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Numerical methods

Numerical identification of the dependence of the right side of the wave equation on the spatial variable

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Abstract. The problem of identifying the multiplier of the right side of a one-dimensional wave equation depending on a spatial variable is considered. As additional information, the condition of the final redefinition is set. A discrete analogue of the inverse problem is constructed using the finite difference method. To solve the resulting difference problem, a special representation is proposed, with the help of which the difference problem splits into two independent difference problems. As a result, an explicit formula is obtained for determining the approximate value of the desired function for each discrete value of a spatial variable. The presented results of numerical experiments conducted for model problems demonstrate the effectiveness of the proposed computational algorithm.

Keywords: wave equation, identification of the right side of the wave equation, inverse problem, final redefinition, difference problem.

1. Introduction

It is known that inverse problems of determining the right-hand sides of wave equations arise in the mathematical modeling of many physical processes in geophysics, seismics, electrodynamics, thermophysics, medicine and many other fields of science and technology [1-4]. In these inverse problems, in addition to solving the wave equation, it is necessary to determine the dependence of the right parts on time or on spatial coordinates. Usually, when considering the problems of identifying the right-hand sides of partial differential equations, two independent tasks are distinguished: identification of dependence on time and identification of dependence on a spatial variable. Most publications devoted to the problems of identifying the right-hand sides of wave equations mainly consider the case when the right-hand side depends on time and investigate the issues of correctness, existence, unambiguous solvability of problems, etc. [5-11].

In this paper, we propose a numerical method for solving problems of identifying the right-hand sides of wave equations, in cases where the dependence of the right-hand side on a spatial variable is unknown.

2. Problem statement and solution method

Let a one- dimensional wave equation be considered

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}(k(x)\frac{\partial u(x,t)}{\partial x}) + p(x)f(x,t), \qquad 0<x<1,\ 0<t\le T, \tag{1}$$

with the following initial

$$u(x,0) = \varphi(x), \tag{2}$$

$$\frac{\partial u(x,0)}{\partial t} = \psi(x), \tag{3}$$

and boundary conditions

$$u(0,t) = q(t), \tag{4}$$

$$u(1,t) = v(t). \tag{5}$$

It is known that the direct problem for equation (1) consists in determining a function $u(x,t)$ from equation (1) with a given coefficient $k(x)$, the right side $p(x)f(x,t)$ and conditions (2)–(5).

Suppose that in addition to the function $u(x,t)$, the function $p(x)$ is also unknown and it is required to restore this function according to the following condition of the final redefinition

$$u(x,T) = r(x), \tag{6}$$

where $r(x)$ is the given function.

Thus, the task is to determine the functions $u(x,t)$ and $p(x)$ satisfying equation (1) and conditions (2) – (6). It should be noted that the mathematical description of the action of external forces in wave processes can always be reduced to a representation $p(x)f(x,t)$ (in particular cases it may be $f(x,t) = const$). At the same time, the separation of the function $f(x,t)$ in time and space is not considered necessary.

The problem belongs to the class of inverse problems associated with the restoration of the right parts of partial differential equations. We assume that the formulated inverse problem (1)–(6) is uniquely solvable. Note that the unique solvability of this class of problems was studied in detail in [12–14].

To solve the problem (1)–(6), we first construct its discrete analogue using the finite difference method. We introduce a uniform difference grid

$$\bar{\omega} = \{(x_i, t_j) : x_i = i\Delta x,\ t_j = j\Delta t,\ i = 0,1,2,...n,\ j = 0,1,2,...m\}$$

in the rectangular area $\{0 \le x \le 1,\ 0 \le t \le T\}$ with the increment $\Delta x = 1/n$ of the variable x and the increment $\Delta t = T/m$ of the time t. To equation (1) in the inner nodes of the grid $\bar{\omega}$, we will match an implicit difference scheme

$$\frac{u_i^{j+1}-2u_i^j+u_i^{j-1}}{\Delta t^2}=\frac{1}{\Delta x}\left[k_{i+1/2}\frac{u_{i+1}^{j+1}-u_i^{j+1}}{\Delta x}-k_{i-1/2}\frac{u_i^{j+1}-u_{i-1}^{j+1}}{\Delta x}\right]+p_i f_i^{j+1}, \quad (7)$$

$i=\overline{1,n-1}, \quad j=\overline{1,m-1}.$

The difference analogs of the initial conditions (2), (3) and boundary conditions (4), (5) are written as

$$u_i^0=\varphi_i, \quad (8)$$

$$\frac{u_i^1-u_i^0}{\Delta t}=\psi_i, \quad (9)$$

$$u_0^{j+1}=q^{j+1}, \quad (10)$$

$$u_n^{j+1}=v^{j+1}, \quad (11)$$

where $u_i^j\approx u(x_i,t_j)$, $p_i\approx p(x_i)$, $q^{j+1}=q(t_{j+1})$, $v^{j+1}=v(t_{j+1})$, $\psi_i=\psi(x_i)$, $\varphi_i=\varphi(x_i)$, $f_i^{j+1}=f(x_i,t_{j+1})$, $k_{i\pm1/2}=k(x_i\pm\Delta x/2)$.

The difference analogue of the final redefinition condition (6) is represented as

$$u_i^m=r_i, \qquad i=0,\ 1,\ 2,\ ...,\ n, \quad (12)$$

where $r_i=r(x_i)$.

The constructed difference problem (7)–(12) is a system of linear algebraic equations in which the approximate values of the desired functions $u(x,t)$ and $p(x)$ in the nodes of the difference grid $\overline{\omega}$ act as unknowns, i.e. u_i^j, p_i, $i=0,1,\ 2,\ ...,\ n$, $j=0,1,\ 2,\ ....,m$.

To split the system of difference equations (6)-(12) into mutually independent subsystems, each of which can be solved independently, the solution of this system for each fixed value $j=0,1,2,....,m$ is represented as [15–17]

$$u_i^j=w_i^j+p_i\,\theta_i^j, \qquad i=0,1,\ 2,\ ...,\ n, \quad (13)$$

where w_i^j, θ_i^j are unknown variables. Substituting the expression u_i^j into equation (7), we obtain

$$\frac{w_i^{j+1}+p_i\,\theta_i^{j+1}-2w_i^j-2p_i\,\theta_i^j+w_i^{j-1}+p_i\,\theta_i^{j-1}}{\Delta t^2}=$$

$$=\frac{1}{\Delta x}\left[k_{i+1/2}\frac{w_{i+1}^{j+1}+p_{i+1}\theta_{i+1}^{j+1}-w_i^{j+1}-p_i\,\theta_i^{j+1}}{\Delta x}-k_{i-1/2}\frac{w_i^{j+1}+p_i\,\theta_i^{j+1}-w_{i-1}^{j+1}-p_{i-1}\theta_{i-1}^{j+1}}{\Delta x}\right]+$$

$$+p_i f_i^{j+1}.$$

Accepting $p_{i-1} \approx p_{i+1} \approx p_i$ the last ratio is written as

$$\left[\frac{w_i^{j+1} - 2w_i^j + w_i^{j-1}}{\Delta t^2} - \frac{1}{\Delta x}\left(k_{i+1/2}\frac{w_{i+1}^{j+1} - w_i^{j+1}}{\Delta x} - k_{i-1/2}\frac{w_i^{j+1} - w_{i-1}^{j+1}}{\Delta x}\right)\right] +$$

$$+ p_i\left[\frac{\theta_i^{j+1} - 2\theta_i^j + \theta_i^{j-1}}{\Delta t^2} - \frac{1}{\Delta x}\left(k_{i+1/2}\frac{\theta_{i+1}^{j+1} - \theta_i^{j+1}}{\Delta x} - k_{i-1/2}\frac{\theta_i^{j+1} - \theta_{i-1}^{j+1}}{\Delta x}\right) - f_i^{j+1}\right] = 0. \qquad (14)$$

Substituting representation (13) into (8)–(11), give

$$w_i^0 + p_i\,\theta_i^0 = \varphi_i, \qquad (15)$$

$$\frac{w_i^1 + p_i\,\theta_i^1 - w_i^0 - p_i\,\theta_i^0}{\Delta t} = \psi_i, \qquad (16)$$

$$w_0^{j+1} + p_0\,\theta_0^{j+1} = q^{j+1}, \qquad (17)$$

$$w_n^{j+1} + p_n\,\theta_n^{j+1} = v^{j+1}. \qquad (18)$$

Assume that the auxiliary variables w_i^j and θ_i^j, $j = 0, 1, 2, \ldots, m$, $i = 0, 1, 2, \ldots, n$ are solutions to the following two independent difference problems:

$$\frac{w_i^{j+1} - 2w_i^j + w_i^{j-1}}{\Delta t^2} - \frac{1}{\Delta x}\left(k_{i+1/2}\frac{w_{i+1}^{j+1} - w_i^{j+1}}{\Delta x} - k_{i-1/2}\frac{w_i^{j+1} - w_{i-1}^{j+1}}{\Delta x}\right) = 0, \qquad (19)$$

$$i = \overline{1, n-1}, \quad j = \overline{1, m-1},$$

$$w_i^0 = \varphi_i, \qquad (20)$$

$$\frac{w_i^1 - w_i^0}{\Delta t} = \psi_i, \qquad (21)$$

$$w_0^{j+1} = q^{j+1}, \qquad (22)$$

$$w_n^{j+1} = v^{j+1}. \qquad (23)$$

$$\frac{\theta_i^{j+1} - 2\theta_i^j + \theta_i^{j-1}}{\Delta t^2} - \frac{1}{\Delta x}\left(k_{i+1/2}\frac{\theta_{i+1}^{j+1} - \theta_i^{j+1}}{\Delta x} - k_{i-1/2}\frac{\theta_i^{j+1} - \theta_{i-1}^{j+1}}{\Delta x}\right) - \gamma_i\theta_i^{j+1} - f_i^{j+1} = 0, \qquad (24)$$

$$i = \overline{1, n-1}, \quad j = \overline{1, m-1},$$

$$\theta_i^0 = 0, \tag{25}$$

$$\frac{\theta_i^1 - \theta_i^0}{\Delta t} = 0, \tag{26}$$

$$\theta_0^{j+1} = 0, \tag{27}$$

$$\theta_n^{j+1} = 0. \tag{28}$$

It is obvious that in this case equation (14) and conditions (15)–(18) are fulfilled automatically. The difference problems (19)–(23) and (24)–(28) for each fixed value $j = 1, 2, \ldots, m-1$ are systems of linear algebraic equations with a tridiagonal matrix, the solutions of which, regardless of p_i, can be found by the Thomas method [15]. Having determined the values of the variables w_i^j, θ_i^j, $i = 1, 2, ..., n-1$, $j = 1, .2 \ldots, m$ and substituting the representation (13) into (12), we have

$$w_i^m + p_i \theta_i^m = r_i, \quad i = 1, 2, ..., n-1. \tag{29}$$

From condition (29) follows the calculation formula for determining the value of the desired function $p(x)$ for each fixed value $x = x_i$

$$p_i = \frac{r_i - w_i^m}{\theta_i^m}, \quad i = 1, 2, ..., n-1 \tag{30}$$

Thus, the computational algorithm for solving the difference problem (7)–(12) by definition u_i^j, p_i, $i = 0, 1, 2, ..., n$, $j = 1, 2, 3, ..., m-1$, consists of the following stages:

solutions of two independent difference problems (19)–(23) and (24)–(28) with respect to auxiliary variables w_i^j, θ_i^j, $i = 1, 2, ..., n$, $j = 1, .2 \ldots, m$ are determined;

according to formula (30), approximate values of the desired function $p(x)$ are determined for $x = x_i$, i.e. p_i, $i = 1, 2., n-1$;

the values of the variables u_i^j, $i = 0, 1, 2, ..., n$, $j = 1, 2, 3, ..., m-1$, are calculated by the formula (13).

It should be noted that the approximate values of the desired function $p(x)$ at the boundary points $x_0 = 0$ and $x_n = 1$ cannot be determined by formula (30) due to the fulfillment of the conditions $\theta_0^{j+1} = 0$, $\theta_n^{j+1} = 0$. Therefore, the values of the desired function $p(x)$ at the boundary points can be determined by interpolation.

3. Numerical examples

To find out the effectiveness of the proposed computational algorithm, numerical experiments were carried out for model problems. Calculations were carried out on a space-time difference grid with steps $\Delta x = 0.05$, $\Delta t = 0.0001$. In order to identify the influence of the final moment of time T on the solution of the inverse problem, numerical calculations were carried out at $T = 0.05$ and $T = 0.1$.

Task A. Find the functions $u(x,t)$ and $p(x)$, satisfying the following conditions

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} - p(x)\sin t, \qquad 0 < x < 1, \;\; 0 < t \le T,$$

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 5e^{-2x}, \quad u(0,t) = 5\sin t, \quad u(1,t) = 5e^{-2}\sin t, \quad u(x,T) = 5e^{-2x}\sin T.$$

This problem has an exact solution

$$u(x,t) = 5e^{-2x}\sin t, \quad p(x) = 25e^{-2x}.$$

The results of the numerical solution of problem A are shown in Table 1; in it x_i– the spatial coordinate, $\overline{p}_i$ and $\tilde{p}_i$ –respectively, the exact and calculated values of the function $p(x)$.

Table 1. Numerical results for problem A

x_i	$p(x)$		
	$\overline{p}_i$	$\tilde{p}_i$	
		T=0.05	T=0.1
0.05	22.621	22.586	22.582
0.10	20.468	20.463	20.440
0.15	18.520	18.520	18.554
0.20	16.758	16.758	16.793
0.25	15.163	15.163	15.195
0.30	13.720	13.720	13.749
0.35	12.415	12.415	12.441
0.40	11.233	11.233	11.257
0.45	10.164	10.164	10.186
0.50	9.197	9.197	9.216
0.55	8.322	8.322	8.339
0.60	7.530	7.530	7.546
0.65	6.813	6.813	6.828
0.70	6.165	6.165	6.178
0.75	5.578	5.578	5.590
0.80	5.047	5.047	5.058
0.85	4.567	4.567	4.578
0.90	4.132	4.133	4.153
0.95	3.739	3.750	3.732

Task B. Find the functions $u(x,t)$ and $p(x)$, satisfying the following conditions

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 2\frac{\partial^2 u(x,t)}{\partial x^2} + p(x)\cos t, \qquad 0 < x < 1, \;\; 0 < t \le T,$$

$$u(x,0) = 10\sin x, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad u(0,t) = 0, \quad u(1,t) = 10\sin 1\cos t, \quad u(x,T) = 10\sin x\cos T.$$

The exact solution to this problem has the form

$u(x,t)=10\sin x\cos t, \quad p(x)=10\sin x.$

The results of the numerical solution of problem B are shown in Table 2.

Table 2. Numerical results for problem B

x_i	$p(x)$		
	$\overline{p}_i$	$\widetilde{p}_i$	
		T=0.05	T=0.1
0.05	0.500	0.499	0.501
0.10	0.998	1.003	1.002
0.15	1.494	1.491	1.501
0.20	1.987	1.981	1.983
0.25	2.474	2.467	2.467
0.30	2.955	2.947	2.946
0.35	3.429	3.419	3.418
0.40	3.894	3.883	3.882
0.45	4.350	4.337	4.336
0.50	4.794	4.781	4.779
0.55	5.227	5.212	5.211
0.60	5.646	5.630	5.629
0.65	6.052	6.035	6.033
0.70	6.442	6.424	6.422
0.75	6.816	6.797	6.795
0.80	7.174	7.153	7.150
0.85	7.513	7.491	7.476
0.90	7.833	7.826	7.828
0.95	8.134	8.099	8.120

The results of numerical experiments show that the values of the desired functions $u(x,t)$ and $p(x)$ are determined with a sufficiently high accuracy. Moreover, the maximum relative error in determining the desired function $p(x)$ in both problems does not exceed 0.5%. Numerical results obtained for two different values of T indicate the stability of the solution with respect to T.

Analysis of the results of numerical experimentation indicates that it is sufficient to use small steps of the difference grid to increase the accuracy of solutions.

4. Conclusion

The problem of identifying the right-hand side of a one-dimensional wave equation depending on a spatial variable according to an additional condition of final redefinition is considered. The proposed computational algorithm, based on the discretization of the problem and the use of a special representation for solving the difference problem, allows us to find by an explicit formula the approximate value of the desired function for each discrete value of the spatial variable.

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