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Dynamic economic models

Dynamics of technological development: innovation, imitation, depreciation

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Abstract. The nonlinear dynamical system, describing the dynamics of sector capital distribution over two levels of technological development, low and high, proposed. The dynamics is determined by the interaction of innovation and imitation process with the process of depreciation taking into account. The qualitative behavior of the system trajectories depending on the relationship of parameters determining the rates of innovation, imitation and depreciation processes is studied. The invariant set which all trajectories enter in a finite time is found. In the case of nonzero rate of innovation process, the uniqueness of equilibrium is proved. On the basis of proposed constructive geometrical method the sufficient conditions of its global stability are obtained. In order to study the behavior of isoclines the special curve parametrization, connected with the invariant set geometry, is proposed. It is shown that in the absence of innovation process the existence of two equilibria, stable and unstable, is possible. In addition, the transition to a high technological level, corresponding to a stable equilibrium, may not occur even under sufficiently small depreciation process rate. The bifurcation value of the imitation rate parameter is found.

Keywords: nonlinear dynamical system, global stability, Schumpeterian dynamics, bifurcation.

1 Introduction

In [1],[2] K.Iwai proposed a model describing the evolution of economical system through innovation and imitation. It was one of the first attempts to create the mathematical interpretation of well-known Schumpeterian concept of endogenous economic growth, caused by interacting processes of innovation and imitation [3].

It is worth to note that Schumpeterian concept of economic development through innovation and imitation attracts considerable attention because there exists a large number of empirical confirmations of its main principles. Subsequently, different approaches to mathematical modeling of of its main principles are proposed [4, 5, 6, 7, 8, 9].

V.M. Polterovich and G.M. Henkin, invoking the Iwai's model, proposed the economic growth model [10, 11], based on Burgers type difference-differential equations used to study the fluid processes [12].

Extending these results (see also [13]), in [14, 15] the authors introduced the notion of economic niche volume in order to take into account the boundedness of economic growth, and under this condition, the dynamics of capital distribution over efficiency levels was studied. In addition, in [16] was considered the influence of depreciation process on the capital distribution dynamics taking into account the innovation process only. It is worth to note that the result presented in [16] is the partial solution of the problem of the influence of depreciation on Schumpeterian dynamics, which was formulated in [10]. In this paper, the study of the dynamics of capital distribution over efficiency levels in presence of depreciation is presented. Unlike [16], in the proposed model not only the innovation process is taken into account but the imitation process as well. The latter circumstance considerably complicates the study of dynamics of corresponding two-dimensional dynamic system with sufficiently large number of parameters, namely six. The main difficulty is due to the right hand sides which may be reduced to the polynomials of third degree, depending in two variables. It is worth to note that the qualitative theory of the differential equations used in this paper is widely applied for the mathematical modeling of the economic processes (for example, see [17, 18, 19, 20]).

The paper is organized as follows. Section 2 describes the model of the capital distribution dynamics with innovation, imitation and depreciation processes.

In Section 2 the construction of invariant set Ω is given,and, it is proved that all positive half-trajectories enter and do not leave Ω. Besides, the existence of an unique equilibrium C^* of the presented two-dimensional dynamic system is proved.

Sections 3, 4 provide the analysis of isoclines. The sufficient condition on economic niche volume V, under which the arcs of isoclines, belonging to Ω, present the graphs of monotonically decreasing functions, is obtained. The main tool to prove these result is the special parametrization of the arcs of isoclines, which is due to the geometric structure of Ω.

In Section 5 the main result is proved: the equilibrium of the system is globally stable for sufficiently large economic niche volume V. The proof is constructive and is based on the parameterized set of nested polygons $P(d)$ shrinking to the equilibrium $C^* = (C_1^*, C_2^*)$ as $d \to d^*$, where $d^* = C_1^* + C_2^* - V$. In addition, all trajectories enter and do not leave each polygon.

Section 6 deals with the analysis of capital distribution dynamics without an innovation process, i.e. only an imitation is taken into account. It is shown that in this case there appears the second equilibrium $(V, 0)$, which is not stable, if along with $(V, 0)$ there exists the first equilibrium C^*. It is shown that the technological development, i.e. the transition to a higher technological level, via an imitation only, without an innovation, may be impossible.

2 The model

Let $C_i = C_i(t)$ be the integrated capital of the sector firms at the i-th technological level (one firm can have the capital at different levels) at a time t, $i = 1, 2$. Consider the following system of ordinary differential equations, describing the dynamics of capital distribution over the technological, low ($i = 1$) and high ($i = 2$), levels

$$\dot{C}_1 = \frac{1}{\lambda_1}(1 - \varphi(C_1, C_2))C_1(V - C_1 - C_2) + \mu C_2, \tag{1}$$

$$\dot{C}_2 = \frac{1}{\lambda_2}C_2(V - C_1 - C_2) - \mu C_2 + \varphi(C_1, C_2)C_1, \tag{2}$$

where

$$\varphi(C_1, C_2) = \alpha + \beta\frac{C_2}{C_1 + C_2}. \tag{3}$$

Here, $\varphi(C_1, C_2)$ is the share of capital of the firms at the first, low, level intended to the developing of the production at the second, high, level. The structure of φ

is as follows: α is the capital share due to innovation firms's activity and $\beta\frac{C_2}{C_1+C_2}$ is the share due to imitation activity. The latter, as one can see, depends on the share of the firms's capital engaging in the development of the second, high, technological level. The term μC_2 describes the depreciation process. $0 < V$ is the economic niche volume. The presence of the multiplier $V - C_1 - C_2$ in the right-hand sides permits to take into account the boundedness of the economic growth [14],[15]. Further, $0 < \lambda_i$ is the unit prime cost at the i-th level (i.e. the unit goods production cost per unit time), $i = 1, 2$. All parameters, $\alpha,\ \beta,\ \mu > 0,\ V, \lambda_i,\ i = 1, 2$, are constants. Denote $C = (C_1, C_2) \in \mathbb{R}^2_+ \setminus \{O\}$, where $\{O\} = (0, 0)$.

In [16] the system (1)-(2) without imitation process was considered, i.e. it was supposed $\beta = 0$. The global stability of a unique equilibrium in $\mathbb{R}^2_+ \setminus \{O\}$ was proved. Here we suppose that

$$\beta > 0, \quad \alpha \geqslant 0, \tag{4}$$

which means that the model involves the imitation process, while the innovation process will be absent in some cases considered below. Besides, suppose that

$$\lambda_1 > \lambda_2, \quad \alpha + \beta < 1, \quad \mu > 0. \tag{5}$$

The first condition means that the unit prime cost at the low technological level is greater than at the high one.

The inequality $\alpha + \beta < 1$ implies that

$$\varphi(C_1, C_2) \in (0, 1). \tag{6}$$

Remark 1 *It easy to show that $\mathbb{R}^2_+ \setminus \{O\}$ is invariant set. Thus, further, the phase space is this invariant set $\mathbb{R}^2_+ \setminus \{O\}$.*

3 Invariant sets, equilibria ($\alpha > 0,\ \beta > 0$)

Denote by $\overline{v} \cdot \overline{w}$ the scalar (inner) product of vectors $\overline{v} = (v_1, v_2)$ and $\overline{w} = (w_1, w_2)$. Then $\overline{v} \cdot \overline{w} = v_1 w_1 + v_2 w_2$.

Definition 1 *Let $\gamma(t, x_0), \gamma(0, x_0) = x_0$ be the solution of the differential equation $\dot{x} = F(x)$. The positive half-trajectory corresponding to $\gamma(t, x_0)$ is the set $\{\gamma(t, x_0), t \geqslant 0\}$.*

Denote

$$d_1 = \frac{\lambda_1(\alpha+\beta)}{1-\alpha-\beta}.$$

Lemma 1 *All positive half-trajectories of the system (1)-(2), with initial points in* $\mathbb{R}^2_+ \setminus \{O\}$*, enter and do not leave the set* $\Omega = \{(C_1, C_2) \in \mathbb{R}^2_+ \setminus \{O\} : C_1 + C_2 \in [V, V + d_1]\}$.

Proof

Let $\overline{n} = (1,1)$ be the normal vector of the straight line $C_1 + C_2 = V + d$, where d is any nonnegative constant, $f = (f_1, f_2)$, where f_1, f_2 are the righthand sides of (1), (2), respectively. Then

$$f \cdot \overline{n} = (a_1 C_1 + a_2 C_2)(V - C_1 - C_2) + \varphi(C_1, C_2) C_1,$$

where $a_1 = \frac{1-\varphi(C_1,C_2)}{\lambda_1}$, $a_2 = \frac{1}{\lambda_2}$. Hence,

$$f \cdot \overline{n}|_{C_1+C_2=V} = \varphi(C_1, V - C_1) C_1 > 0, \tag{7}$$

where the left-hand side is the value of inner product on the straight line. Next

$$f \cdot \overline{n}|_{C_1+C_2=V+d} = -\frac{d}{\lambda_1}\big(1-\varphi(C_1, V+d-C_1)\big)C_1 + a_2(V+d-C_1) + \varphi(C_1, V+d-C_1)C_1,$$

and after obvious transformations we obtain for $d \geqslant d_1$

$$f \cdot \overline{n}|_{C_1+C_2=V+d} = \big(-\frac{d}{\lambda_1}(1-\alpha-\beta)+\alpha+\beta\big)C_1 - (d+1)\frac{\beta C_1^2}{V+d} - d a_2 C_2 < 0. \tag{8}$$

The conclusion of lemma follows from (7), (8). □

Remark 2 *The set* Ω *is a quadrangle, namely, a trapezoid.*

Lemma 2 *Assume* $\alpha > 0$. *The system* (1)-(2) *has the unique equilibrium* $C^* \in \mathbb{R}^2_+ \setminus \{O\}$.

Proof

To determine an equilibrium consider the system

$$f_i(C_1, C_2) = 0, \quad i = 1, 2. \tag{9}$$

If $C_1 = 0$ then (1) implies that $C_2 = 0$ but $(0,0)$ does not belong to $\mathbb{R}^2_+ \setminus \{O\}$. Thus, suppose $C_1 \neq 0$. Dividing the equations (9) by C_1 and denoting $x = \frac{C_2}{C_1}$, $y = V - C_1 - C_2$, we obtain the system of equations

$$\frac{1}{\lambda_1}\big(1 - \alpha - \beta\frac{x}{1+x}\big)y + \mu x = 0, \tag{10}$$

$$\frac{1}{\lambda_2}xy - \mu x + \alpha + \beta\frac{x}{1+x} = 0. \tag{11}$$

From (10), (11) we obtain the functions $g_1(x),\ g_2(x)$

$$y = -\frac{\lambda_1\mu x}{1-\alpha-\beta\frac{x}{1+x}} = g_1(x), \quad y = \lambda_2(\mu - \frac{\alpha}{x} - \frac{\beta}{1+x}) = g_2(x).$$

Let us prove that the curves $y = g_1(x)$ and $y = g_2(x)$ have the unique common point if $x > 0$. The following properties of g_1, g_2 are valid for $x > 0$

$$g_1(x) < 0, \quad g_1(0) = 0, \quad g_1'(x) = -\lambda_1\mu(1-\alpha-\beta\frac{x^2}{(1+x)^2})(1-\alpha-\frac{\beta x}{1+x})^{-2} < 0;$$

$$\lim_{x\to+0} g_2(x) = -\infty, \quad \lim_{x\to+\infty} g_2(x) = \mu\lambda_2 > 0, \quad g_2'(x) = \lambda_2(\frac{\alpha}{x^2} + \frac{\beta}{(1+x)^2}) > 0.$$

Denoting $G(x) = g_1(x) - g_2(x)$ we have for $x > 0 :\ G'(x) < 0,\ \lim_{x\to+0} G(x) = +\infty,\ \lim_{x\to+\infty} G(x) = -\infty$, that implies the existence of the unique point $x^* > 0$ such that $G(x^*) = 0$, i.e. the unique point (x^*, y^*) of intersection of the curves $y = g_1(x),\ y = g_2(x)$, for $x > 0$. Therefore, the system (1)-(2) has a unique equilibrium $C^* \in \mathbb{R}^2_+ \setminus \{0,0\}$, where $C_1^* = \frac{V-y^*}{1+x^*}$, $C_2^* = C_1^* x^*$. □

Remark 3 *It follows from Lemma 1 that $C^* \in \Omega$.*

4 Isoclines ($\alpha > 0,\ \beta > 0$)

In this section, we prove that for sufficiently large value of V the arcs of isoclines, belonging to the invariant set Ω, are the graphs of the monotonically decreasing functions.

It is worth to note that after multiplying the equations (9) by $C_1 + C_2 \neq 0$ we obtain that isoclines are the cubic curves, which equations contain six parameters: $\alpha,\ \beta,\ \mu,\ \lambda_1,\ \lambda_2,\ V$. The latter fact produces supplementary difficulties while proving the global stability.

Consider the isocline $f_1(C_1, C_2) = 0$

$$\frac{1}{\lambda_1}(1-\alpha-\beta\frac{C_2}{C_1+C_2})C_1(V - C_1 - C_2) + \mu C_2 = 0.$$

This isocline has no common points with C_2-coordinate axis if $C_2 > 0$, and $(V, 0)$ is its unique point with C_1-coordinate axis. Next, prove some properties of the isocline $f_1(C_1, C_2) = 0$.

Lemma 3 *The isocline $f_1(C_1, C_2) = 0$ has one and only one common point $C(d) = (C_1(d), C_2(d))$ with the straight line $C_1 + C_2 = V + d$, where $d \geqslant 0$. If $d \in [0, d_1]$ then $C(d) \in \Omega$.*

Proof

Consider the equation $f_1(C_1, V + d - C_1) = 0$, or

$$\frac{1}{\lambda_1}\Big(1 - \alpha - \beta(1 - \frac{C_1}{V+d})C_1(-d)\Big) + \mu(V + d - C_1) = 0.$$

Dividing it by $V + d$ and denoting $z = \frac{C_1}{V+d}$ we obtain the equation

$$\beta d z^2 + (\lambda_1\mu + d(1 - \alpha - \beta))z - \lambda_1\mu = 0$$

with the unique positive solution $z(d) > 0$, if $d \geqslant 0$, and, therefore, the isocline $f_1(C_1, C_2) = 0$ has one and only one common point $C(d) = (C_1(d), C_2(d))$ with the straight line $C_1 + C_2 = V + d$, if $d \geqslant 0$, such that $C_1(d) = z(d)(V + d) > 0$.

It is clear from the above equation that $z(0) = 1$. Next, if $d > 0$, then $z(d) < 1$, or

$$z(d) = \frac{1}{2\beta}\Bigg(-\Big(1-\alpha-\beta+\frac{\lambda_1\mu}{d}\Big)+\sqrt{\Big(1-\alpha-\beta+\frac{\lambda_1\mu}{d}\Big)^2+\frac{4\lambda_1\mu\beta}{d}}\Bigg) < 1 \quad (12)$$

Really, it is easy to show, that this inequality is equivalent to the obvious one $\beta d + d(1 - \alpha - \beta) > 0$. Therefore, $C_1(d) = z(d)(V + d) < V + d$ and $0 < C_2(d) = V + d - C_1(d) = (V + d)(1 - z(d)) < V + d$. If $d = 0$, then $C_1 = V$, and $C(0) = (V, 0) \in \Omega$. Thus, $C(d) \in \Omega$, if $d \in [0, d_1]$. □

Denote by M and R the common points of the isocline $f_1 = 0$ and the straight line $C_1 + C_2 = V + d_1$ and the C_1-coordinate axis, respectively, i.e.

$$M = \{C : f_1(C) = 0\} \cap \{C : C_1 + C_2 = V + d_1\}, \quad R = (V, 0).$$

We shall write $M = (C_1(M), C_2(M)) = (C_1(d_1), C_2(d_1)) = C(d_1)$.

Let RM be the ark of the isocline $f_1 = 0$, with endpoints R and M. Lemma 3 implies the existence of the unique arc of isocline $f_1 = 0$ belonging to Ω.

Corollary 1 $RM \in \Omega$.

Lemma 4 *The isocline $f_1(C_1, C_2) = 0$ and the straight line $C_2 = q$, where $q \in [0, V + d_1]$, have no more than one common point in Ω.*

Proof

Consider the equation $f_1(C_1, q) = 0$, or

$$s_1(C_1) = a_1\big(1 - \alpha - \frac{\beta q}{C_1 + q}\big) = -\frac{\mu q}{C_1(V - C_1 - q)} = s_2(C_1).$$

Using standard methods of analysis we can easily prove that the graphs of functions $s_1,\ s_2$ have an unique common point if $q > 0$. This point may belong or not to Ω if $q \in [0, V + d_1]$ which implies the conclusion of Lemma. □

Lemma 5 *If $d > 0$ then*

(1) $C_2'(d) > 0$;

(2) *there exists* $\widehat{d} > 0$ *such that for* $d \in (0, \widehat{d})$:

$C_1'(d) > 0$ *if* $V < \frac{\lambda_1\mu}{1-\alpha}$, $\qquad C_1'(d) < 0$ *if* $V > \frac{\lambda_1\mu}{1-\alpha}$.

Proof

From the proof of Lemma 3: $C_1(d) = (V + d)z(d)$, $C_1'(d) = z(d) + (V + d)z'(d)$. Differentiating $z(d)$, given by (12), we obtain

$$z'(d) = \frac{1}{2\beta}\Big(-\frac{\lambda_1\mu}{d^2}\Big)\Big(-1 + \frac{\frac{\lambda_1\mu}{d} + (1-\alpha-\beta) + 2\beta}{\sqrt{D}}\Big),$$

where $D = (a + \frac{b}{d})^2 + \frac{4\beta b}{d}$, $a = 1 - \alpha - \beta$, $b = \lambda_1\mu$. It easy to show that

$$z'(d) = \frac{\lambda_1\mu}{d^2\sqrt{D}}\big(z(d) - 1\big),$$

and, therefore, taking into account (12) we have: $z'(d) < 0$. Further,

$$C_1'(d) = z(d) + \frac{b(z(d) - 1)}{d^2\sqrt{D}}(V + d),$$

$$C_2'(d) = 1 - C_1'(d) = (1 - z(d))\big(1 + \frac{b}{d^2\sqrt{D}}\big) > 0.$$

Let us find $\lim_{d\to+0} C_1'(d)$. As a result of simple transformations of $z'(d)$ we have

$$2\beta z'(d)(V + d) = -\frac{4e(ab + e)}{b}\cdot$$

$$\frac{1}{\sqrt{(ad + b)^2 + 4ed}\cdot(ad + b + \frac{2ed}{b} + \sqrt{(ad + b)^2 + 4ed})}\cdot(V + d),$$

where $a = 1 - \alpha - \beta,\ \ b = \lambda_1\mu,\ e = \lambda_1\mu\beta.$ Hence,

$$\lim_{d\to+0} z'(d)(V+d) = -\frac{e(ab+e)V}{\beta b^3}.$$

Further, from (12)

$$2\beta z(d) = -(a+\frac{b}{d}) + \sqrt{D} = \frac{D-(a+\frac{b}{d})^2}{D+(a+\frac{b}{d})^2}$$

we obtain

$$2\beta \lim_{d\to+0} z(d) = \lim_{d\to+0} \frac{4e}{d\sqrt{D}+ad+b} = \frac{2e}{b}.$$

Taking into account that $C_1'(d) = z(d) + z'(d)(V+d)$, we have

$$\lim_{d\to+0} C_1'(d) = \frac{e}{\beta b}\Big(1 - \frac{V(ab+e)}{b^2}\Big).$$

Thus, $C_1'(d) > 0$ if $V < \frac{b^2}{ab+e}$, $C_1'(d) < 0$ if $V > \frac{b^2}{ab+e}$. Taking into account the continuity of $C_1'(d)$ and that $\frac{b^2}{ab+e} = \frac{\lambda_1\mu}{1-\alpha}$, we obtain the conclusion of Lemma. □

Lemma 6 *Assume* $V > \frac{\lambda_1\mu(1-\alpha)}{(1-\alpha-\beta)^2}$. *Then the equation* $f_1(C_1, C_2) = 0$ *determines the unique monotonically decreasing function* $C_2 = h(C_1)$ *on the segment* $[C_1(M), V]$.

Proof

Let us show, that each straight line $C_1 = p$, where $p \in (0, V+d_1]$, intersects the isocline, or the arc RM, at a unique point, if $V > \frac{\lambda_1\mu(1-\alpha)}{(1-\alpha-\beta)^2}$. Substituting $C_1 = p$ in $f_1(C_1, C_2) = 0$ we obtain

$$\frac{1}{\lambda_1}\big(1-\alpha-\beta\frac{C_2}{p+C_2}\big)p(V-p-C_2) + \mu C_2 = 0,$$

and denoting $r = p + C_2$ it is easy to obtain the following equation from the above one

$$\big(\frac{\mu\lambda_1}{p} + \alpha + \beta - 1\big)r^2 + (V(1-\alpha-\beta) - \beta p - \mu\lambda_1)r + \beta p V = 0, \qquad (13)$$

where $r \in [V, V+d_1]$. Let us study the conditions under which (13) has 0, 1 or 2 positive solutions.

Consider the case $\frac{\mu\lambda_1}{p} + \alpha + \beta - 1 > 0$, or

$$p < \frac{\lambda_1\mu}{1-\alpha-\beta}. \quad (14)$$

Suppose, the discriminant of (13) is nonnegative. Then this equation has two solutions, both positive or negative. Let us show, that (13) cannot have two positive solutions. Two positive solutions exist if and only if $V(1-\alpha-\beta) - \beta p - \mu\lambda_1 < 0$, or $p > \frac{1}{\beta}(V(1-\alpha-\beta) - \lambda_1\mu)$. Taking into account (14), we obtain

$$\frac{1}{\beta}(V(1-\alpha-\beta) - \lambda_1\mu) < p < \frac{\lambda_1\mu}{1-\alpha-\beta},$$

which implies $\frac{1}{\beta}(V(1-\alpha-\beta) - \lambda_1\mu) < \frac{\lambda_1\mu}{1-\alpha-\beta}$ or $V < \frac{\lambda_1\mu(1-\alpha)}{(1-\alpha-\beta)^2}$ that contradicts to the assumption of Lemma.

Consider the case $\frac{\mu\lambda_1}{p} + \alpha + \beta - 1 = 0$, or $p = \frac{\lambda_1\mu}{1-\alpha-\beta}$. Then from (13)

$$r = -\frac{\beta p V}{V(1-\alpha-\beta) - \beta p - \lambda_1\mu},$$

and, as $r > 0$, it is necessary that $V(1-\alpha-\beta) - \beta p - \lambda_1\mu < 0$, or $\frac{\lambda_1\mu}{1-\alpha-\beta} = p > \frac{1}{\beta}(V(1-\alpha-\beta) - \lambda_1\mu)$, which contradicts to the assumption of Lemma as well.

Consider the case $\frac{\mu\lambda_1}{p} + \alpha + \beta - 1 < 0$, or $p > \frac{\lambda_1\mu}{1-\alpha-\beta}$. Then, the inequality $\beta p V > 0$ implies that (13) has one positive solution.

Further, taking into account Corollary 1, we obtain, that under condition $V > \frac{\lambda_1\mu(1-\alpha)}{(1-\alpha-\beta)^2}$, the isocline $f_1 = 0$ and the straight line $C_1 = p$ have one common point. Thus, the equation $f_1 = 0$ determines the unique function $C_2 = h(C_1)$ with the graph RM. The monotonicity of h follows from Lemma 4.

Taking into account that

$$V > \frac{\lambda_1\mu(1-\alpha)}{(1-\alpha-\beta)^2} = \frac{\lambda_1\mu}{1-\alpha-\beta}\cdot(1+\frac{\beta}{1-\alpha-\beta}) > \frac{\lambda_1\mu}{1-\alpha-\beta} > \frac{\mu\lambda_1}{1-\alpha},$$

we obtain from Lemma 5 that $C_1'(d) < 0$ for sufficiently small $d > 0$. Therefore, $C_2'(d) = 1 - C_1'(d) > 0$ and for sufficiently small $d > 0$ we have

$$h'(C_1) = \frac{C_2'(d)}{C_1'(d)} < 0.$$

The monotonicity of h permits to assert the validity of the latter inequality for $d \in [0, d_1]$. □

Next, consider the isocline $f_2(C_1, C_2) = 0$.

Lemma 7 *The isocline $f_2(C_1, C_2) = 0$ has one and only one common point $\widehat{C}(d) = (\widehat{C}_1(d), \widehat{C}_2(d))$ with the straight line $C_1 + C_2 = V + d$, where $d \geqslant 0$. If $d \in [0, d_1]$ then $\widehat{C}(d) \in \Omega$.*

Proof

Consider the equation $f_2(V + d - C_2, C_2) = 0$:

$$-da_2C_2 - \mu C_2 + (\alpha + \beta\frac{C_2}{V+d})(V + d - C_2) = 0,$$

where $a_2 = \frac{1}{\lambda_2}$. Dividing the latter equation by $V + d$ and substituting $\widehat{z} = \frac{C_2}{V+d}$ we obtain the equation

$$\beta\widehat{z}^2 + (a_2d + \mu + \alpha - \beta)\widehat{z} - \alpha = 0$$

with one positive solution $\widehat{z}(d)$:

$$\widehat{z}(d) = \frac{1}{2\beta}\big(- (a_2d + \mu + \alpha - \beta) + \sqrt{\widehat{D}(d)}\big) < 1, \tag{15}$$

where$\widehat{D}(d) = (a_2d + \mu + \alpha - \beta)^2 + 4\alpha\beta$. The inequality $\widehat{z}(d) < 1$ is valid because it is equivalent to the obvious one $0 < a_2d + \mu$. Therefore, $0 < \widehat{C}_2(d) = \widehat{z}(d)(V + d) < V + d$, and $\widehat{C}_1(d) = V + d - \widehat{C}_2(d) = (V + d)(1 - \widehat{z}(d))$, i.e. $\widehat{C}_1(d) \in (0, V + d)$. Thus, $\widehat{C}(d) = (\widehat{C}_1(d), \widehat{C}_2(d)) \in \Omega$ if $d \in [0, d_1]$. □

Denote by K and N the common points of the isocline $f_2 = 0$ and the straight lines $C_1 + C_2 = V$ and $C_1 + C_2 = V + d_1$, respectively

$$K = \{C : f_2(C) = 0\} \cap \{C : C_1+C_2 = V\}, \quad N = \{C : f_2(C) = 0\} \cap \{C : C_1+C_2 = V+d_1\}$$

We shall write$K = \widehat{C}(0)$, $N = \widehat{C}(d_1)$.

Let KN be the arc of the isocline $f_2 = 0$, with the endpoints K and N.

Corollary 2 $KN \in \Omega$.

Lemma 8 $C_1'(d) > 0$.

Proof

Differentiating (15), we obtain

$$z'(d) = \frac{1}{2\beta}\Big(- a_2 + \frac{(a_2d + \mu + \alpha - \beta)a_2}{\sqrt{D(d)}}\Big) = -\frac{a_2\widehat{z}(d)}{\sqrt{\widehat{D}}}.$$

From $\widehat{C}_2(d) = (V + d)\widehat{z}(d)$ we obtain $\widehat{C}_2'(d) = \widehat{z}(d) + \widehat{z}'(d)(V + d) = \widehat{z}(d)\big(1 - \frac{a_2(V+d)}{\sqrt{\widehat{D}}}\big)$. Hence $\widehat{C}_1'(d) = 1 - \widehat{C}_2'(d) = (1 - \widehat{z}(d)) + \frac{a_2(V+d)}{\sqrt{\widehat{D}}} > 0$. □

Corollary 3 $C_1(K) < C_1(N)$.

Lemma 9 *Assume* $V > \lambda_2\widehat{z}(0)\sqrt{\widehat{D}(0)}$. *Then* $\widehat{C}_2'(d) < 0$.

Proof

Using the expression for $\widehat{C}_1'(d)$ from Lemma 8, we obtain

$$\widehat{C}_2'(d) = 1 - \widehat{C}_1'(d) = \widehat{z}(d) - \frac{a_2(V+d)}{\sqrt{\widehat{D}(d)}}.$$

Therefore, the inequality $\widehat{C}_2'(d) < 0$ is equivalent to $V > \frac{\widehat{z}(d)\sqrt{\widehat{D}(d)}}{a_2} - d = G(d) - d$, where $G(d) = \frac{\widehat{z}(d)\sqrt{\widehat{D}(d)}}{a_2}$. If we show that $G'(d) < 0$, then $G(0) - 0 > G(d) - d$ for $d > 0$, and we will obtain the conclusion of Lemma. Using the expression for $\widehat{z}$ from (15), we obtain

$$G(d) = \frac{\lambda_2}{2\beta}\Big(-(a_2 d + \mu + \alpha - \beta) + \sqrt{\widehat{D}(d)}\Big)\sqrt{\widehat{D}(d)}.$$

Denote $x = x(d) = a_2 d + \mu + \alpha - \beta$ and consider the following function $H(x)$:

$$H(x) = \frac{\lambda_2}{2\beta}\Big(-x + \sqrt{x^2 + 4\alpha\beta}\Big)\sqrt{x^2 + 4\alpha\beta}.$$

After simple transformations we get: $H'(x) = \frac{-3x^2 - 4\alpha\beta + 2x\sqrt{x^2+4\alpha\beta}}{\sqrt{x^2+4\alpha\beta}}$.

Let us show that the numerator of the above fraction is negative, or $2x\sqrt{x^2+4\alpha\beta} < 3x^2 + 4\alpha\beta$. If $x < 0$, this inequality is obvious. If $x \geq 0$, then, after squaring, it is equivalent to the obvious one $0 < 5x^4 + 8x^2\alpha\beta + 16(\alpha\beta)^2$. Hence, $H'(x) < 0$. Noting that $H(x(d)) = G(d)$, we obtain $G'(d) = H'(x(d)) = H'(x)x'(d) = H'(x)a_2 < 0$. □

Lemma 10 *If* $V > \lambda_2\widehat{z}(0)\sqrt{\widehat{D}(0)}$, *then the equation* $f_2(C_1, C_2) = 0$ *determines the unique monotonically decreasing function* $C_2 = g(C_1)$ *for* $C_1 \in [C_1(K), C_1(N)]$.

Proof

The proof follows from Lemmas 8, 9 and the following equality $g'(C_1) = \frac{\widehat{C}_2'(d)}{\widehat{C}_1'(d)} < 0$. □

Denote

$$V_{\max} = \max\Big(\frac{\lambda_1\mu(1-\alpha)}{(1-\alpha-\beta)^2},\ \lambda_2\widehat{z}(0)\sqrt{\widehat{D}(0)}\Big).$$

Corollary 4 *If $V > V_{\max}$, then $C_1(M) < C_1(N)$.*

Proof

The proof follows from Lemmas 2, 6, 10. □

5 Global stability

In this section we consider the system (1)-(2), where $\alpha > 0$, that is a process of technological development involves an innovation process (along with imitation).

Denote by $C(t, C^0)$ a solution of the system (1)-(2) such that $C(0, C^0) = C^0$.

Definition 2 *The equilibrium C^* of the system* (1)-(2) *is called globally stable if $\lim_{t\to+\infty} C(t, C^0) = C^*$ for each $C^0 \in \mathbb{R}^2_+ \setminus \{O\}$.*

Denote

$$d^* = C_1^* + C_2^* - V,$$

i.e. the value of the parameter $d = d^*$ corresponds to the straight line $C_1 + C_2 = C_1^* + C_2^*$, or $C_1 + C_2 = V + d^*$, containing the equilibrium C^*.

The main result is the following.

Theorem 1 (about global stability) *If $V > V_{\max}$ then C^* is a globally stable equilibrium of* (1)-(2).

Remark 4 *The establishment of global stability for nonlinear dynamical systems is rather difficult procedure. The method of Lyapunov functions does not give any constructive receipt for its application. Here, we present a geometrical approach to prove the global stability based on the construction of parameterized nested polygons, shrinking to C^*, and such that each positive half-trajectory with initial point in $\mathbb{R}^2_+ \setminus \{O\}$ enters each polygon. The difficulties of implementing the proposed method are due to the impossibility of obtaining the explicit expression for C^* and sufficiently large number of parameters.*

Proof

The method of proof is based on the results of Section 4 (see Lemmas 3, 6 7, 10) which implies that the arcs of isoclines, RM, KN, belonging to Ω, have no more than one common point with the straight lines $C_1 = p$, $C_2 =$

$q,\ C_1+C_2=V+d$, where $p,q\in[0,V+d_1],\ d\in[0,d_1]$. Taking the latter into view, we construct the continuous parameterized set of convex polygons $P(d)$ which sides are parallel to the straight lines $C_1=p,\ C_2=q,\ C_1+C_2=V+d$.

In addition, the polygons $P(d)$ compose the nested set, i.e. $P(\widetilde{d}_1)\supset P(\widetilde{d}_2)$, if $\widetilde{d}_1<\widetilde{d}_2$, and $C^*\in P(d)$ for any $d\in[0,d^*)$. As a result, $P(d)\to C^*$ as $d\to d^*$-0, i.e. $P(d)$ shrinks to C^*, which means that all vertexes of $P(d)$ tends to C^*. Then, we will show that all trajectories enter each polygon $P(d)$ and do not leave it, which means the global stability of C^*.

The partition of Ω. Denote by R,R',E,E' the vertexes of the quadrangle $\Omega=RR'EE'$

$$R=(V,0),\quad R'=(V+d_1,0),\quad E=(0,V),\quad E'=(0,V+d_1).$$

Let $l_0\ l_1$ be the following straight lines

$$l_0=\{C:C_1+C_2=V\},\quad l_1=\{C:C_1+C_2=V+d_1\}.$$

The invariant set Ω is divided by the arcs $RM,\ KN$ of the isoclines into four simply connected subsets $\Omega_i,\ \Omega=\cup_{i=1}^4\Omega_i$. The boundaries $\partial\Omega_i$ of Ω_i are as follows

$$\partial\Omega_1=KR\cup RC^*\cup C^*K;\quad \partial\Omega_3=C^*N\cup NM\cup MC^*;$$

$$\partial\Omega_2=VR'\cup R'N\cup NC^*\cup C^*R;\quad \partial\Omega_4=KC^*\cup C^*M\cup ME'\cup E'E\cup EK,$$

where $EK,\ KR$ are the segments of the straight line l_0, $R'N,\ NM,\ ME'$ are the segments of the straight line l_1, $C^*R,\ MC^*$ are the arcs of the isocline $f_1=0$, KC^*,NC^* are the arcs of the isocline $f_2=0$. Here, we consider the non-oriented arcs: $KC^*=C^*K$, etc.

Note, that $\partial\Omega_i\cap\partial\Omega_j$ is the boundary component of Ω_i and Ω_j if $i\neq j$.

Analyzing the signs of the righthand sides of (1), (2) it is easy to obtain the following

$$f_1>0,\ f_2>0\quad \text{if } C\in \operatorname{int}\Omega_1;\qquad f_1<0,\ f_2<0\quad \text{if } C\in \operatorname{int}\Omega_3;$$

$$f_1<0,\ f_2>0\quad \text{in } C\in \operatorname{int}\Omega_2;\qquad f_1>0,\ f_2<0\quad \text{in } C\in \operatorname{int}\Omega_4,$$

where $\operatorname{int}\Omega_j$ is the interior of Ω_j.

The construction of polygon $P(d)$. Consider the relative position of the points $R,\ N,\ M,\ K$. For points $A,B\in\mathbb{R}^2_+$ we write $A<B$ if $C_1(A)<C_1(B)$. From the above study we have $K<R,\ K<N,\ M<N,\ M<R$. Below, we consider four possible cases:

1) $K < M < R < N$; 2) $M < K < R < N$;

3) $K < M < N < R$; 4) $M < K < N < R$.

In what follows, the boundary components of polygons $P(d)$ are the segments of straight lines with normal vectors

$$\overline{u} = (1, 0), \quad \overline{v} = (0, 1), \quad \overline{w} = (1, 1).$$

Below, we represent the detailed proof in the first case. In the other cases, the character of arguments is analogous to the first one.

1)Assume $K < M < R < N$.

All trajectories enter and do not leave the parallelogram M_1RR_1M, where

$$M_1 = \{C : C_1 = C_1(M)\} \cap l_0, \quad R_1 = \{C : C_1 = C_1(R) = V\} \cap l_1.$$

Indeed, consider the segment $Seg(p) = \{C : C_1 = p\} \cap \Omega$. If $p \in [0, C_1(M)]$, then $f \cdot \overline{u} = f_1 > 0$ on $Seg(p) \subset \Omega_1 \cup \Omega_4$, except for the point M: $f_1(M) = 0$. But Lemma 1 implies that M is the ingress point for M_1RR_1M. If $p \in [V, V + d_1]$ then $f \cdot \overline{u} = f_1 < 0$ on $Seg(p) \subset \Omega_2 \cup \Omega_3$, except for the point R: $f_1(R) = 0$, but as it was noted above, R is the ingress point for M_1RR_1M as well.

Let us construct the parameterized set of polygons $P(d) \subset M_1RR_1M$, $d \in [0, d^*)$. Below, we represent the sequence of steps providing the vertexes and sides of $P(d)$ (fig. 1).

a. *Vertex* $R(d)$. Let us begin with the point $R(d) \in C^*R$, which will be called the *generating vertex*. It means that $R(d)$ initialize $P(d)$. Namely, we will obtain $P(d)$, constructing its boundary, moving from $R(d)$ clockwise and counterclockwise, determining the sequence of segments, sides of $P(d)$, till they meet at some final point.

b. *Vertex* $Q(d)$. Denote $Q(d) = \{C : C_1 + C_2 = C_1(R(d)) + C_2(R(d))\} \cap M_1C^*$, where M_1C^* is the segment with endpoints M_1, C^*. Note that $M_1C^* \subset \widetilde{K}C^*\widetilde{C}$, where $\widetilde{K}C^*\widetilde{C}$ is a triangle with vertexes $\widetilde{K} = \{C : C_2 = C_2(C^*)\} \cap l_0$, $\widetilde{C} = \{C : C_1 = C_1(C^*)\} \cap l_0$ and C^*. Moreover, due to the monotone decreasing of the functions $C_2 = h(C_1)$, $C_2 = g(C_1)$, the arcs KC^* and C^*R do not belong to $\widetilde{K}C^*\widetilde{C}$, except for C^*. Hence, M_1C^* has no common points with the isoclines, except for C^*.

Let the segment $R(d)Q(d)$ be the side of $P(d)$, parallel to l_0.

c. *Vertex* $N(d)$. Denote $N(d) = \{C : C_1 = C_1(R(d))\} \cap C^*N$.

Let the segment $R(d)N(d)$ be the side of $P(d)$, parallel to $C_1 = 0$.

d. *Vertexes* $L(d)$, $M(d)$*: "L-procedure".*

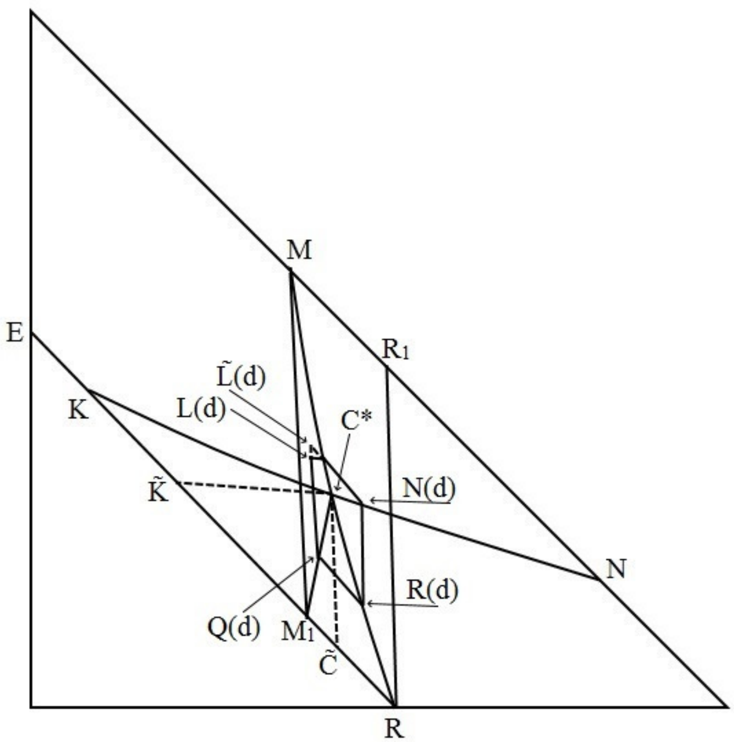

Figure 1: $K < M < R < N$.

Denote by $Ray(A, \overline{e})$ the ray with initial point A and directing vector $\overline{e}$. Consider the point $\widetilde{L}(d) = Ray(Q(d), \overline{v}) \cap Ray(N(d), \overline{w}_1)$, where $\overline{w}_1 = (-1, 1)$. Obviously, $\widetilde{L}(d) \in M_1RR_1M$.

We have three cases of $\widetilde{L}(d)$ position: $\widetilde{L}(d) \in \Omega_3$, $\widetilde{L}(d) \in \Omega_4$, $\widetilde{L}(d) \in MC^*$ (fig. 2).

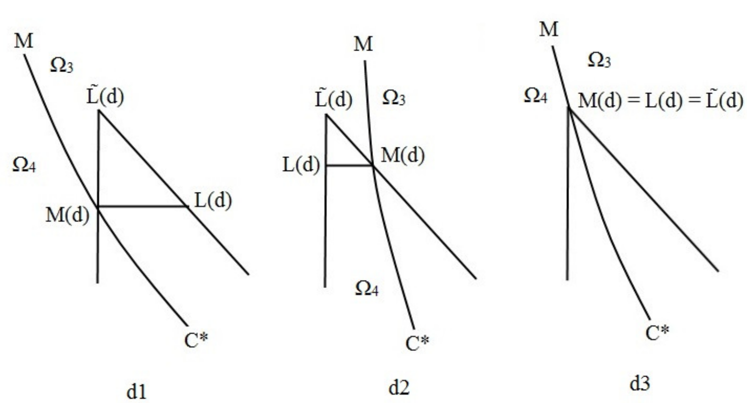

Figure 2: Position of $\widetilde{L}(d)$.

d1. $\widetilde{L}(d) \in \Omega_3$.

Consider the point $M(d) = Ray(Q(d), \overline{v}) \cap MC^*$. Note, that $C_1(M(d)) = C_1(\widetilde{L}(d))$ and $C_2(M(d)) < C_2(\widetilde{L}(d))$. Then, obtain the point $L(d) = \{C : C_2 = C_2(M(d))\} \cap Ray(N(d), \overline{w}_1) \in \Omega_3$. Thus, we obtain the polygon $P(d) = R(d)Q(d)M(d)L(d)N(d)$.

d2. $\widetilde{L}(d) \in \Omega_4$.

Consider the point $M(d) = Ray(N(d), \overline{w}_1) \cap MC^*$. Then, obtain the point $L(d) = \{C : C_2 = C_2(M(d))\} \cap Ray(Q(d), \overline{v}) \in \Omega_4$. Thus, we obtain the polygon $P(d) = R(d)Q(d)L(d)M(d)N(d)$.

d3. $\widetilde{L}(d) \in MC^*$.

Now, $M(d) = L(d) = \widetilde{L}$. Thus, we obtain the polygon $P(d) = R(d)Q(d)M(d)N(d)$.

The procedure of obtaining of the vertexes $M(d) = L(d)$, comprising items d1, d2, d3, will be used below. Let us call it, for brevity, the *"L-procedure"*.

According to the construction, in all cases, $P(\widetilde{d}_2) \subset P(\widetilde{d}_1)$ if $\widetilde{d}_1, \widetilde{d}_2 \in [0, d^*)$ and $\widetilde{d}_1 < \widetilde{d}_2$. Moreover, $C^* \in P(d)$ for any $d \in [0, d^*)$. Thus, $P(d)$ shrinks to the equilibrium C^*, as $d \to d^*$, which means that all its vertexes tends to C^*.

Let us show that all trajectories enter $P(d)$ and do not leave it. We have

$f \cdot \overline{w} = f_1 + f_2 > 0$ on $R(d)Q(d) \subset \Omega_1$,

$f \cdot \overline{u} = f_1 > 0$ on $Q(d)M(d) \subset \Omega_4$, if $\widetilde{L}(d) \in \Omega_3$,

$f \cdot \overline{u} = f_1 > 0$ on $Q(d)L(d) \subset \Omega_4$, if $\widetilde{L}(d) \in \Omega_4$,

$f \cdot \overline{v} = f_2 < 0$ on $M(d)L(d) \subset \Omega_3$, if $\widetilde{L}(d) \in \Omega_3$,

$f \cdot \overline{v} = f_2 < 0$ on $L(d)M(d) \subset \Omega_4$, if $\widetilde{L}(d) \in \Omega_4$,

$f \cdot \overline{w} = f_1 + f_2 < 0$ on $M(d)L(d) \subset \Omega_3$, if $\widetilde{L}(d) \in \Omega_3$,

$f \cdot \overline{w} = f_1 + f_2 < 0$ on $L(d)M(d) \subset \Omega_4$, if $\widetilde{L}(d) \in \Omega_4$.

Thus, C^* is globally stable.

2)Assume $M < K < R < N$.

Similarly to the previous case 1, it is easy to show that all positive half-trajectories enter and do not leave the polygon (pentagon) $KRR_1\widetilde{D}\widetilde{M}$, where $\widetilde{M} = \{C : C_1 = C_1(K)\} \cap MC^*$, $\widetilde{D} = \{C : C_2 = C_2(\widetilde{M}) \cap l_1$.

The procedure of constructing of polygons $P(d) \subset KRR_1\widetilde{D}\widetilde{M}$, $d \in [0, d^*)$, is absolutely similar to the one described in the previous case 1 (fig. 3).

3) $K < M < N < R$.

Let us introduce the following points: $\widetilde{R} = \{C : C_1 = C_1(N)\} \cap C^*R$,

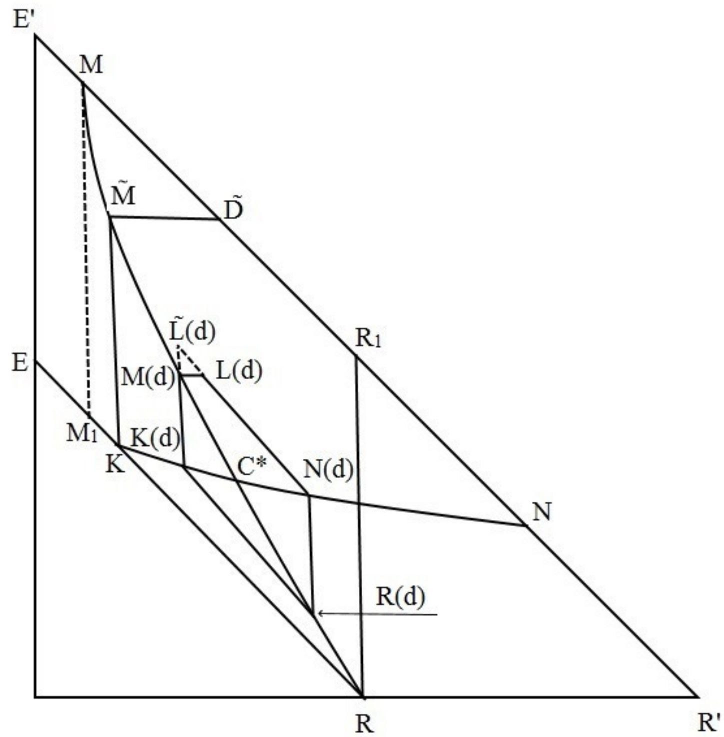

Figure 3: $M < K < R < N$.

$\widetilde{E} = \{C : C_2 = C_2(\widetilde{R})\} \cap l_0$.

Consider three possible cases of the relative position of the points M_1, $\widetilde{C}$, $\widetilde{E}$ belonging to l_0: $M_1 < \widetilde{C} < \widetilde{E}$ (3_1), $M_1 < \widetilde{E} < \widetilde{C}$ (3_2), $\tilde{E} < M_1 < \tilde{C}$ (3_3).

3_1. $M_1 < \widetilde{C} < \widetilde{E}$ (fig. 4).

Similarly to the case 1, it is easy to show, that all trajectories enter and do not leave the polygon $M_1\widetilde{E}\widetilde{R}NM$.

Denote $\widehat{R} = \{C : C_1 = C_1(\widetilde{E})\} \cap C^*\widetilde{R}$, where $C^*\widetilde{R}$ is the arc: $C^*\widetilde{R} \subset C^*R$. Consider two segments: $\widetilde{E}\widehat{R}$ and M_1C^*, which will contain the vertexes of $P(d)$. As it was shown in the case 1, M_1C^* has no common points with the isoclines, except for C^*. Obviously, $\widetilde{E}\widehat{R}$ also has no common points with the isoclines.

Let $R(d) \in C^*\widetilde{R}$ be the generating point.

We have two cases: $R(d) \in \widehat{R}\widetilde{R}$, $R(d) \in C^*\widehat{R}$.

If $R(d) \in \widehat{R}\widetilde{R}$ then $E(d) = \{C : C_2 = C_2(R(d))\} \cap \widetilde{E}\widehat{R}$, $Q(d) = \{C : C_1 + C_2 = C_1(E(d)) + C_2(E(d))\} \cap M_1C^*$.

If $R(d) \in C^*\widehat{R} \setminus C^*$ then $Q(d) = \{C : C_1 + C_2 = C_1(R(d)) + C_2(R(d))\} \cap M_1M$.

Next, after $Q(d)$, in both cases, we obtain $N(d) = \{C : C_2 = C_2(R(d))\} \cap$

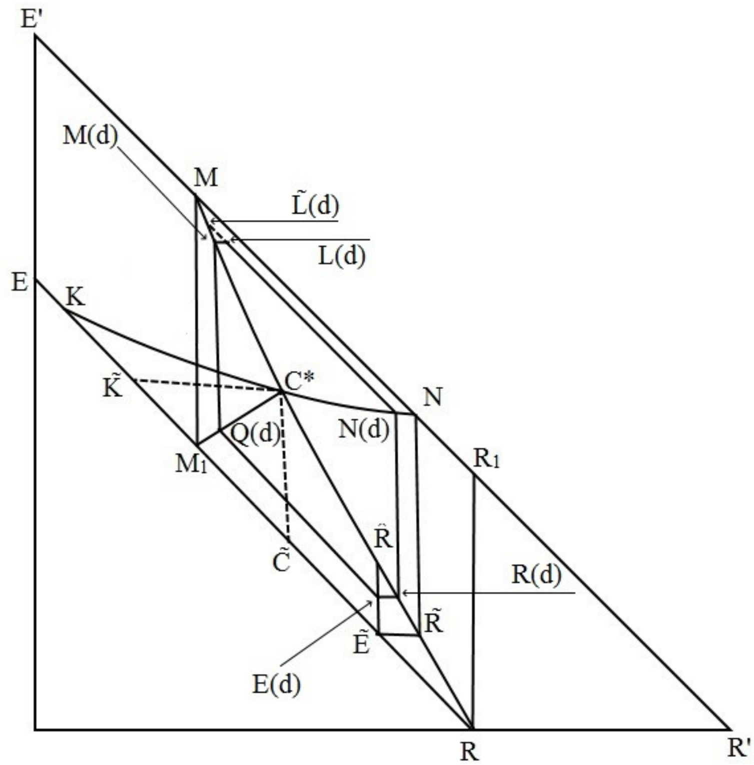

Figure 4: $K < M < N < R$, $M_1 < \widetilde{C} < \widetilde{E}$.

C^*N.

Then, as in the case 1, we have three cases of $\widetilde{L}(d)$ position: $\widetilde{L}(d) \in \Omega_3$, $\widetilde{L}(d) \in \Omega_4$, $\widetilde{L}(d) \in MC^*$. Using the *L-procedure*, we finish the polygons constructing:

$P(d) = R(d)E(d)Q(d)L(d)(or M(d))M(d)(or L(d))N(d)$, if $R(d) \in \hat{R}\widetilde{R}$,

$P(d) = R(d)Q(d)L(d)(or M(d))M(d)(or L(d))N(d)$, if $R(d) \in C^*\widehat{R}$;

3_2. $M_1 < \widetilde{E} < \widetilde{C}$ (fig. 5).

Consider the segment $\widetilde{E}C^* \subset \widetilde{K}C^*\widetilde{C}$. Similarly to M_1C^*, $\widetilde{E}C^*$ has no common points with the isoclines, except for C^*.

Let $R(d) \in C^*\widetilde{R}$ be the generating point.

$E(d) = \{C : C_2 = C_2(R(d))) \cap \widetilde{E}C^*$.

$Q(d) = \{C : C_1 + C_2 = C_1(E(d)) + C_2(E(d))\} \cap M_1C^*$.

Then, to obtain $N(d)$, $M(d), L(d)$ we use the same construction as in case 3_1.

Thus, we obtain the polygon $P(d) = R(d)N(d)M(d)(or L(d))L(d)(or M(d))Q(d)$.

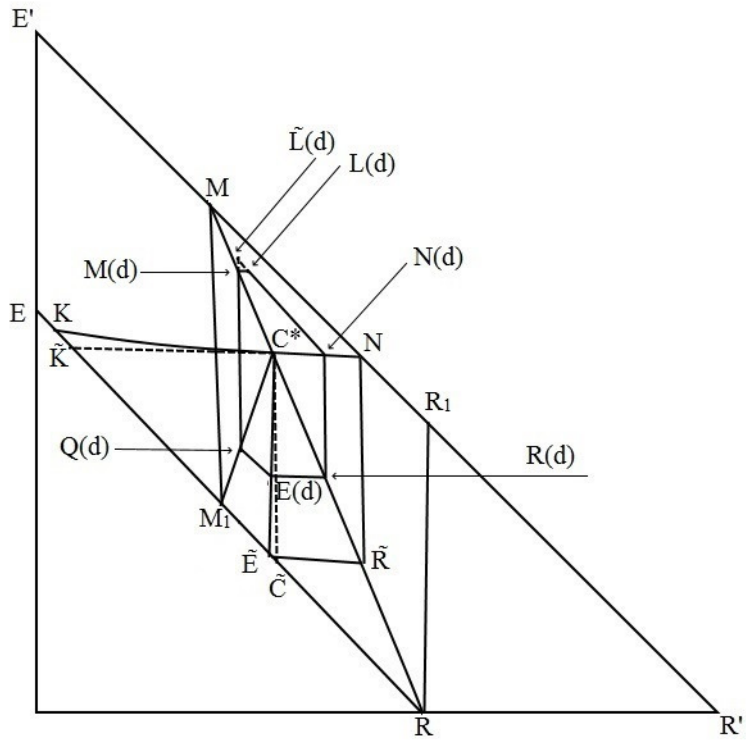

Figure 5: $K < M < N < R$, $M_1 < \widetilde{E} < \widetilde{C}$.

3_3. $\widetilde{E} < M_1 < \widetilde{C}$ (fig. 6).

Denote $\widehat{E} = \{C : C_2 = C_2(\widetilde{R})\} \cap \{C : C_1 = C_1(M)\}$. Similarly to the previous arguments, it is easy to show, that all trajectories enter the polygon $\widehat{E}\widetilde{R}NM$.

Consider the segment $\widehat{E}C^* \subset \widetilde{K}C^*\widetilde{C}$, hence, having no common points with the isoclines, except for C^*.

Let $R(d) \in C^*\widetilde{R}$ be the generating point.

Next, obtain the point $E(d) = \{C : C_1+C_2 = C_1(R(d))+C_2(R(d))\} \cap \widehat{E}C^*$.

Then, to obtain $N(d),\ M(d), L(d)$ we use the same construction as in the case 3_1.

Thus, we obtain the polygon $P(d) = R(d)N(d)M(d)(or L(d))L(d)(or M(d))E(d)$.

4) $M < K < N < R$.

Using the same arguments as in the previous cases, it can be shown that trajectories enter and do not leave the polygon $K\widetilde{E}\widetilde{R}N\widetilde{N}\widetilde{M}$, where $\widetilde{N} = \{C : C_2 = C_2(\widetilde{M})\} \cap l_1$, $\widetilde{M},\ \widetilde{E},\ \widetilde{R}$ are determined in the previous cases 1 and 3.

Consider two cases: $C^* < \widetilde{E}$, $\widetilde{E} < C^*$.

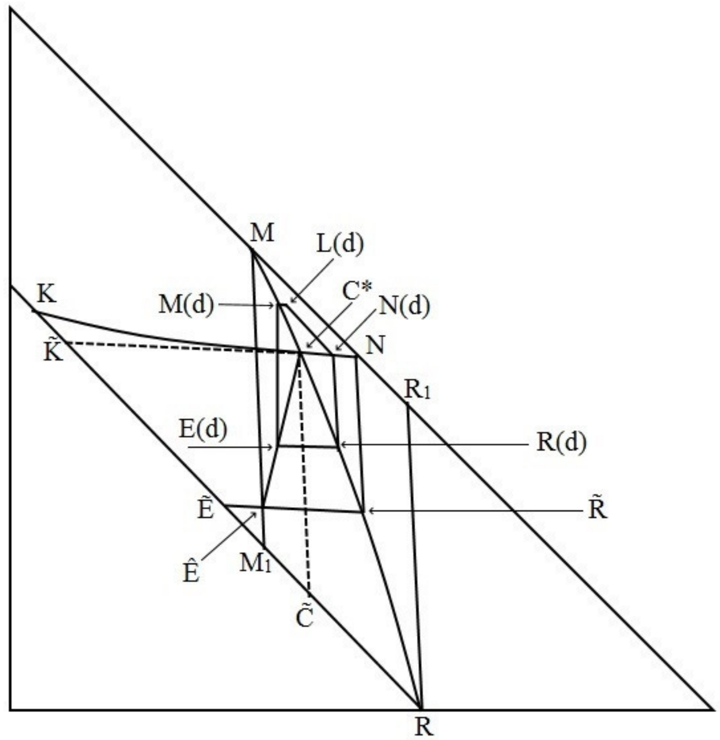

Figure 6: $K < M < N < R$, $\widetilde{E} < M_1 < \widetilde{C}$.

Assume $C^* < \widetilde{E}$ (fig. 9).

Let $R(d) \in C^*\widetilde{R}$ be the generating point.

If $R(d) \in \widetilde{R}\widehat{R}$, then $E(d) = \{C : C_2 = C_2(R(d))\} \cap \widetilde{E}\widehat{R}$, where $\widehat{R}$ is determined in the previous case 3.

$K(d) = \{C : C_1 + C_2 = C_1(E(d)) + C_2(E(d))\} \cap KC^*$.

$N(d) = \{C : C_1 = C_1(R(d))\} \cap C^*N$.

$\widetilde{L}$, $L(d)$ $M(d)$ are determined by the L-procedure (see the case 1).

Thus we obtain the polygon $P(d) = R(d)N(d)L(d)M(d)(or M(d)L(d))K(d)E(d)$.

If $R(d) \in \widehat{R}C^*$, then we obtain $K(d) = \{C : C_1 + C_2 = C_1(R(d)) + C_2(R(d))\} \cap KC^*$.

$N(d) = \{C : C_1 = C_1(R(d))\} \cap C^*N$.

$\widetilde{L}$, $L(d)$ $M(d)$ are determined by the L-procedure (see the case 1).

Thus, we obtain the polygon $P(d) = R(d)N(d)L(d)M(d)(or M(d)L(d))K(d)$.

Assume $\widetilde{E} < C^*$ (fig. 10). Let $R(d) \in C^*\widetilde{R}$ be the generating point.

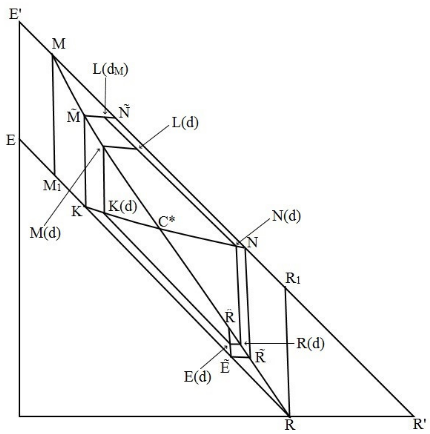

Figure 7: $M < K < N < R$, $C^* < \widetilde{E}$.

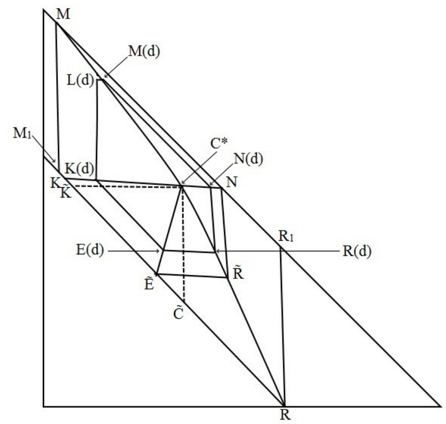

Figure 8: $M < K < N < R$, $\widetilde{E} < C^*$.

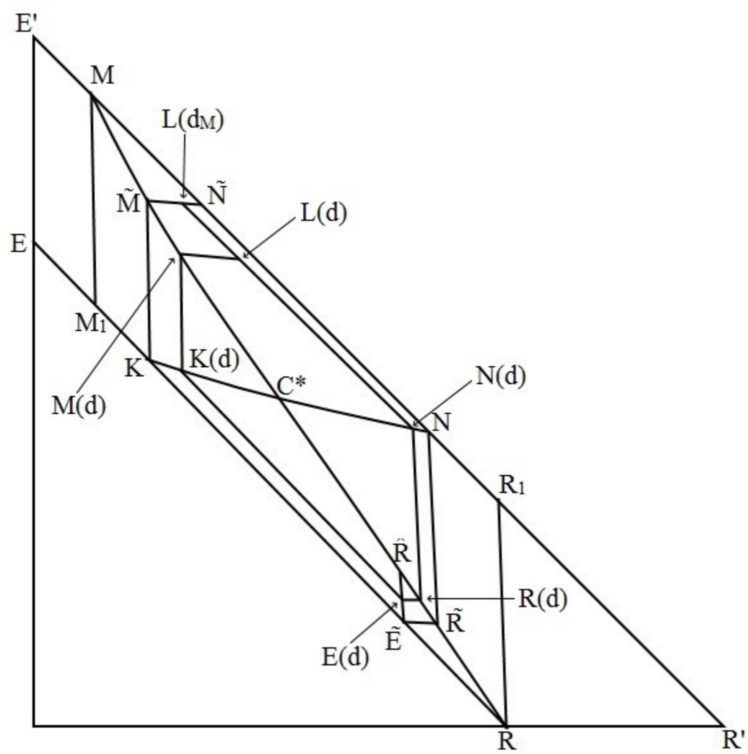

Figure 9: $M < K < N < R$, $C^* < \widetilde{E}$.

$E(d) = \{C : C_2 = C_2(R(d))\} \cap \widetilde{E}C^*$ (see the case 3).

$K(d) = \{\{C : C_1 + C_2 = C_1(E(d)) + C_2(E(d))\}\} \cap KC^*$.

$N(d) = \{C : C_1 = C_1(R(d))\} \cap C^*N$.

$\widetilde{L}$, $L(d)$ $M(d)$ are determined by the L-procedure (see the case 1).

Thus, we obtain the polygon $P(d) = R(d)N(d)L(d)M(d)(or M(d)L(d))K(d)E(d)$.

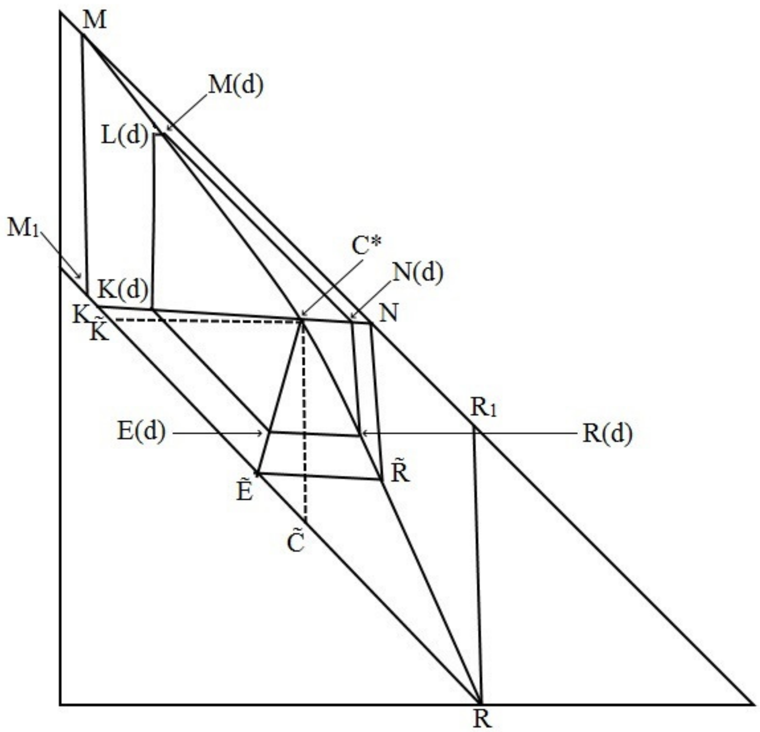

Figure 10: $M < K < N < R$, $\widetilde{E} < C^*$.

Thus, Theorem is proved. □

6 Global stability, bifurcation ($\alpha = 0$)

Suppose, $\alpha = 0$, which means that the system (1)-(2) describes the dynamics of capital distribution over the technological levels without innovation process. The simple analysis of the system (10)-(11) shows that we have two equilibria, C^* and $R = (V, 0)$ if $\beta > \mu$, or one equilibrium R if $\beta \leq \mu$.

Let us formulate results describing the long run dynamics of capital distribution over efficiency levels in the case $\alpha = 0,\ \beta > 0$.

Theorem 2 *If* $\alpha = 0,\ \beta > \mu$ *then the system* (1)-(2) *has two equilibria, namely* C^* *and* R*. In addition, if* $V > V_{\max}$ *then* C^* *is globally stable in* $\mathbb{R}^2_+ \setminus \{C : C_2 = 0\}$ *and* R *is unstable.*

Proof

The proof of global stability is analogous to the proof of Theorem 1. The remaining part of the statement is obvious. □

Theorem 3 *If* $\alpha = 0,\ \beta \leqslant \mu$ *then the system* (1)-(2) *has the unique equilibrium* R*, which is globally stable in* $\mathbb{R}^2_+ \setminus \{O\}$.

Proof

The uniqueness of equilibrium $(V, 0)$ may be obtained by the use of arguments of Lemma 2.

To prove the global stability of $(V, 0)$ in $\mathbb{R}^2_+ \setminus \{O\}$ it is sufficient to note that $(V, 0) \in \Omega$ and for $C \in \mathbb{R}^2_+ \setminus \{C : C_2 = 0\}$

$$\dot{C}_2 = C_2(a_2(V - C_1 - C_2) - \mu + \frac{\beta C_1}{c_1 + C_2}) < C_2\Big(a_2(V - C_1 - C_2) - \mu + \beta\Big) < 0.$$

Besides, there are two trajectories, namely, $\{C(t, (C_1(0), 0) : C_1(0) \in (0, V),\ t \in \mathbb{R}\}$ and $\{C(t, (C_1(0), 0) : C_1(0) > V,\ t \in \mathbb{R}\}$, such that $C(t, (C_1(0), 0) \to (V, 0)$ as $t \to \infty$. □

Suppose μ is fixed, and let us decrease β. If $\beta > \mu$ then the system (1)-(2) has two equilibria, C^* and R, globally stable in $\mathbb{R}^2_+ \setminus \{C : C_2 = 0\}$ and unstable, respectively. If $\beta = \mu$ then $C^* = R$, which is globally stable in $\mathbb{R}^2_+ \setminus \{O\}$. If $\beta < \mu$ then R is the unique, globally stable, equilibrium in $\mathbb{R}^2_+ \setminus \{O\}$, and C^* goes into a set $\{C : C_2 < 0\}$, which has no economic sense in the considered problem. Thus, we may say that μ is the bifurcation value of β.

Let us formulate the economic sense of the obtained results. If $\alpha = 0$ the appearance of the equilibrium $R = (V, 0)$ implies the situation when the second, more high, level of technology efficiency may not emerge in a finite time. Really, if the rate of depreciation is sufficiently large, i.e. $\beta \leqslant \mu$, then $R = (V, 0)$ is globally stable. If $\beta > \mu$, then $R = (V, 0)$ is the saddle and, as it is easy to show, has two stable separatrices, belonging to the set $\{C : C_2 = 0\}$. Hence, the theorem of integral continuity implies that for any small $\varepsilon > 0$ and any large $T > 0$ there exists $\delta > 0$ such that $C_2(t, C^0) < \varepsilon$ if $C_2(0) < \delta$ and $t \in [0, T]$.

The latter means that under the condition $\alpha = 0$, which means the absence of innovation process in the economical system, the imitation process only cannot initialize the emergence of the second, high, technological level within a reasonable time. The proposed model permits to obtain the estimations of parameters for which the imitation efforts do not lead to an economic system modernization.

Conclusions

The process of technological development, based on Schumpeterian classical approach, is modeled by the two-dimensional nonlinear dynamical system. The influence of innovation, imitation and depreciation processes interaction is studied. The results concerning the existence and stability of equilibria which depends on the parameters's relationships are proved. The bifurcation value of the imitation rate parameter, in the absence of innovation process, is found.

The future investigations will be dedicated to the weakening of the global stability condition. Also, it is interesting to consider the system with three technological levels.

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