h_i(0)}^{0} \|e^{\alpha s}\phi(s)\|^2 ds \leq \|\phi\|^2 \sum_{i=1}^{m} \int_{-h_i(0)}^{0} e^{2\alpha s} ds$$
$$= \frac{1}{2\alpha} \sum_{i=1}^{m} (1 - e^{-2\alpha h_i(0)})\|\phi\|^2,$$

it follows that

$$\|y(t)\|^2 \leq \frac{1}{\epsilon}\{< P(0)y(0), y(0) > +\epsilon\|y(0)\|^2 + \frac{1}{2\alpha} \sum_{i=1}^{m} (1 - e^{-2\alpha h_i(0)})\|\phi\|^2\}.$$

Therefore the solution $y(t,\phi)$ of the system (3.2) is bounded, returning to the solution $x(t,\phi)$ of system (2.1) and noting that

$$\|y(0)\| = \|x(0)\| = \phi(0) \leq \|\phi\|,$$

we have

$$\|x(t,\phi)\| \leq \xi(\|\phi\|)e^{-\alpha t}, \text{ for all } t \in \mathbb{R}^+,$$

where

$$\xi(\|\phi\|) := [\frac{1}{\epsilon}\{\|P(0)\|\|\phi\|^2 + \epsilon\|\phi\|^2 + \frac{1}{2\alpha}\sum_{i=1}^{m}(1 - e^{-2\alpha h_i(0)})\|\phi\|^2\}]^{\frac{1}{2}}.$$

This implies that system (2.1) is α-stable, which completes the proof.

Theorem 3.2. *Linear non-autonomous time-varying system with multiple delays (2.1) is α-stable, if there exists a symmetric matrix function $P(t) > 0$, such that the following condition LMI holds*

$$\begin{pmatrix} \chi(t) & P(t)A_{1,\alpha}(t) + \epsilon A_{1,\alpha}(t) & ... & P(t)A_{m,\alpha}(t) + \epsilon A_{m,\alpha}(t) \\ A^{\tau}_{1,\alpha}(t)P(t) + \epsilon A^{\tau}_{1,\alpha}(t) & -(1-\delta_1)I & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A^{\tau}_{m,\alpha}(t)P(t) + \epsilon A^{\tau}_{m,\alpha}(t) & 0 & ... & -(1-\delta_m)I \end{pmatrix} < 0, \quad (3.7)$$

where

$$\chi(t) = \dot{P}(t) + A^{\tau}_{0,\alpha}(t)P(t) + P(t)A_{0,\alpha}(t) + \epsilon\{A_{0,\alpha}(t) + A^{\tau}_{0,\alpha}(t)\} + mI.$$

Then the system (2.1) *is α-stable.*

Proof. Regarding to the above condition, we can reset relation (3.4) as follows:

$$\dot{V}(t, y(t)) \leq < [\dot{P}(t) + A^{\tau}_{0,\alpha}(t)P(t) + P(t)A_{0,\alpha}(t) + mI]y(t), y(t) >$$
$$+ \epsilon < [A_{0,\alpha}(t) + A^{\tau}_{0,\alpha}(t)]y(t), y(t) > +2\sum_{i=1}^{m} < P(t)A_{i,\alpha}(t)y(t - h_i(t)), y(t) >$$
$$+ 2\epsilon\sum_{i=1}^{m} < A_{i,\alpha}(t)y(t - h_i(t)), y(t) > - \sum_{i=1}^{m}(1 - \delta_i)\|y(t - h_i(t))\|^2$$
$$= Z^{\tau}(t) \begin{pmatrix} \chi(t) & P(t)A_{1,\alpha}(t) + \epsilon A_{1,\alpha}(t) & ... & P(t)A_{m,\alpha}(t) + \epsilon A_{m,\alpha}(t) \\ A^{\tau}_{1,\alpha}(t)P(t) + \epsilon A^{\tau}_{1,\alpha}(t) & -(1-\delta_1)I & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A^{\tau}_{m,\alpha}(t)P(t) + \epsilon A^{\tau}_{m,\alpha}(t) & 0 & ... & -(1-\delta_m)I \end{pmatrix} Z(t),$$

where

$$Z(t) := [y(t),\ y(t-h_1(t)),\ \cdots,\ y(t-h_m(t))].$$

Therefore by condition (3.7) there is a constant $\beta > 0$ such that

$$\dot{V}(t, y(t)) \leq -\beta \|Z(t)\|^2, \text{ for all } t \in \mathbb{R}^+.$$

Since $\|Z(t)\|^2 \geq \|y(t)\|^2$ we have

$$\dot{V}(t, y(t)) \leq -\beta \|y(t)\|^2, \text{ for all } t \in \mathbb{R}^+. \tag{3.8}$$

By integrating both sides of (3.8) from 0 to t we get

$$V(t, y(t)) - V(0, y(0)) \leq -\beta \int_0^t \|y(s)\|^2 ds.$$

Hence

$$< P(t)y(t), y(t) > +\epsilon \|y(t)\|^2 + \sum_{i=1}^m \int_{t-h_i(t)}^t \|y(s)\|^2 ds - < P(0)y(0), y(0) >$$

$$- \epsilon \|y(0)\|^2 - \sum_{i=1}^m \int_{-h_i(0)}^0 \|y(s)\|^2 ds \leq -\beta \int_0^t \|y(s)\|^2 ds.$$

Since

$$P_0 = P(0) > 0, \quad < P(t)y(t), y(t) > \geq 0, \ \sum_{i=1}^m \int_{t-h_i(t)}^t \|y(s)\|^2 ds \geq 0,$$

$$\epsilon \|y(t)\|^2 \geq 0, \ \sum_{i=1}^m \int_{-h_i(0)}^0 \|y(s)\|^2 ds \leq \frac{1}{2\alpha} \sum_{i=1}^m (1 - e^{-2\alpha h_i(0)}) \|\phi\|^2,$$

it follows that

$$\int_0^t \|y(s)\|^2 ds \leq \frac{1}{\beta} [< P(0)y(0), y(0) > +\epsilon \|y(0)\|^2 + \frac{1}{2\alpha} \sum_{i=1}^m (1 - e^{-2\alpha h_i(0)}) \|\phi\|^2].$$

Letting $t \to \infty$ and noting that $P(0) > 0$, we obtain that $\int_0^\infty \|y(s)\|^2 ds < \infty$, which proves that $y(t) \in L_2([0, \infty), \mathbb{R}^n)$ and hence the solution $y(t, \phi)$ which is a continuously differentiable function of linear system (3.2) is bounded, there exists a function
$\xi(.) : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|y(t, \phi)\| \leq \xi(\|\phi\|), \quad \text{for all } t \geq 0.$$

Returning to the solution $x(t, \phi)$ of system (2.1) and noting that

$$\|y(0)\| = \|x(0)\| = \phi(0) \leq \|\phi\|,$$

we have

$$\|x(t, \phi)\| \leq \xi(\|\phi\|)e^{-\alpha t}, \text{ for all } t \in \mathbb{R}^+,$$

Then the system (2.1) is α-stable.
Thus the proof of Theorem 3.2 is now complete.

Example 3.1. Consider the following linear non-autonomous time-varying delay system

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t - h_1(t)), \quad t \in \mathbb{R}^+,$$

with any initial function $\phi \in C([-\frac{1}{2}, 0], \mathbb{R}^2)$, time-varying delay $h_1(t) = \frac{1}{2}sin^2\frac{t}{2}$ and

$$A_0(t) = \begin{pmatrix} -\frac{1}{2} - 6e^{-t}sin^2t - e^t & 0 \\ 0 & -\frac{1}{2} - \frac{3}{2}e^{-t}cos^2t - e^t \end{pmatrix},$$

$$A_1(t) = \begin{pmatrix} \sqrt{3}e^{-\frac{1}{2}sin^2\frac{t}{2}}sint & 0 \\ 0 & \sqrt{\frac{3}{2}}e^{-\frac{1}{2}sin^2\frac{t}{2}}cost \end{pmatrix}.$$

Therefore we have

$$m = 1, \quad h = \frac{1}{2} \text{ and } \delta_1 = \frac{1}{2}.$$

For $\alpha = 1$, we obtain

$$A_{0,\alpha}(t) = \begin{pmatrix} \frac{1}{2} - 6e^{-t}sin^2t - e^t & 0 \\ 0 & \frac{1}{2} - \frac{3}{2}e^{-t}cos^2t - e^t \end{pmatrix},$$

$$A_{1,\alpha}(t) = \begin{pmatrix} \sqrt{3}sint & 0 \\ 0 & \sqrt{\frac{3}{2}}cost \end{pmatrix}.$$

Hence we find

$$\mu(A_{0,\alpha}) = -2, \quad \eta(A_{1,\alpha}) = \sqrt{3} \text{ and}$$

$$Q_1(t) = \begin{pmatrix} 12sin^2t & 0 \\ 0 & 6cos^2t \end{pmatrix}.$$

Let $\epsilon = \frac{1}{2}$, we have $\epsilon_1 = 2$.
Then the solution of the following Riccati differential equation

$$\dot{P}(t) + A^{\tau}_{0,\alpha}(t)P(t) + P(t)A_{0,\alpha}(t) + P(t)Q_1P(t) + \epsilon_1 I = 0,$$

is found by

$$P(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix} > 0, \quad \text{for all } t \in \mathbb{R}^+.$$

Thus, by Theorem 3.1, the system is 1-stable.

4 Application to control problem

We apply the above results to the strong stabilizability problem of the following linear non-autonomous multiple time-varying delays system with control problem

$$\dot{x}(t) = A_0(t)x(t) + \sum_{i=1}^{m} A_i(t)x(t - h_i(t)) + B(t)u(t), \quad t \in \mathbb{R}^+, \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}^r$ is the control, $A_0(t), A_i(t) \in M^{n\times n}; i = 1, 2, ..., m$ and $B(t) \in M^{n\times r}$ are given matrix functions continuous on $\mathbb{R}^+$.
Strong stabilizability means that for every given number $\alpha > 0$, there exists a delay-free feedback control $u(t) = K(t)x(t)$, $K(t) \in M^{r\times n}$ such that the closed-loop system

$$\dot{x}(t) = [A_0(t) + B(t)K(t)]x(t) + \sum_{i=1}^{m} A_i(t)x(t - h_i(t)),$$

is exponentialy stable with the rate convergence α.

Recently the stabilization problem has been studied by using Lyapunov functionals method, for example, [2, 3], etc.

As a direct consequence of Theorem 3.1 and Theorem 3.2 we obtain the following sufficient conditions for the strong stabilizability in terms of Linear matrix inequality and the solution of Riccati differential equation.

Theorem 4.1. *Linear non-autonomous time-varying control system with multiple delays* (4.1) *is α-stabilizable, if there exists a symmetric matrix function $P(t) > 0$ satisfying the following Riccati differential equation*

$$\dot{P}(t) + A^{\tau}_{0,\alpha}(t)P(t) + P(t)A_{0,\alpha}(t) - \sum_{i=1}^{m} P(t)\bar{Q}_i(t)P(t) + \sum_{i=1}^{m} \bar{\epsilon}_i I = 0, \tag{4.2}$$

where

$$\|B\| = \sup_{t\in\mathbb{R}^+} \|B(t)\|,$$

$$\sum_{i=1}^{m} \bar{Q}_i(t) = B(t)B^\tau(t) - \sum_{i=1}^{m} \frac{2}{1-\delta_i} A_{i,\alpha}(t)A^\tau_{i,\alpha}(t),$$

$$\sum_{i=1}^{m} \bar{\epsilon}_i = m + \epsilon^2\|B\|^2 + 2\epsilon\mu(A_{0,\alpha}) + \sum_{i=1}^{m} \frac{2\epsilon^2}{1-\delta_i}\eta^2(A_{i,\alpha}).$$

Moreover the feedback control is determined by

$$u(t) = -\frac{1}{2}B^\tau(t)[P(t) - \epsilon I]x(t),$$

where $P(t)$ is the solution of Riccati differential equation (4.2).

Proof. Let us make a variable transformation

$$y(t) = e^{\alpha t}x(t), \quad t \ge 0.$$

Under the feedback control $u(t) = K(t)x(t)$, where $K(t) = -\frac{1}{2}B^\tau(t)[P(t) - \epsilon I]$, the system (4.1) is reduced to

$$\begin{aligned} \dot{y}(t) &= [A_{0,\alpha}(t) + B(t)K(t)]y(t) + \sum_{i=1}^{m} A_{i,\alpha}(t)y(t - h_i(t)), \\ y(t) &= e^{\alpha t}\phi(t), \quad t \in [-h, 0]. \end{aligned} \tag{4.3}$$

For the system (4.3), we apply the same Lyapunov-Krasovskii functional (3.3). Then for the derivative of $V(t, y(t))$ in t along the solution $y(t)$ of system (4.3) we have

$$\begin{aligned} \dot{V}(t, y(t)) \le &< [\dot{P}(t) + A^\tau_{0,\alpha}(t)P(t) + P(t)A_{0,\alpha}(t) - P(t)B(t)B^\tau(t)P(t) \\ &+ mI]y(t), y(t) > + < [\epsilon^2 B(t)B^\tau(t) + \epsilon(A_{0,\alpha}(t) + A^\tau_{0,\alpha}(t))]y(t), y(t) > \\ &\quad + 2\epsilon \sum_{i=1}^{m} < A_{i,\alpha}(t)y(t - h_i(t)), y(t) > \\ &\quad + 2\sum_{i=1}^{m} < P(t)A_{i,\alpha}(t)y(t - h_i(t)), y(t) > \\ &\quad - \sum_{i=1}^{m}(1 - \delta_i) < y(t - h_i(t)), y(t - h_i(t)) > . \end{aligned} \tag{4.4}$$

From the inequalities (3.5) and (3.6), then we can obtain

$$\dot{V}(t,y(t)) \le < [\dot{P}(t) + A^{\tau}_{0,\alpha}(t)P(t) + P(t)A_{0,\alpha}(t) - \sum_{i=1}^{m} P(t)\bar{Q}_i(t)P(t) \\ + mI]y(t), y(t) > +\epsilon^2\|B\|^2 < y(t), y(t) > +2\epsilon\mu(A_{0,\alpha})\|y(t)\|^2 \\ + \sum_{i=1}^{m} \frac{2\epsilon^2}{1-\delta_i}\eta^2(A_{i,\alpha})\|y(t)\|^2.$$

Let

$$\sum_{i=1}^{m} \bar{\epsilon}_i = m + \epsilon^2\|B\|^2 + 2\epsilon\mu(A_{0,\alpha}) + \sum_{i=1}^{m} \frac{2\epsilon^2}{1-\delta_i}\eta^2(A_{i,\alpha}).$$

Thus we get

$$\dot{V}(t,y(t)) \le < [\dot{P}(t) + A^{\tau}_{0,\alpha}(t)P(t) + P(t)A_{0,\alpha}(t) - \sum_{i=1}^{m} P(t)\bar{Q}_i(t)P(t) \\ + \sum_{i=1}^{m} \bar{\epsilon}_i I]y(t), y(t) > .$$

Since $P(t)$ is a solution of (4.2), therefore

$$\dot{V}(t,y(t)) \le 0, \qquad \text{for all } t \in \mathbb{R}^+.$$

Since the completion of this proof is similar to that of Theorem 3.1, which implies that system (4.1) is α-stabilizable.

Theorem 4.2. *Linear non-autonomous time-varying control system with multiple delays* (4.1) *is α-stabilizable, if there exists a symmetric matrix function $P(t) > 0$, such that the following condition LMI holds*

$$\begin{pmatrix} \bar{\chi}(t) & P(t)A_{1,\alpha}(t) + \epsilon A_{1,\alpha}(t) & \dots & P(t)A_{m,\alpha}(t) + \epsilon A_{m,\alpha}(t) \\ A^{\tau}_{1,\alpha}(t)P(t) + \epsilon A^{\tau}_{1,\alpha}(t) & -(1-\delta_1)I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A^{\tau}_{m,\alpha}(t)P(t) + \epsilon A^{\tau}_{m,\alpha}(t) & 0 & \dots & -(1-\delta_m)I \end{pmatrix} < 0, \tag{4.5}$$

where

$$\|B\| = \sup_{t\in\mathbb{R}^+} \|B(t)\|,$$

$$\tilde{\epsilon}_1 = m + \epsilon^2\|B\|^2,$$

$$\begin{aligned}\bar{\chi}(t) =& \dot{P}(t) + A^{\tau}_{0,\alpha}(t)P(t) + P(t)A_{0,\alpha}(t) - P(t)B(t)B^{\tau}(t)P(t)\\ &+ \epsilon\{A_{0,\alpha}(t) + A^{\tau}_{0,\alpha}(t)\} + \tilde{\epsilon}_1 I.\end{aligned}$$

Moreover the feedback control is determined by

$$u(t) = -\frac{1}{2}B^{\tau}(t)[P(t) - \epsilon I]x(t).$$

Proof. Regarding to the above condition, we can write the relation (4.4) in the following form:

$$\begin{aligned}\dot{V}(t, y(t)) \leq &< [\dot{P}(t) + A^{\tau}_{0,\alpha}(t)P(t) + P(t)A_{0,\alpha}(t) - P(t)B(t)B^{\tau}(t)P(t)\\ &+ \epsilon(A_{0,\alpha}(t) + A^{\tau}_{0,\alpha}(t)) + \epsilon^2\|B\|^2 + mI]y(t), y(t) >\\ &+ 2\epsilon\sum_{i=1}^{m} < A_{i,\alpha}(t)y(t-h_i(t)), y(t) > +2\sum_{i=1}^{m} < P(t)A_{i,\alpha}(t)y(t-h_i(t)), y(t) >\\ &- \sum_{i=1}^{m}(1-\delta_i) < y(t-h_i(t)), y(t-h_i(t)) >\\ &= Z^{\tau}(t)\end{aligned}$$

$$\begin{pmatrix} \bar{\chi}(t) & P(t)A_{1,\alpha}(t) + \epsilon A_{1,\alpha}(t) & \ldots & P(t)A_{m,\alpha}(t) + \epsilon A_{m,\alpha}(t)\\ A^{\tau}_{1,\alpha}(t)P(t) + \epsilon A^{\tau}_{1,\alpha}(t) & -(1-\delta_1)I & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots\\ A^{\tau}_{m,\alpha}(t)P(t) + \epsilon A^{\tau}_{m,\alpha}(t) & 0 & \ldots & -(1-\delta_m)I \end{pmatrix}$$

$$Z(t),$$

where

$$Z(t) := [y(t),\ y(t-h_1(t)),\ \cdots,\ y(t-h_m(t))].$$

Therefore by condition (4.5) there is a constant $\beta > 0$ such that

$$\dot{V}(t, y(t)) \leq -\beta\|Z(t)\|^2, \text{ for all } \ t \in \mathbb{R}^+.$$

Since the completion of this proof is the same to the proof of Theorem 3.2, then the linear non-autonomous time-varying control system with multiple delays (4.1) is α-stabilizable.

Example 4.1. Consider the following linear non-autonomous time-varying delay system with control problem

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-h_1(t)) + B(t)u(t), \ \ t \in \mathbb{R}^+,$$

with any initial function $\phi \in C([-\frac{1}{2}, 0], \mathbb{R}^2)$, the time-varying delay $h_1(t)$ is the same of Example 3.1, and

$$A_0(t) = \begin{pmatrix} -\frac{1}{2} + 4e^{-t}sin^2t - \frac{5}{2}e^t & 0 \\ 0 & -\frac{1}{2} + e^{-t}cos^2t - \frac{5}{2}e^t \end{pmatrix},$$

$$A_1(t) = \begin{pmatrix} \sqrt{2}e^{-\frac{1}{2}sin^2\frac{t}{2}}sint & 0 \\ 0 & \sqrt{\frac{3}{2}}e^{-\frac{1}{2}sin^2\frac{t}{2}}cost \end{pmatrix} \text{ and } B(t) = \begin{pmatrix} 4sint & 0 \\ 0 & 2\sqrt{2}cost \end{pmatrix}.$$

For $\alpha = 1$, therefore we obtain

$$m = 1,\ h = \frac{1}{2},\ \delta_1 = \frac{1}{2},\ \mu(A_{0,\alpha}) = -2,\ \eta(A_{1,\alpha}) = \sqrt{2} \text{ and } \|B\| = 4.$$

Let $\epsilon = \frac{1}{2}$, we find $\bar{\epsilon}_1 = 4$ and $\bar{Q}_1(t) = \begin{pmatrix} 4sin^2t & 0 \\ 0 & 2cos^2t \end{pmatrix}$.
Then the solution of the following Riccati differential equation

$$\dot{P}(t) + A^{\tau}_{0,\alpha}(t)P(t) + P(t)A_{0,\alpha}(t) - P(t)\bar{Q}_1(t)P(t) + \bar{\epsilon}_1 I = 0,$$

is found by

$$P(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix} > 0, \quad \text{for all } t \in \mathbb{R}^+.$$

Thus, by Theorem 4.1, the system is 1-stabilizable with feedback control

$$u(t) = \begin{pmatrix} (1 - 2e^{-t})sint & 0 \\ 0 & \frac{1}{\sqrt{2}}(1 - 2e^{-t})cost \end{pmatrix}.$$

5 Conclusion

In this paper we have presented sufficient conditions for the α-stable (exponential stability with a given rate α) of a class of linear non-autonomous time-varying system with multiple delays. The conditions are derived in terms of the solution of Riccati differential equation or of linear matrix inequality. The results are applied to obtain α-stabilizable conditions for linear non-autonomous time-varying control system with multiple delays.

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