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Ordinary differential equations
Numerical methods

Singularly perturbed boundary-value problems: Sundman-type transformations, test problems, exact solutions, and numerical integration

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Abstract. Solutions of singularly perturbed boundary-value problems with a small parameter are characterized by large gradients in very narrow regions (boundary layers). This circumstance sharply limits the use of standard finite-difference methods with a fixed stepsize in such problems due to significant calculation errors or possible loss of stability. This paper presents an effective method for numerical integration of singularly perturbed boundary-value problems based on replacing the spatial variable with a new independent variable of the Sundman-type, which depends on the derivatives of the unknown function. The use of such non-local transformations, which satisfy a simple asymptotic

condition, makes it possible to automatically stretch the boundary-layer region. The resulting problem turns out to be much simpler than the original one in the sense that standard (classical) numerical methods with a fixed stepsize can already be applied to solve it. Several new multiparameter nonlinear and linear singularly perturbed boundary-value problems for second-order reaction-diffusion type ODEs having monotonic and non-monotonic exact or asymptotic solutions, expressed in terms of elementary functions, are constructed. A comparison of numerical solutions with exact and asymptotic solutions is presented. The numerical results show that the method based on Sundman-type transformations for solving boundary-layer problems gives high accuracy. As a result of an extensive analysis of the obtained results, recommendations are given for the choice of regularizing functions that determine the most effective Sundman-type transformations. The difference between regularizing functions in boundary-layer problems and blow-up problems is discussed. The test problems formulated in this paper can be used to estimate the accuracy of any other numerical methods for solving two-point singularly perturbed boundary-value problems with a small parameter.

Keywords: singularly perturbed boundary-value problems, boundary layer, Sundman-type transformations, non-local transformations, nonlinear ODEs, multiparameter test problems, exact solutions, numerical integration.

Contents

1. Introduction

Singularly perturbed linear and nonlinear boundary-value problems with its highest order derivative multiplied by a small parameter are often used for mathematical modeling of various phenomena and processes in natural science and engineering (such problems are often encountered, for example, in hydro- and aerodynamics [1, 2, 3] and in the theory of convective heat and mass transfer [3, 4, 5]). An important qualitative feature of such problems is that for the zero value of a small parameter the order of the differential equation under consideration decreases and some boundary conditions cannot be satisfied.

    Singularly perturbed boundary-value problems have large gradients of solutions in narrow regions (boundary layers), which leads to a loss of convergence

of standard finite-difference methods and makes them of little use for solving problems of this type (in [6], examples of exotic incorrect numerical solutions of singularly perturbed problems obtained by ordinary non-specialized methods are given). One of the first works, where in full measure it was said on the inadmissibility of classical finite-difference schemes and the need to develop special schemes, possessing the property of convergence regardless of the value of the small parameter, there were publications [7, 8], in which the foundations of two different approaches to the solution of boundary-value problems with a boundary layer were laid. In [7], the classical central-difference scheme was proposed with a grid, thickening near boundary layers and having a uniform error in the approximation of nodes (such a grid has uniformly second-order accuracy with respect to a small parameter). In [8], an exponential fitting scheme was developed, the coefficients of which were chosen in such a way that the scheme is asymptotically exact in the boundary-layer region (this approach allows us to construct uniformly convergent finite-difference schemes on a uniform grid).

Various methods of the numerical integration for linear and nonlinear singularly perturbed boundary-value problems are considered, for example, in [9, 10, 11, 12, 13, 14, 15, 34, 16, 17, 18, 20, 19, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. For numerical solution of such problems various authors apply methods with a piecewise-uniform grid (two-grid methods) characterizing by a small stepsize in the boundary layer and a large stepsize outside the boundary layer (for example, see [11, 13, 21, 22, 23, 26, 29]). It is important to note that in the methods based on the use of a piecewise-uniform grid (as well as the methods developed in [7, 8]), a priori information on the structure and the rate of damping of asymptotic solutions in the boundary layer is taken into account (explicitly or implicitly).

In [35] (see also [36]), for numerical integration of singularly perturbed boundary-value problems with a single boundary layer, it was proposed to apply non-local transformations at the initial stage, which allows further integration of the reduced problem by standard numerical methods applying a uniform grid for the computational domain. This paper is a further deepening and development of the approach [35], which is based on the application of non-local transformations (generalized Sundman transformations) for the numerical integration of singularly perturbed boundary-value problems. Extensive testing of this method has been carried out on new transformations and new, more complex nonlinear multiparameter test problems, the solutions of which are expressed in elementary functions. Comparison of numerical, exact, and

asymptotic solutions showed the high efficiency of the Sundman transformations method for solving singularly perturbed problems with a boundary layer (even for problems with degeneracy, when the highest derivative vanishes at the boundary).

Remark 1. *Generalized Sundman transformations (non-local transformations) were used in [37, 38, 39, 40, 41, 42] for numerical integration of Cauchy problems with monotonic and non-monotonic blow-up solutions that have very large gradients in a neighborhood of a priori unknown point. Comparison of exact and numerical solutions of several test problems for ODEs of the first, second, third and fourth orders, as well as systems of nonlinear coupled ODEs, showed high efficiency of this method for numerical integration of blow-up problems. In [42], non-local transformations were used for numerical integration of blow-up problems for nonlinear PDEs.*

Remark 2. *In [44, 45, 46, 47, 48], generalized Sundman transformations of special type were applied to obtain exact solutions, first integrals, and also to linearize some second-order ODEs.*

Remark 3. *In [1, 49, 50, 51, 52, 34, 53, 54, 55], asymptotic methods are described that make it possible to find approximate analytical solutions to boundary-layer problems.*

Note that hypersingular nonlinear boundary-value problems are not considered in this paper (in hypersingular boundary-value problems with a small parameter ε, super-thin boundary layers arise, and the derivative at the boundary layer can have very large values of the order of $e^{1/\varepsilon}$ and more [43] (in ordinary problems with boundary layers, the derivative at the boundary usually has the order of ε^{-1} or $\varepsilon^{-1/2}$)).

2. Qualitative features of singularly perturbed boundary-layer problems with a small parameter

2.1. Illustrating a three-parameter linear problem. Exact and asymptotic solutions. Boundary-layer region

Let us analyze qualitative features of singularly perturbed boundary-value problems with a small parameter at the highest derivative, that have solutions of the boundary-layer type, on an example of a specific problem.

Test problem 1. We consider the following three-parameter boundary-value

problem for the second-order linear equation with constant coefficients [35]:

$$\varepsilon y''_{xx} + y'_x + y = 0 \quad (0 < x < 1); \quad y(0) = a, \quad y(1) = b, \quad (1)$$

where a, b, and ε are free parameters. A boundary layer for this problem develops near the point $x = 0$ as $\varepsilon \to 0$ ($\varepsilon > 0$).

Depending on the values of the determining parameters, problem (1) can have both monotonic and non-monotonic solutions. The exact solution of the problem is determined by the formulas

$$y = \frac{a \exp(\lambda_2) - b}{\exp(\lambda_2) - \exp(\lambda_1)} \exp(\lambda_1 x) + \frac{b - a \exp(\lambda_1)}{\exp(\lambda_2) - \exp(\lambda_1)} \exp(\lambda_2 x),$$
$$\lambda_1 = \frac{-1 - (1 - 4\varepsilon)^{1/2}}{2\varepsilon}, \quad \lambda_2 = \frac{-1 + (1 - 4\varepsilon)^{1/2}}{2\varepsilon}. \quad (2)$$

If ε is small, we find $\lambda_1 \simeq -\varepsilon^{-1}$, $\lambda_2 \simeq -1$, $y'_x(0) \simeq -\varepsilon^{-1}(a - eb)$, and $y''_{xx}(0) \simeq \varepsilon^{-2}(a - eb)$. From the above relation for the first derivative, it follows that when using direct numerical methods to solve such singularly perturbed boundary-value problems with a small parameter, the shooting numerical procedure should begin with large values of the derivative (of order ε^{-1}), which is a complicating factor for numerical methods.

Let us assume $a \geq 0$ and $b \geq 0$. If $a > eb$, then solution (2) decreases monotonically and if $a < eb$, then solution (2) increases monotonically, and very quickly, in a narrow region $0 \leq x < x_*$, where

$$x_* \simeq \varepsilon \ln(1/\varepsilon), \quad y_* \simeq eb \quad \text{(for case } a = 0\text{)}. \quad (3)$$

In the remaining region $x_* \leq x \leq 1$, the solution is a monotonically decreasing function that changes slowly enough.

The exact solutions (2) of problem (1) are shown in Fig. 1 by the solid lines for two sets of numerical values of the determining parameters: *a)* $a = 1$, $b = 0$, $\varepsilon = 0.005$ and *b)* $a = 0$, $b = 1$, $\varepsilon = 0.005$. One can see that for the second set of values the solution in the region $0 \leq x \leq x_* \approx 0.026709653$ increases rapidly (the maximum value $y_* \approx 2.646247631$), and in the region $x_* \leq x \leq 1$ it decreases slowly. For comparison in Fig. 1, we also present the curves obtained by using the exact solution (2) for $\varepsilon = 0.05$ and $\varepsilon = 0.5$ (the dashed and dash-dotted lines, respectively). It is seen that as the parameter ε decreases near the point $x = 0$, the curved lines become steeper and steeper, approaching the vertical line.
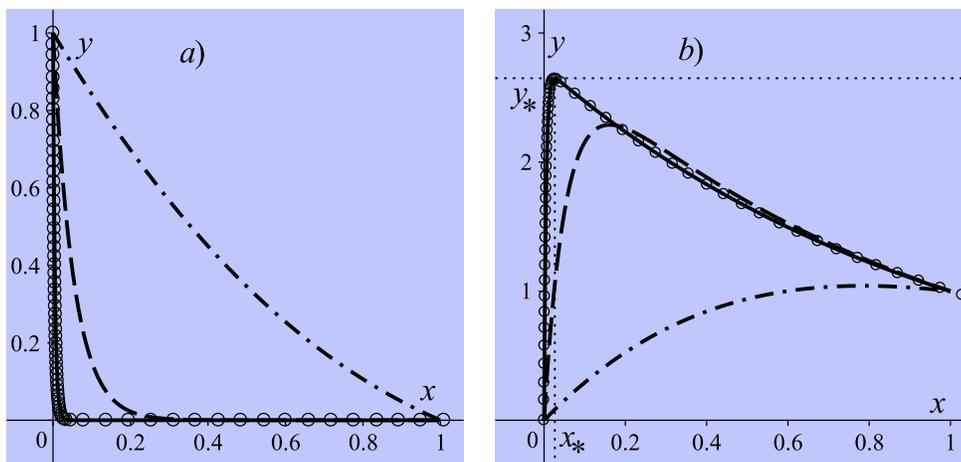
Figure 1: Exact solutions (2) of problem (1) (solid lines) and numerical solutions of the transformed problem (8) (points) with $g = 1 + |z| + |f|^{1/2}$ for two sets of numerical values of the determining parameters: *a*) $a = 1$, $b = 0$, $\varepsilon = 0.005$ and *b*) $a = 0$, $b = 1$, $\varepsilon = 0.005$ (the dashed and dash-dotted lines are obtained by using formula (2) for $\varepsilon = 0.05$ and $\varepsilon = 0.5$, respectively).

When applying direct numerical methods in such problems, to take into account the singularities of the solution in the boundary-layer region, it is necessary to take sufficiently many points in a small neighborhood of the left boundary. Therefore, the use of uniform grids throughout the domain of variation of the independent variable x is connected with the necessity of partitioning the region $0 \leq x \leq 1$ into a large number of intervals of integration.

2.2. Order relation between the first and second derivatives

Let us assume that $|a - eb| = O(1)$. Then, taking into account the estimates of the first and second derivatives given after solution (2), we obtain the following order relation between the derivatives on the left boundary:

$$|y''_{xx}| \simeq |y'_x|^2. \tag{4}$$

Following [35], we will now show that under certain assumptions about the local behavior of the solutions, the order relation between derivatives (4) is sufficiently general and is valid if, inside the boundary layer region, the leading term of the asymptotic expansion of the solution as $\varepsilon \to 0$ has the form $y = \varphi(x/\delta)$, where $\varphi = \varphi(z)$ is a smooth function, the first and second derivatives of which are bounded and non-vanishing in some neighborhood of the point $z = 0$, and $\delta = \delta(\varepsilon)$ is a function with the property $\delta \to 0$ as $\varepsilon \to 0$. Indeed, differentiating twice the function $y = \varphi(x/\delta)$ with respect to x, we find the

derivatives $y'_x = \delta^{-1}\varphi'_z(z)$ and $y''_{xx} = \delta^{-2}\varphi''_{zz}(z)$. Eliminating δ from these formulas, we obtain the order relation (4), which we will need later.

3. Numerical integration of boundary-value problems based on Sundman transformations

3.1. Description of the solution method

Let us consider two-point problems for second-order nonlinear ODEs with boundary conditions of the first kind (Dirichlet boundary conditions):

$$y''_{xx} = f(x, y, y'_x) \quad (0 < x < 1); \tag{5}$$
$$y(0) = a, \quad y(1) = b, \tag{6}$$

where the function f can also depend on the small parameter $\varepsilon > 0$. For singularly perturbed boundary-value problems we have $f = \varepsilon^{-1}F(x, y, y'_x, \varepsilon)$, where $F(x, y, z, 0)$ is a smooth function of three arguments.

We introduce a new non-local (Sundman-type) independent variable ξ by means of the first-order differential equation and the initial condition [35] (see, also [37, 38, 39]):

$$\xi'_x = g(x, y, y'_x), \quad \xi(0) = 0, \tag{7}$$

where $g = g(x, y, y'_x)$ is a regularizing function that must satisfy some simple conditions described below.

We introduce the notation $z = y'_x$. It is not difficult to show that the original boundary-value problem (5)–(6) after passing to the Sandman-type variable (7) is transformed to the following problem for the system of three first-order coupled ODEs [35]:

$$x'_\xi = \frac{1}{g(x, y, z)}, \quad y'_\xi = \frac{z}{g(x, y, z)}, \quad z'_\xi = \frac{f(x, y, z)}{g(x, y, z)} \quad (0 < \xi < \xi_1); \tag{8}$$
$$x(0) = 0, \quad y(0) = a, \quad y(\xi_1) = b,$$

where the value ξ_1 is determined during the solution process when the final condition $x(\xi_1) = 1$ is reached.

Let us consider the simplest regularizing function: $g \equiv 1$. In this case, the first ODE of system (8), taking into account the initial condition $x(0) = 0$, gives $\xi = x$ and numerical integration of the remaining two ODEs is equivalent to the integration of the original problem (5)–(6).

For the general case, the regularizing function $g = g(x, y, z)$, successfully selected, itself will determine the position and the density of points of integration with respect to the original variables x and y and allow more accurately to solve problem (8) using the shooting method or other standard fixed-step numerical methods with respect to the new non-local variable ξ [56, 57, 58, 59, 60, 61].

3.2. Regularizing functions. The asymptotic condition. Examples

For numerical solution of singularly perturbe boundary-value problems, as well as for solving blow-up Cauchy problems, it is reasonable to use regularizing functions of the form [37, 38, 39]:

$$g = G(|z|, |f|) \equiv G(|y'_x|, |y''_{xx}|), \tag{9}$$

where $f = f(x, y, z)$ is the right-hand side of equation (5) and $z = y'_x$. We impose the following conditions on the function $G = G(u, v)$:

$$G > 0; \quad G_u \geq 0, \quad G_v \geq 0; \quad G \to \infty \quad \text{as} \quad u + v \to \infty; \quad G(0, 0) = 1 \tag{10}$$

with $u \geq 0$ and $v \geq 0$.

When using regularizing functions of the form (9) for flat and rectilinear segments of the curve $y = y(x)$, on which $y'_x = \text{const}$, a fixed stepsize in ξ gives a fixed stepsize in x. Note that if equation (5) is autonomous (that is, the equation does not depend explicitly on x) and the regularizing function is chosen in the form (9), then the second and the third equations of system (8) form an independent subsystem that is integrated independently of the first equation.

If the right-hand side of the singularly perturbed ODE (5) has the form

$$f(x, y, y'_x) = \varepsilon^{-1} F(x, y, y'_x),$$

when choosing regularizing functions, besides the simple conditions (9)–(10), additional considerations should be taken into account.

If the Sundman transformations are not applied, i.e. $g = 1$, and $\varepsilon \to 0$, the right-hand sides of the last two ODEs in (8) will tend to infinity (since $|z| \to \infty$ and $|f| \to \infty$) in the boundary layer. In this region, we also have $|f| = O(z^2)$, which follows from the order relation (4). The specified qualitative features of such problems significantly complicate the use of standard numerical methods

for their integration and lead to the necessity of proportional refinement of the stepsize as ε decreases.

To avoid refining the grid as $\varepsilon \to 0$ and to use a fixed stepsize with respect to the Sundman-type variable ξ, one should choose suitable regularizing functions satisfying the following asymptotic condition [35]:

$$|z|/g = O(1) \quad \text{as} \quad \varepsilon \to 0. \tag{11}$$

Then in the boundary layer the right-hand side of the second ODE in (8) will not have singularities as $\varepsilon \to 0$, and the third ODE in (8) will have a substantially smaller singularity than for $g = 1$. It follows from (10) that it is possible to choose regularizing functions which have asymptotics

$$g \to m_1|z| \quad \text{as} \quad |z| \to \infty \quad \text{or} \quad g \to m_2|f|^{1/2} \quad \text{as} \quad |f| \to \infty, \tag{12}$$

where m_1 and m_2 are positive constants of the order of unity. Such asymptotic conditions (as well as the normalization condition in (10)) can be satisfied, for example, if we take simple regularizing functions of the form

$$g = 1 + |z| \quad \text{or} \quad g = (1 + |f|)^{1/2}. \tag{13}$$

Let us consider in more detail the first regularizing function (13). In this case, the inequality $|z|/g < 1$ holds. From the second equation of system (8) we get $|y'_\xi| \leq 1$. In the boundary-layer region, where the derivative z is large as $\varepsilon \to 0$, we have $|y'_\xi| \approx 1$, which leads to a linear dependence $y \approx \pm\xi + \text{const}$. The right-hand side of the third equation of system (8), taking into account relation (4) (that is $|f| \sim z^2$), becomes linear with respect to z in this region. Outside the boundary layer, system (8) does not have qualitative features. The second regularizing function (13) has similar qualitative features.

Thus, the use of regularizing functions (13) allows us to completely suppress the unbounded growth of the right-hand side of the second equation of system (8) in the boundary-layer region as $\varepsilon \to 0$ and to reduce (in comparison with $g = 1$) the right-hand side of the third equation. Further, by using concrete examples, it will be shown that regularizing functions of the form (13) have a limited scope.

It is also advisable to use more complicated regularizing functions of the mixed form

$$g = (1 + k_1 z^2 + k_2|f|)^{1/2}, \tag{14}$$

$$g = 1 + k_1|z| + k_2|f|^{1/2}, \tag{15}$$

$$g = 1 + k \max(|z|, |f|^{1/2}), \tag{16}$$

which satisfy the asymptotic condition (11) as well as the normalization condition in (10). Formulas (14)–(16) include the parameters $k_1 \geq 0$, $k_2 \geq 0$, and $k > 0$ which can vary (with $k_1 + k_2 = O(1)$ and $k = O(1)$). Note that the regularizing function (14) was proposed in [35]. It will be further shown that regularizing functions (15)–(16) are more universal than functions (13) and (14) and lead to much more accurate numerical solutions.

3.3. Additional explanations and comments

$1°$. From equation (7) and formula (9), for small increments of the argument Δx, neglecting terms of the order of $o(\Delta x)$, we obtain

$$\Delta \xi = G(|z|, |f|) \Delta x,$$

where $z = y'_x$ and $f = y''_{xx}$.

It follows that the choice of a fixed stepsize for a new variable $\Delta \xi = h$ is equivalent to using a variable stepsize for the original independent variable $\Delta x = h/G$. Suppose that $G = G(u) > 0$, $G(0) = 1$, $G'_u > 0$, and $u = |z|$. Then an increase in the derivative z with respect to x leads to a decrease in the stepsize Δx. Therefore, the application of the generalized Sundman transformations corresponds to the automatic choice of the stepsize with respect to x. When using formulas (13) and (14) the stepsize Δx is small in the boundary-layer region.

$2°$. It is useful to additionally transform problem (8), which was obtained after the transition to a new (Sundman-type) independent variable, by introducing a normalized derivative by the formula $\bar{z} = \varepsilon z$. In this case, for the regularizing functions (13) and (14), the right-hand sides of the second and third equations obtained from (8) (and written in the form resolved with respect to derivatives) will be bounded in the boundary-layer region.

$3°$. Methods, based on the use of a piecewise-uniform grid (see, for example, [22, 23]) are particular degenerate cases of the method based on generalized Sundman transformations with a piecewise-smooth regularizing function of the form $g = \begin{cases} c_1 & \text{if } 0 \leq x \leq \sigma, \\ c_2 & \text{if } \sigma < x \leq 1, \end{cases}$, where c_1 and c_2 are constants, $\sigma = \sigma(\varepsilon)$, and $\sigma \to 0$ as $\varepsilon \to 0$.

$4°$. For moderately small values of ε ($\varepsilon_{\min} < \varepsilon \ll 1$), regularizing functions that do not satisfy the asymptotic condition (11) can be used. However, such

functions usually lead to significant computation errors at sufficiently small $\varepsilon \ll \varepsilon_{\min}$.

5°. In this paper, we apply a combination of the classical explicit Runge–Kutta method of the fourth-order of approximation with a fixed stepsize and a specific shooting procedure with Maple implementation [60]. For this purpose, an auxiliary Cauchy problem described by the transformed equations (8) with the first two conditions at the left extreme point and an additional initial condition $z(0) = s$ is numerically integrated. The value of the parameter s is determined by satisfying the boundary condition at the right extreme point $x = 1$ (see the last boundary condition in (8)). Similarly, after applying a generalized Sundman transformation to the original problem in the first stage, any effective numerical methods with a fixed stepsize (for example, explicit and implicit Runge–Kutta methods, explicit Adams–Bashforth and implicit Adams–Moulton methods [62, 63, 64]) can be used in the final stage.

The general flow chart showing the main steps needed to implement the proposed method is displayed in Fig. 2.

4. Singularly perturbed linear boundary-value problems. Numerical and analytical solutions

4.1. Linear simple test problem. Numerical solutions obtained by using various regularizing functions

First, we illustrate the characteristic features of the proposed method on simple linear problems.

The results of numerical integration of the transformed problem (8), used for solving the original problem (1), for $f = -\varepsilon^{-1}(z + y)$ with the regularizing function $g = 1 + |z| + |f|^{1/2}$, which are obtained by the shooting method (from the point $x = 0$) with the fixed stepsize $h = 0.01$ by using Maple [60], are shown in Fig. 1 by points for two sets of numerical values of the determining parameters. The first set is $a = 1$, $b = 0$, $\varepsilon = 0.005$ (monotonic solution) and the second set is $a = 0$, $b = 1$, $\varepsilon = 0.005$ (non-monotonic solution).

It can be seen that there is a good coincidence between the numerical solutions and the corresponding exact solutions, which are determined by formula (2) and are represented by solid lines.

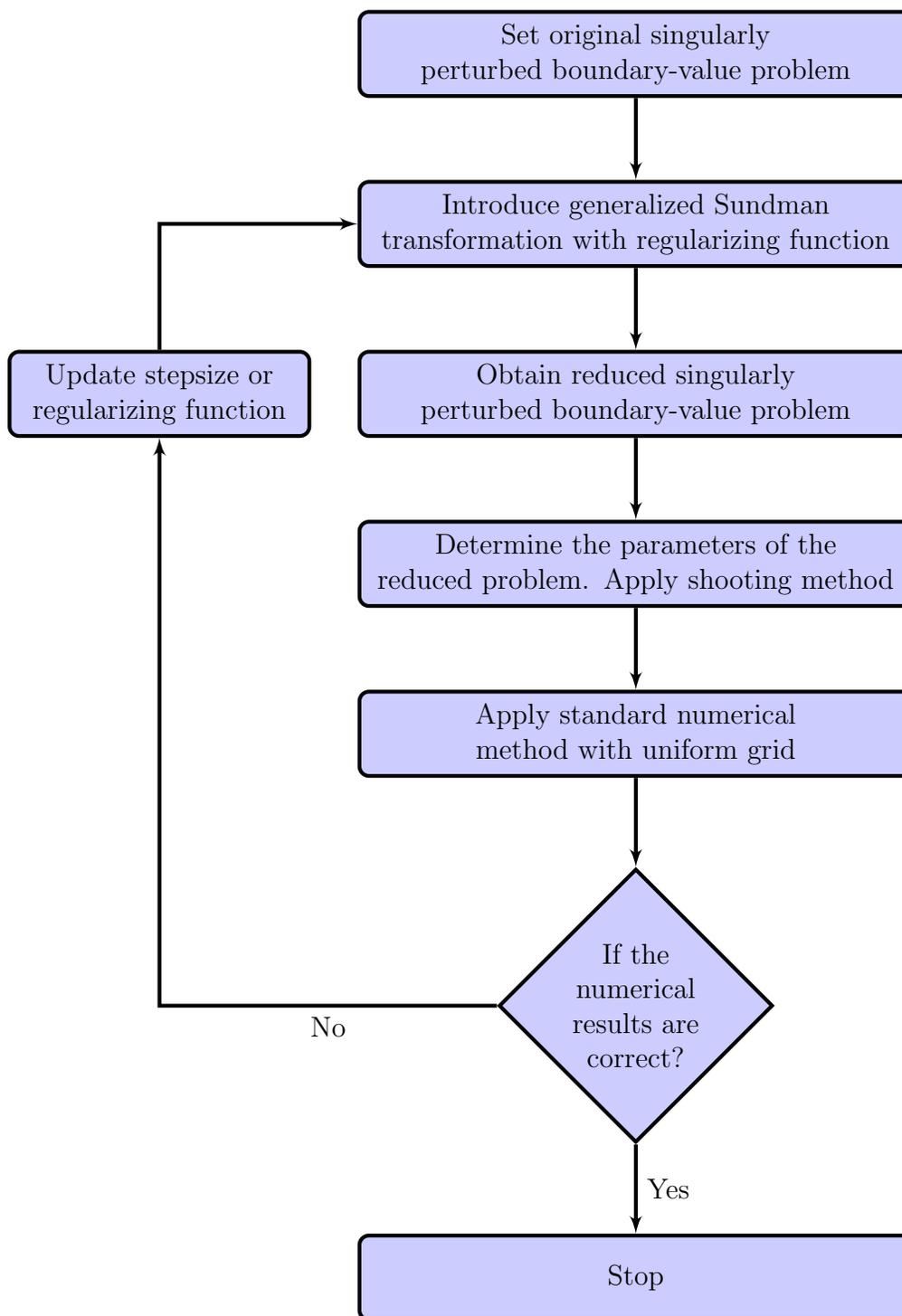Table 1 shows the maximum absolute errors of numerical solutions of the

Figure 2: The flow chart of the proposed method described in Section 3.

transformed problem (8) used for numerical integration of the original problem (1) with $\varepsilon = 0.005$ obtained by the method based on generalized Sundman transformations for $a = 1$, $b = 0$ and $a = 0$, $b = 1$ with three stepsizes h and eight different regularizing functions g. For comparison, similar data are also indicated for the case $g = 1$, which corresponds to the direct numerical integration (without using Sundman-type transformations) with the same stepsize with respect to x.

The maximum absolute error of the numerical solutions for $a = 1$, $b = 0$				
No.	Regularizing function	Stepsize 0.1	Stepsize 0.05	Stepsize 0.01
1	$g=1+|z|$	0.017119347	0.006702741	0.000137030
2	$g=(1+|f|)^{1/2}$	0.000707586	0.000160259	0.000001602
3	$g=(1+|z|+|f|)^{1/2}$	0.000611528	0.000146118	0.000001741
4	$g=(1+z^2+|f|)^{1/2}$	0.000900004	0.000204128	0.000001775
5	$g=(1+z^4+f^2)^{1/4}$	0.000886025	0.000193071	0.000002601
6	$g=1+|z|+|f|^{1/2}$	0.000512010	0.000112509	0.000000410
7	$g=(1+R_1)^{1/2}$	0.000707586	0.000160259	0.000001602
8	$g=1+R_2$	0.000550849	0.000119910	0.000000414
9	$g=1$	process diverges	process diverges	0.193331173
The maximum absolute error of the numerical solutions for $a = 0$, $b = 1$				
No.	Regularizing function	Stepsize 0.1	Stepsize 0.05	Stepsize 0.01
1	$g=1+|z|$	0.047029578	0.013710597	0.000713696
2	$g=(1+|f|)^{1/2}$	0.000824707	0.000249922	0.000001663
3	$g=(1+|z|+|f|)^{1/2}$	0.000570299	0.000115649	0.000000554
4	$g=(1+z^2+|f|)^{1/2}$	0.000559160	0.000109360	0.000000180
5	$g=(1+z^4+f^2)^{1/4}$	0.000630398	0.000136417	0.000000390
6	$g=1+|z|+|f|^{1/2}$	0.000265927	0.000025385	0.000000017
7	$g=(1+R_1)^{1/2}$	0.000592523	0.000122175	0.000000346
8	$g=1+R_2$	0.000602708	0.000090517	0.000000145
9	$g=1$	process diverges	process diverges	0.528189578

Table 1: Comparison of the accuracy of numerical solutions of the transformed problem (8), which is used to solve the original problem (1), for various regularizing functions g with $\varepsilon = 0.005$ and three stepsizes h. Here $R_1 = \max(z^2, |f|)$, $R_2 = \max(|z|, |f|^{1/2})$.

It can be seen that the seven regularizing functions (Nos. 2–8) make it possible to obtain numerical solutions in the entire region with high accuracy even with a sufficiently large stepsize (with respect to ξ) equal to $h = 0.1$.

It is important to note that the first five regularizing functions (Nos. 1–5) were used in [35], and the three functions (Nos. 6–8) are new. It can be seen that the best results are obtained by using the new regularizing function No. 6: for a

nonmonotonic solution and the stepsize 0.01, it provides an order of magnitude of the numerical solution more accurate than the other regularizing functions. Function No. 8 also provides high accuracy of calculations. Further comparison of numerical and exact solutions in more complex nonlinear test boundary layer problems will lead to the same conclusions: these two regularizing functions lead to very good results in the vast majority of cases. Here and further on, for greater clarity, the two best results in each column of all tables are highlighted in lilac.

The maximum absolute error of the numerical solutions for $a = 1$, $b = 0$				
No.	Regularizing function	N=100	N=200	N=500
1	$g = 1 + \|z\|$	0.002126935	0.000129392	0.000001946
2	$g = (1 + \|f\|)^{1/2}$	0.000183256	0.000007810	0.000000141
3	$g = (1 + \|z\| + \|f\|)^{1/2}$	0.000227354	0.000007881	0.000000140
4	$g = (1 + z^2 + \|f\|)^{1/2}$	0.000216955	0.000012022	0.000000216
5	$g = (1 + z^4 + f^2)^{1/4}$	0.000242947	0.000013334	0.000000289
6	$g = 1 + \|z\| + \|f\|^{1/2}$	0.000322285	0.000010201	0.000000132
7	$g = (1 + R_1)^{1/2}$	0.000188884	0.000007943	0.000000139
8	$g = 1 + R_2$	0.000152543	0.000002787	0.000000035
9	$g = 1$	0.193331172	0.006948616	0.000105565
The maximum absolute error of the numerical solutions for $a = 0$, $b = 1$				
No.	Regularizing function	N=100	N=200	N=500
1	$g = 1 + \|z\|$	0.022065809	0.001390730	0.000470727
2	$g = (1 + \|f\|)^{1/2}$	0.000685290	0.000129855	0.000006104
3	$g = (1 + \|z\| + \|f\|)^{1/2}$	0.000481694	0.000019363	0.000000765
4	$g = (1 + z^2 + \|f\|)^{1/2}$	0.000762107	0.000039963	0.000000667
5	$g = (1 + z^4 + f^2)^{1/4}$	0.000751407	0.000081003	0.000000685
6	$g = 1 + \|z\| + \|f\|^{1/2}$	0.001389189	0.000027408	0.000000479
7	$g = (1 + R_1)^{1/2}$	0.000729929	0.000060206	0.000000643
8	$g = 1 + R_2$	0.000617123	0.000016893	0.000000338
9	$g = 1$	0.528189578	0.018983935	0.000288408

Table 2: Comparison of the accuracy of numerical solutions of the transformed problem (8), which is used to solve the original problem (1), for various regularizing functions g with $\varepsilon = 0.005$ for a different number of grid points N. Here $R_1 = \max(z^2, |f|)$, $R_2 = \max(|z|, |f|^{1/2})$.

Table 2 shows the results allowing to compare the efficiency of various regularizing functions that are used for numerical solution of the transformed problem (8) with $\varepsilon = 0.005$ for $a = 1$, $b = 0$ and $a = 0$, $b = 1$ for a different number of grid points $N = \xi_1/h$ (here ξ_1 is the length of the interval of numerical integration of the transformed equations).

Let us compare, for example, the maximum absolute errors of the numerical solutions obtained for a given ε with the regularizing function No. 8 and function No. 9 (the latter solution is obtained without using transformations) with the same number of grid points. In this case, for $N = 100$, the use of generalized Sundman transformations makes it possible to increase the accuracy of the numerical solution by more than 1200–3000 times, and for $N = 500$, approximately 850 times (as the number of grid points increases, the error of the numerical solution, obtained without using transformations, gradually decreases).

It can be seen that for all regularizing functions considered in Table 2, the accuracy of the numerical integration of problem (1) with a monotone solution for $a = 1$, $b = 0$ is significantly higher than the accuracy of the numerical integration of problem (1) with a non-monotonic solution for $a = 0$, $b = 1$.

It can also be seen that function No. 1 is low effective for the non-monotonic solution. This is due to the fact that near the boundary layer there is a sharp extremum at the point $x = x_*$ (see Fig. 1b) where the derivative vanishes, $y'_x|_{x=x_*} = z|_{x=x_*} = 0$. Using formula (2) with $a = 0$ and $b = 1$, we find the curvature at the extremum point

$$k_* = \frac{|y''_{xx}|}{(1 + |y'_x|^2)^{3/2}}\bigg|_{x=x_*} = |y''_{xx}|_{x=x_*} \simeq e\left(\frac{1}{\varepsilon} - 1\right). \tag{17}$$

Thus, the curvature of k_* tends to infinity as $\varepsilon \to 0$. Therefore, in the vicinity of the point x_*, where the most dramatic qualitative changes of the solution take place, it is necessary to make a small stepsize to obtain high accuracy of numerical calculations. When choosing the regularizing function No. 1 in Table 2, we have $g = 1 + |z| \approx 1$ in the vicinity of the extremum, so the stepsize in x (for a fixed N) here will be maximal (whereas a small stepsize is required for this region with a large curvature). This circumstance leads to a large calculation error in the extremum region for the function $g = 1 + |z|$.

To explain the above more clearly, let us make a linear-fractional transformation of the independent variable

$$X = \frac{(\beta + 1)x}{x + \beta}, \quad \beta = \frac{x_*}{1 - 2x_*}, \tag{18}$$

which stretches the boundary layer area and translates the point x_* in the x, y plane to the point $X_* = 1/2$ in the X, y plane (and leaves the endpoints of the considered interval motionless).

In Fig. [3](a), the exact solution (2) of problem (1) is shown by the thick solid line for $a = 0$, $b = 1$, $\varepsilon = 0.005$ in the X, y plane. Also in Fig. [3](b), we present the behavior of the absolute error E of the numerical solution of the transformed problem (8) for the regularizing function $g = 1 + |z|$ with the stepsize 0.01 and the same values of the parameters. It can be seen that the maximum absolute error in this case is located in the transition area between the boundary layer and the outer region (near the extremum of the function y).
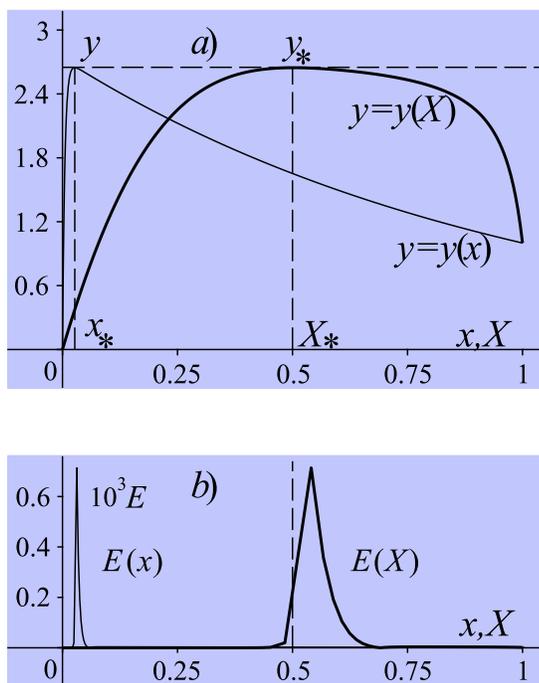

Figure 3: *a*) Exact solution (2) of problem (1) with $a = 0$, $b = 1$, and $\varepsilon = 0.005$ in the x, y plane (thin solid line) and in the X, y plane (thick solid line); *b*) absolute error E of the numerical solution of the transformed problem (8) with the regularizing function $g = 1 + |z|$ and the same values of the parameters in the x, y plane (thin solid line) and in the X, y plane (thick solid line).

The regularizing functions of mixed type Nos. 3–5 in Tables [1](1) and [2](2), in addition to $z = y'_x$ also including $f = y''_{xx}$, satisfy the asymptotic condition (11) in the boundary-layer region.

The inclusion of the second derivative in these formulas allows us to take into account the large curvature in the vicinity of the extremum point (see (17)), and accordingly, reduces the stepsize (with respect to x) in this area. Therefore, the accuracy of numerical solutions obtained by using the regularizing functions Nos. 3–5 is much higher than by using function No. 1.

For the non-monotonic solution (with $a = 0$, $b = 1$) the best results are given by the regularizing function No. 4.

In Fig. 4, the numerical integration results of the transformed problem (8), used for solving the original problem (1), are shown for $a = 0$, $b = 1$, $\varepsilon = 0.005$, $h = 0.01$, and $g = (1 + z^2 + |f|)^{1/2}$. For clarity, solutions are given using the original independent variable x and normalized Sundman-type variable $\bar{\xi} = \xi/\xi_1$, which stretches the boundary-layer region.

Precisely such a stretching of the region with large gradients, which occurs automatically after choosing a suitable regularizing function, reduces the stepsize in x and ensures high accuracy of numerical solutions in the boundary layer.

Further, we will consider only boundary-layer type problems that have non-monotonic solutions (since such problems present the greatest difficulties for numerical integration).
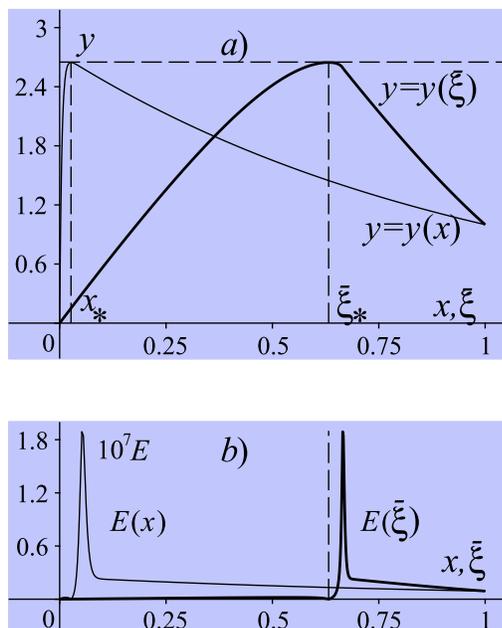
Figure 4: *a*) Numerical solutions of the transformed problem (8), which is used to solve the original problem (1) with $a = 0$, $b = 1$, and $\varepsilon = 0.005$, for $g = (1+z^2+|f|)^{1/2}$ and $h = 0.01$: $y(x)$ (thin solid line) and $y(\bar{\xi})$, where $\bar{\xi} = \xi/\xi_1$ (thick solid line); *b*) absolute errors $E(x)$ and $E(\bar{\xi})$ of numerical solutions of the transformed problem (8) for the same values of the parameters and the regularizing function g.

4.2. Multiparameter linear boundary-value problem having solutions with several extrema

Test problem 2. Consider now the more complex five-parameter linear boundary-value problem

$$\varepsilon y''_{xx} + y'_x + c\cos(\lambda x) = 0 \quad (0 < x < 1); \quad y(0) = a, \quad y(1) = b, \qquad (19)$$

where a, b, c, λ, and ε are free parameters. Depending on the values of the parameters, this problem can have one, two and more extrema or cannot have them at all.

It is easy to show that the exact solution of problem (19) is determined by the formulas

$$
y = A + Be^{-x/\varepsilon} + S(x), \quad S(x) = \frac{c[\varepsilon\lambda\cos(\lambda x) - \sin(\lambda x)]}{\lambda(1 + \varepsilon^2\lambda^2)},
$$
$$
A = \frac{b - S(1) + [S(0) - a]e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}, \quad B = \frac{a - b + S(1) - S(0)}{1 - e^{-1/\varepsilon}}.
\qquad (20)
$$

The corresponding asymptotic solution as $\varepsilon \to 0$ has the form

$$y = b + c\frac{\sin\lambda}{\lambda} + \left(a - b - c\frac{\sin\lambda}{\lambda}\right)e^{-x/\varepsilon} - \frac{c}{\lambda}\sin(\lambda x). \qquad (21)$$

Let us consider in more detail a special case of $a = 0$, $b = c = 1$, and $\lambda = \pi n$ ($n = 1, 2, \dots$). Substituting the indicated values of the parameters in (21), we obtain

$$y = 1 - e^{-x/\varepsilon} - \frac{1}{\lambda}\sin(\lambda x), \quad \lambda = \pi n. \qquad (22)$$

The exact solutions of problem (19), which are described by formulas (20) for $a = 0$, $b = c = 1$, and $\varepsilon = 0.005$, are represented in Fig. 5 by solid lines for two values of the parameter λ: $\lambda = \pi$ and $\lambda = 2\pi$. For these parameter values, the solution has two and three extrema, respectively (for $\lambda = \pi$ at $x_* = 0.0265534145$ the solution reaches a maximum value of $y_* = 0.978476138$). Results of numerical solutions of the transformed problem (8), used for solving problem (19) with the regularizing function $g = (1 + z^2 + |f|)^{1/2}$ and the fixed stepsize $h = 0.01$ for the same values of the determining parameters, are shown by points. It can be seen that there is a good coincidence between the numerical and exact solutions. The maximum absolute error of the numerical solutions for $\lambda = \pi$ is $E = 0.000000926$ and $\lambda = 2\pi$ is $E = 0.009993346$. The maximum absolute error of the asymptotic solutions (22) for $\lambda = \pi$ is $E = 0.009966909$ and $\lambda = 2\pi$ is $E = 0.009992805$.
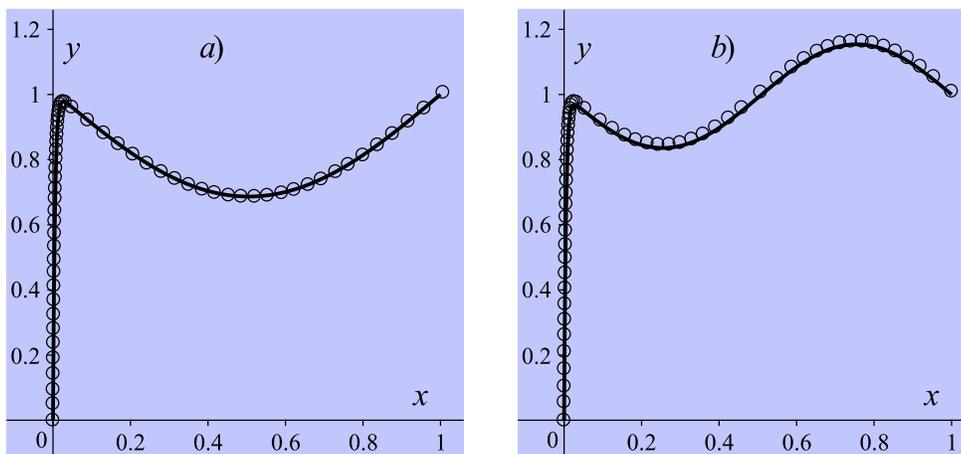
Figure 5: Exact solutions (20) of problem (19) with $a = 0$, $b = c = 1$, and $\varepsilon = 0.005$ (solid lines) and numerical solutions of the corresponding transformed problem (8) with $g = (1 + z^2 + |f|)^{1/2}$ and stepsize $h = 0.01$ (points) for the two values of λ: *a)* $\lambda = \pi$ and *b)* $\lambda = 2\pi$.

5. Singularly perturbed nonlinear boundary-value problems. Numerical and analytical solutions

5.1. Multiparameter boundary-value problem with quadratic nonlinearity. Exact and numerical solutions

Test problem 3. We consider the five-parameter boundary-value problem for the reaction-diffusion type ODE with quadratic nonlinearity

$$\varepsilon y''_{xx} + (y + px + q)y'_x + p(y + px + q) = 0 \quad (0 < x < 1); \tag{23}$$

$$y(0) = a, \quad y(1) = b, \tag{24}$$

where a, b, p, q, and ε are free parameters.

The substitution $u = y + px + q$ transforms equation (23) to the autonomous equation $\varepsilon u''_{xx} + uu'_x = 0$; by introducing the new variable $v(u) = u'_x$ it is reduced to the first-order linear ODE $\varepsilon v'_u + u = 0$.

As a result, we find the general solution of equation (23) in the explicit form

$$y = c\frac{1 - Ae^{-cx/\varepsilon}}{1 + Ae^{-cx/\varepsilon}} - px - q. \tag{25}$$

The constants of integration A and c are determined from the transcendental

system of equations

$$c\frac{1-A}{1+A} = a+q, \quad c\frac{1-Ae^{-c/\varepsilon}}{1+Ae^{-c/\varepsilon}} = b+p+q, \qquad (26)$$

obtained by substituting function (25) into the boundary conditions (24). As $\varepsilon \to 0$ and $b+p+q > 0$, the asymptotic solution of system of ODEs (26) is given by the expressions

$$A = \frac{b-a+p}{b+a+p+2q}, \quad c = b+p+q. \qquad (27)$$

Remark 4. *The asymptotic solution* (27) *exactly satisfies the first ODE of system* (26)*, and the discrepancy of the second ODE of the same system has the exponentially small order* $e^{-(b+p+q)/\varepsilon}$ *as* $\varepsilon \to 0$.

Figure 6: a) Numerical solution of the transformed problem (8), which is used to solve the original problem (23)–(24), for the regularizing function $g = (1+z^2+|f|)^{1/2}$ with $a = 1$, $b = 1$, $p = 1$, $q = 0$, and $\varepsilon = 0.005$: $x = x(\xi)$ (solid line) and $y = y(\xi)$ (dashed line); b) exact solution (25) (solid line) of problem (23)–(24) with $a = 1$, $b = 1$, $p = 1$, $q = 0$, and $\varepsilon = 0.005$ and numerical solution (circles) of the transformed problem (8) for $g = (1+z^2+|f|)^{1/2}$ with $a = 1$, $b = 1$, $p = 1$, $q = 0$, and $\varepsilon = 0.005$; c) and d) absolute errors E of numerical solutions of the transformed problem (8) for $g = (1+z^2+|f|)^{1/2}$ and the same values of the parameters, respectively, with respect to ξ and x.

The numerical solution $x = x(\xi)$, $y = y(\xi)$ of the transformed problem (8) with the regularizing function $g = (1+z^2+|f|)^{1/2}$ which is used to solve the

original problem (23)–(24) with $a = b = 1$, $p = 1$, $q = 0$, and $\varepsilon = 0.005$ is shown in Fig. 6a. The exact solution to the original problem (23)–(24), which is determined by formula (25) with $A = 1/3$, $c = 2$ for the same values of the parameters a, b, p, q, and ε, is shown by solid line in Fig. 6b (in this case, the difference between the exact and the asymptotic solutions is far beyond the limits of the accuracy of our calculations). The circles represent the results of numerical integration of the corresponding transformed problem (8) with $g = (1 + z^2 + |f|)^{1/2}$, which are obtained by the shooting method with initial point $x = 0$ and the stepsize $h = 0.01$ by using a special Maple procedure. The maximum modulus of the difference between the exact and the numerical solutions is equal to 0.000003261.

No.	Regularizing function	Stepsize 0.1	Stepsize 0.05	Stepsize 0.01
1	$g = 1 + |z|$	0.137389203	0.053399823	0.000857913
2	$g = (1 + |f|)^{1/2}$	0.000937303	0.000228167	0.000005030
3	$g = (1 + |z| + |f|)^{1/2}$	0.000786873	0.000196698	0.000003249
4	$g = (1 + z^2 + |f|)^{1/2}$	0.000607467	0.000154641	0.000003261
5	$g = (1 + z^4 + f^2)^{1/4}$	0.000617535	0.000156509	0.000005624
6	$g = 1 + |z| + |f|^{1/2}$	0.000637870	0.000096382	0.000000429
7	$g = (1 + R_1)^{1/2}$	0.000630415	0.000172091	0.000004600
8	$g = 1 + R_2$	0.000621275	0.000164464	0.000001680
9	$g = 1$	process diverges	process diverges	process diverges

The maximum absolute error of the numerical solutions of problem (23)–(24)

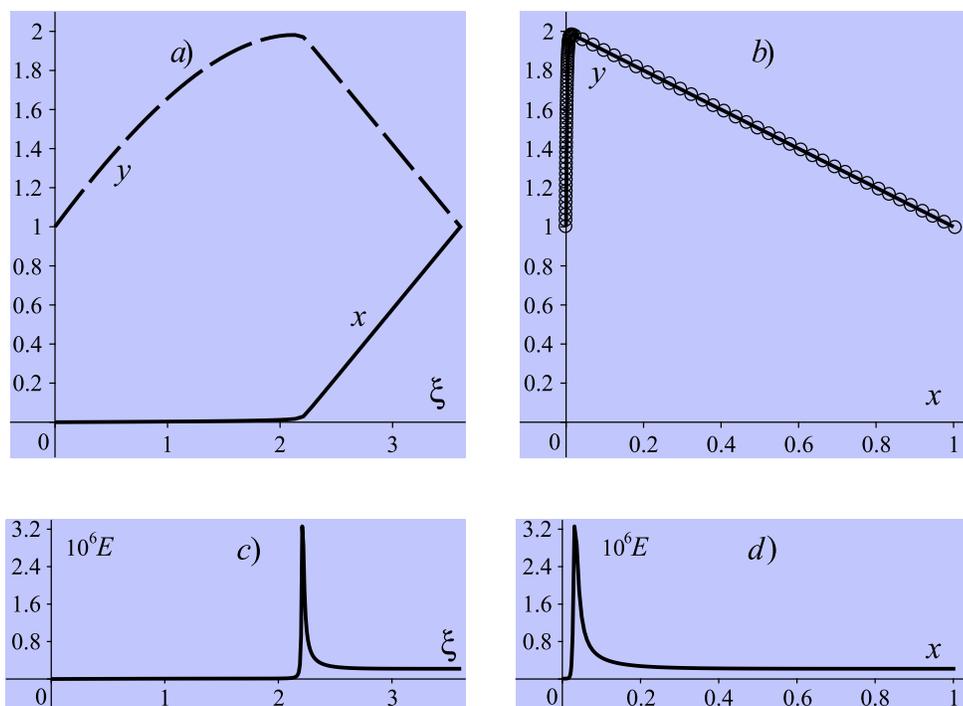Table 3: Comparison of the accuracy of numerical solutions of the transformed problem (8), which is used to solve the original problem (23)–(24), for various regularizing functions g with $a = b = 1$, $p = 1$, $q = 0$, $\varepsilon = 0.005$ and three stepsizes h. Here $R_1 = \max(z^2, |f|)$, $R_2 = \max(|z|, |f|^{1/2})$.

Table 3 shows the maximum absolute errors of numerical solutions of the transformed problem (8) used for numerical solutions of the original problem (23)–(24) for $a = b = 1$, $p = 1$, $q = 0$, $\varepsilon = 0.005$ with three stepsizes h and different regularizing functions g. It can be seen that functions Nos. 2–8 allow one to obtain numerical solutions in the entire region with high accuracy even with a sufficiently large stepsize (with respect to the non-local variable ξ) equal to $h = 0.1$. For stepsize $h = 0.05$ and $h = 0.01$, the best results are provided by function No. 6.

Tables 4 and 5 shows the maximum absolute errors of the numerical solutions of the transformed problem (8), used for the numerical solution of the original problem (23)–(24) for $a = b = 0$, $p = 1$, $q = 0$, and $\varepsilon = 0.005$, for three

stepsizes h, different numbers of grid points N, and nine different regularizing functions g.

No.	Regularized function	Stepsize 0.1	Stepsize 0.05	Stepsize 0.01
	The maximum absolute error of the numerical solutions of problem (23)–(24)			
1	$g = 1 + \|z\|$	0.177592060	0.035246285	0.000212137
2	$g = (1 + \|f\|)^{1/2}$	process diverges	0.376921099	0.021473151
3	$g = (1 + \|z\| + \|f\|)^{1/2}$	0.025249660	0.006467125	0.000196032
4	$g = (1 + z^2 + \|f\|)^{1/2}$	0.000752856	0.000163579	0.000000699
5	$g = (1 + z^4 + f^2)^{1/4}$	0.000627973	0.000154532	0.000002003
6	$g = 1 + \|z\| + \|f\|^{1/2}$	0.000393742	0.000067536	0.000000061
7	$g = (1 + R_1)^{1/2}$	0.000712931	0.000202734	0.000000999
8	$g = 1 + R_2$	0.000663385	0.000119895	0.000000265
9	$g = 1$	process diverges	process diverges	0.019513818

Table 4: Comparison of the accuracy of numerical solutions of the transformed problem (8), which is used to solve the original problem (23)–(24), for various regularizing functions g with $a = b = 0$, $p = 1$, $q = 0$, $\varepsilon = 0.005$ for three stepsizes h. Here $R_1 = \max(z^2, \|f\|)$, $R_2 = \max(\|z\|, \|f\|^{1/2})$.

The special case $g = 1$ corresponds to the direct numerical solution (without using transformations) with the same stepsize with respect to x. It can be seen that functions Nos. 4–8 allow one to obtain numerical solutions in the entire region with high accuracy even with a sufficiently large stepsize (with respect to ξ) equal to $h = 0.1$. For stepsize $h = 0.05$ and $h = 0.01$, the best results are provided by functions No. 6 and No. 8.

Unsatisfactory results for function No. 2 can be explained by the fact that in this case a degeneracy occurs at the initial point, where the second derivative vanishes, $y''_{xx}|_{x=0} = f|_{x=0} = 0$.

Indeed, when choosing the regularizing function No. 2, in the vicinity of the point $x = 0$ we have $g = (1 + \|f\|)^{1/2} \approx 1$ and the asymptotic condition (11) is not satisfied.

Therefore, the function $g = (1 + \|f\|)^{1/2}$ cannot suppress here the growth of the right-hand side of the second equation of the transformed problem (8).

As a result, the stepsize in x (for a fixed N) will be maximum here (whereas for this region with a large first derivative, $y'_x|_{x=0} \approx 2\varepsilon^{-1}$, a small stepsize is required). This circumstance leads to a large calculation error in the boundary-layer region for the function $g = (1 + \|f\|)^{1/2}$.

From Tables 4 and 5 it can be seen that the use of the regularizing function

The maximum absolute error of the numerical solutions of problem (23)–(24)			
No.	Regularized function	N=100	N=200	N=300
1	$g = 1 + \|z\|$	0.000734178	0.000325332	0.000061158
2	$g = (1 + \|f\|)^{1/2}$	process diverges	0.034146715	0.016310528
3	$g = (1 + \|z\| + \|f\|)^{1/2}$	0.004963520	0.000514743	0.000202650
4	$g = (1 + z^2 + \|f\|)^{1/2}$	0.000198725	0.000007921	0.000001328
5	$g = (1 + z^4 + f^2)^{1/4}$	0.000222372	0.000010748	0.000002162
6	$g = 1 + \|z\| + \|f\|^{1/2}$	0.000195161	0.00003566	0.000000433
7	$g = (1 + R_1)^{1/2}$	0.000159026	0.000009546	0.000001646
8	$g = 1 + R_2$	0.000118378	0.000004655	0.000000747
9	$g = 1$	0.019513818	0.001179663	0.000182152

Table 5: Comparison of the accuracy of numerical solutions of the transformed problem (8), which is used to solve the original problem (23)–(24), for various regularizing functions g with $a = b = 0$, $p = 1$, $q = 0$, $\varepsilon = 0.005$ for a different number of grid points N. Here $R_1 = \max(z^2, |f|)$, $R_2 = \max(|z|, |f|^{1/2})$.

No. 3 gives a low accuracy of numerical solutions. This is due to the fact that in this case we have $g = (1 + |z| + |f|)^{1/2}|_{x=0} = O(|z|^{1/2})$, i.e. near the initial point $x = 0$, the asymptotic condition (11) does not hold.

5.2. Multiparameter boundary-value problem with exponential nonlinearity. Exact and numerical solutions

Test problem 4. We now consider the five-parameter boundary-value problem for the reaction-diffusion type ODE with exponential nonlinearity

$$\varepsilon y''_{xx} + e^{y+px+q}y'_x + pe^{y+px+q} = 0 \quad (0 < x < 1); \tag{28}$$

$$y(0) = a, \quad y(1) = b, \tag{29}$$

where a, b, p, q, and ε are free parameters.

The substitution $u = y + px + q$ transforms ODE (28) to the autonomous equation $\varepsilon u''_{xx} + e^u u'_x = 0$; by introducing the new variable $v(u) = u'_x$ it is reduced to the first-order linear ODE $\varepsilon v'_u + e^u = 0$. As a result, we find the general solution of equation (28) in the explicit form

$$y = -\ln\left(ce^{-kx/\varepsilon} + \frac{1}{k}\right) - px - q. \tag{30}$$

The integration constants c and k are determined from the transcendental

system of equations

$$c + \frac{1}{k} = e^{-a-q}, \quad ce^{-k/\varepsilon} + \frac{1}{k} = e^{-b-p-q}, \tag{31}$$

obtained by substituting expression (30) into the boundary conditions (29) and by performing some elementary transformations.

As $\varepsilon \to 0$, the asymptotic solution of system (31) is given by the formulas

$$c = e^{-a-q} - e^{-b-p-q}, \quad k = e^{b+p+q}. \tag{32}$$

No.	Regularized function	Stepsize 0.1	Stepsize 0.05	Stepsize 0.01
The maximum absolute error of the numerical solutions of problem (28)–(29)
1	$g = 1 + |z|$	0.185049898	0.035618317	0.000212182
2	$g = (1 + |f|)^{1/2}$	0.000692372	0.000191043	0.000001852
3	$g = (1 + |z| + |f|)^{1/2}$	0.000699196	0.000182170	0.000000707
4	$g = (1 + z^2 + |f|)^{1/2}$	0.000706940	0.000139584	0.000000656
5	$g = (1 + z^4 + f^2)^{1/4}$	0.000741403	0.000160845	0.000002135
6	$g = 1 + |z| + |f|^{1/2}$	0.000479280	0.000062701	0.000000075
7	$g = (1 + R_1)^{1/2}$	0.000790921	0.000199628	0.000001181
8	$g = 1 + R_2$	0.000492648	0.000109479	0.000000283
9	$g = 1$	process diverges	process diverges	0.016651291

Table 6: Comparison of the accuracy of numerical solutions of the transformed problem (8), which is used to solve the original problem (28)–(29), for various regularizing functions g with $a = b = 0$, $p = 1$, $q = -1$, $\varepsilon = 0.005$ and three stepsizes h. Here $R_1 = \max(z^2, |f|)$, $R_2 = \max(|z|, |f|^{1/2})$.

Tables 6 and 7 shows the maximum absolute errors of the numerical solutions of the transformed problem (8), used for the numerical solution of the original problem (28)–(29) for $a = b = 0$, $p = 1$, $q = -1$, and $\varepsilon = 0.005$, for three stepsizes h, different numbers of grid points N, and different regularizing functions g. It can be seen that functions Nos. 2–8 allow one to obtain numerical solutions in the entire region with high accuracy even with a sufficiently large stepsize (with respect to ξ) equal to $h = 0.1$. For stepsize $h = 0.05$ and $h = 0.01$, the best results are provided by functions No. 6 and No. 8. Note that in this case function No. 1 is low effective; the reason for this is due to the non-monotonicity of the solution and is explained at the end of Section 4.1.

No.	Regularized function	N=100	N=200	N=300
	The maximum absolute error of the numerical solutions of problem (28)–(29)			
1	$g = 1 + |z|$	0.008033009	0.000490903	0.000168976
2	$g = (1 + |f|)^{1/2}$	0.000174781	0.000006417	0.000001164
3	$g = (1 + |z| + |f|)^{1/2}$	0.000146092	0.000005956	0.000000895
4	$g = (1 + z^2 + |f|)^{1/2}$	0.000215039	0.000007804	0.000001386
5	$g = (1 + z^4 + f^2)^{1/4}$	0.000186220	0.000009774	0.000002374
6	$g = 1 + |z| + |f|^{1/2}$	0.000254116	0.000003781	0.000000540
7	$g = (1 + R_1)^{1/2}$	0.000149504	0.000009551	0.000001620
8	$g = 1 + R_2$	0.000096813	0.000004651	0.000000799
9	$g = 1$	0.016651291	0.000385984	0.000062467

Table 7: Comparison of the accuracy of numerical solutions of the transformed problem (8), which is used to solve the original problem (28)–(29), for various regularizing functions g with $a = b = 0$, $p = 1$, $q = -1$, $\varepsilon = 0.005$ for a different number of grid points N. Here $R_1 = \max(z^2, |f|)$, $R_2 = \max(|z|, |f|^{1/2})$.

5.3. Another multiparameter problem with exponential nonlinearity. Asymptotic and numerical solutions

Test problem 5. Let us consider the four-parameter boundary-value problem with exponential nonlinearity

$$\varepsilon y''_{xx} + e^y y'_x + cxe^y = 0 \quad (0 < x < 1); \tag{33}$$

$$y(0) = a, \quad y(1) = b, \tag{34}$$

where a, b, c, and ε are free parameters.

The solution of problem (33)–(34) cannot be represented in a closed analytical form. Therefore, in order to obtain an approximate solution as $\varepsilon \to 0$, we use the method of matched asymptotic expansions [52, 53, 53].

As $\varepsilon \to 0$, near the left boundary $x = 0$, a boundary layer, called the inner region, is formed. In this region, the last term of equation (33) can be neglected. The leading term of the asymptotic expansion of the solution in the boundary layer, satisfying the first boundary condition (34), has the form

$$y_{\mathrm{i}} = -\ln\left[\left(e^{-a} - \frac{1}{k}\right)e^{-k\tau} + \frac{1}{k}\right], \quad \tau = x/\varepsilon, \tag{35}$$

where τ is the boundary-layer (stretched) variable, k is the constant that is determined further in the solution process. In the outer region, $O(\varepsilon) \le x \le 1$, the first term of equation (33) can be neglected and the leading term of the

asymptotic solution of problem (33)–(34) is found from the truncated equation $e^y(y'_x + cx) = 0$. Its solution, satisfying the second boundary condition (34), has the form

$$y_e = b + \frac{1}{2}c - \frac{1}{2}cx^2. \tag{36}$$

The inner and outer solutions, (35) and (36), must satisfy the matching condition $y_i(\tau \to \infty) = y_e(x \to 0)$, which allows us to determine the constant k entering into (35):

$$k = e^{(2b+c)/2}. \tag{37}$$

A uniform applicable (in the entire domain $0 \le x \le 1$) composite expansion of the solution of problem (33)–(34) is defined by the formula

$$\begin{aligned} y &= y_i + y_e - y_e|_{x=0} \\ &= -\ln\left[\left(e^{-a} - \frac{1}{k}\right)e^{-kx/\varepsilon} + \frac{1}{k}\right] - \frac{1}{2}cx^2, \qquad k = e^{(2b+c)/2}. \end{aligned} \tag{38}$$

For non-monotonic solutions (if $a < b$ and $c > 0$), for the maximum value of the required function we obtain $y_* \simeq y_e|_{x=0} = b + \frac{1}{2}c$.
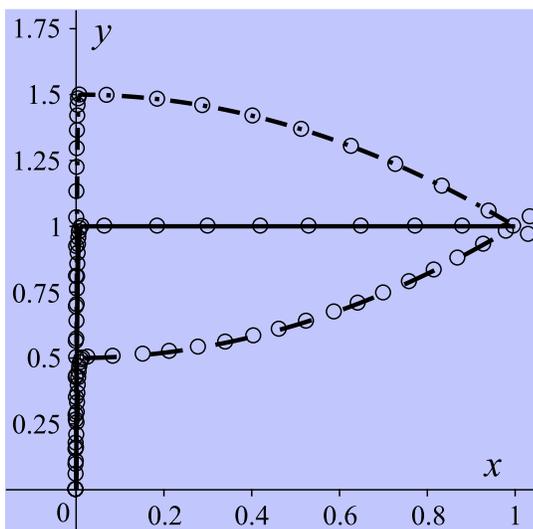

Figure 7: Asymptotic solutions (38) of the original problem (33)–(34) for $a = 0$, $b = 1$, $\varepsilon = 0.005$ and $c = 0$ (solid line), $c = -1$ (dashed line), and $c = 1$ (dash-dotted line), and numerical solutions of the corresponding transformed problem (8) (circles) obtained by generalized Sundman transformations with a regularizing function $g = (1 + z^2 + |f|)^{1/2}$.

In Fig. 7, the asymptotic solutions (38) of problem (33)–(34) are shown for $a = 0$, $b = 1$, $\varepsilon = 0.005$ and the three values of the parameter $c = -1$, 0, 1. The circles represent the results of numerical solutions of the corresponding transformed problem (8) with the regularizing function $g = (1+z^2+|f|)^{1/2}$, obtained by the shooting method (from the point $x = 0$) with the stepsize $h = 0.01$ by using Maple. The maximum modulus of the difference between the asymptotic and numerical solutions at $c = -1$, 0, 1 are, respectively, 0.002612894, 0.000005160, 0.001691475. The maximum modulus of the difference between the numerical solutions for $h = 0.01$ and $h = 0.005$ (stepsize is reduced by half) for the same values ââof c, respectively, are equal to 0.001063206, 0.000004164, 0.000013097.

6. Regularizing functions recommended for calculations. Remarks on blow-up problems

6.1. Regularizing functions recommended for calculations

For all the singularly perturbed linear and nonlinear boundary-value problems with a small parameter considered in this paper, which are described by ODEs of the form $\varepsilon y''_{xx} = F(x, y, y'_x)$, the best numerical results are given by using the composite regularizing functions

$$
\begin{aligned}
g &= 1 + |y'_x| + |y''_{xx}|^{1/2}, \\
g &= 1 + \max(|y'_x|, |y''_{xx}|^{1/2}),
\end{aligned}
\tag{39}
$$

where y''_{xx} can be replaced by $f = \varepsilon^{-1}F(x, y, y'_x)$. These formulas are the most versatile and work well in all cases.

Sufficiently good results are also provided by the regularizing function

$$
g = (1 + |y'_x|^2 + |y''_{xx}|)^{1/2},
$$

recommended in [35].

Recall that earlier, in Sections 4.1 and 5.1, by analyzing the solutions of test problems (1) and (23)–(24) (for some parameter values), it was shown that the regularizing functions (13) (functions Nos. 1 and 2 in the tables) are of limited applicability and do not work well for non-monotonic solutions or if the original equation degenerates at the border of the boundary layer.

6.2. Difference between the choice of regularizing functions in boundary-layer problems and blow-up problems

In Cauchy problems for nonlinear ODEs, the blow-up solutions can be represented in a vicinity of the power-law singular point x_* as

$$y \simeq A(x - x_*)^{-\beta}, \quad \beta > 0, \tag{40}$$

where $0 \le x < x_*$. Since the position of the singular point x_* is unknown in advance, it is difficult to solve such problems by standard numerical methods with a fixed stepsize.

In [39, 40, 41], to solve blow-up problems, it was proposed to use Sandman-type transformations of the form (7), which replaced the unknown interval of variation of the independent variable of the original problem $0 \le x < x_*$ by a given semi-infinite interval $0 \le \xi < \infty$ of the resulting problem. By integrating Eq. (7), we have

$$\xi = \int_0^x g(x, y, y'_x) \, dx, \quad y = y(x). \tag{41}$$

Since the singular point x_* after transformation must go to the infinite point $\xi = \infty$, the condition

$$\lim_{x \to x_*} I = \infty, \quad I = \int_0^x g(x, y, y'_x) \, dx \tag{42}$$

must be satisfied.

For second-order nonlinear ODEs, taking into account the relations (40) and (42), it can be shown that for the simplest regularizing functions of the form $g = g(z)$ and $g = g(f)$ with $z = y'_x$ and $f = y''_{xx}$, the following power-law asymptotic conditions should be used [40, 41]:

$$g(z) \to m_1 |z|^{\frac{1}{\beta+1}} + \varepsilon_1 \quad \text{as} \quad |z| \to \infty, \tag{43}$$

$$g(f) \to m_2 |f|^{\frac{1}{\beta+2}} + \varepsilon_2 \quad \text{as} \quad |f| \to \infty, \tag{44}$$

where $m_1 > 0$, $m_2 > 0$, $\varepsilon_1 \ge 0$, and $\varepsilon_2 \ge 0$. To the conditions (43) and (44), as in the problems with a boundary layer, one should add a condition of the normalization type $g(0) = 1$. Comparative analysis of exact and numerical solutions to blow-up problems showed that in asymptotic relations (43) and (44) one should choose the minimum values $\varepsilon_1 = \varepsilon_2 = 0$ [40, 41].

It is seen that conditions (11) and (12) imposed on regularizing functions in singularly perturbed problems with a small parameter differ significantly from conditions (42)–(44), which must be satisfied by regularizing functions in blow-up problems.

For second-order nonlinear ODEs with monotone and non-monotone blow-up solutions, that have a singularity in the form of a pole of any integer order ($\beta = 1, 2, \dots$), one can recommend to use the composite regularizing function [39, 40, 41]:

$$g = (1 + |y'_x| + |y''_{xx}|)^{1/3}, \tag{45}$$

where the second derivative y''_{xx} can be replaced with the right-hand side of the original equation $y''_{xx} = f(x, y, y'_x)$. It can be seen that the regularizing function (45) differs significantly from functions (39), which give good results in boundary-layer problems.

Note that the composite regularizing functions (39) satisfy the condition (42) for any blow-up solutions of the power-law form (40) with $\beta > 0$. The analysis of the possibility of using these functions for numerical integration of blow-up problems is far beyond the scope of this paper. The authors are going to investigate this issue in the future.

7. Brief conclusions

Singularly perturbed boundary-value problems for second-order ODEs with a small parameter at the highest derivative are considered. Such problems are characterized by narrow boundary layers with large gradients, which greatly limits the applicability of standard fixed-step numerical methods (their use can lead to significant errors). We offer an effective method of numerical integration of singularly perturbed boundary-value problems based on using Sundman-type transformations. As a result, we obtain more convenient reduced problems that allow one to apply standard numerical methods with a fixed stepsize with respect to a new independent variable. An extensive testing of the method is carried out on various multiparameter linear and nonlinear problems with monotonic and non-monotonic solutions. Comparison of numerical, exact, and asymptotic solutions of several singularly perturbed boundary-value problems with a small parameter showed the high accuracy of the method based on Sundman-type transformations.

Acknowledgments

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