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Theory of ordinary differential equations

A method for obtaining an explicit solution of second-order matrix ODE based on diagonalization the matrices and the Kronecker matrix algebra

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Abstract. A method for obtaining an explicit solution of matrix differential equations in second-order ordinary derivatives with constant matrices is considered. The method allows one to reduce the initial system of interconnected differential equations to a system of independent differential equations that are easily solved analytically. The method developed in the article is based on the diagonalization of all matrices included in the equation, which is carried out using the spectral decomposition of the matrices and Kronecker matrix algebra. An example of the application of the developed method is given.

Keywords: matrix differential equations in ordinary derivatives, diagonalization of matrices, spectral decomposition of a matrix, Kronecker matrix algebra.

1. Introduction

Mathematical modeling of processes of various physical nature [1, 2, 3, 4], such as, oscillations in mechanical, electrical, electronic and hydraulic systems, dynamics of mechanical systems, wave propagation in thermoelastic media, thermal stresses, aerodynamics of aircraft, control of engineering systems, etc., results in a system of differential equations in second-order ordinary derivatives, which takes the following form in matrix notation:

$$A\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Cx(t) = f(t), \tag{1}$$

$$\frac{dx(0)}{dt} = x_0', \;\; x(0) = x_0,$$

where A, B, C – some time-independent $m \times m$-matrices; $x(t) = (x_1(t), x_2(t), \dots, x_m(t))^T$ – m-vector of the required functions $x_i(t)$, $i = 1,2,\dots,m$; $f(t) = (f_1(t), f_2(t), \dots, f_m(t))^T$ – m-vector of the specified functions; t – time; $x_0 = (x_{01}, x_{02}, \dots, x_{0m})^T$ and $x_0' = (x_{01}', x_{02}', \dots, x_{0m}')^T$ – m-vectors of the initial conditions of the required functions $x_i(t)$ and their first-order derivatives $dx_i(t)/dt$ at the initial moment of time $t = 0$; $(\cdot)^T$ – operation of transposition.

Despite a fairly large number of methods for solving systems of various-order ordinary differential equations, including numerical ones [4, 5, 6, 7, 8], the importance of developing methods for explicit analytical solutions remains relevant and in demand when carrying out any substantial analysis of the investigated physical phenomena and processes, as well as mathematical models that describe them.

To obtain analytical solutions of a system of ordinary differential equations, it is necessary to reduce the system of initially coupled equations to a system of independent decoupled equations or, at least, reduce the degree of interrelationship [9, 10, 11]. Obtaining decoupled equations is achieved by reducing all matrices of the system of ordinary differential equations to a completely diagonal form. At the same time, reducing the matrices of the system of equations to "almost diagonal form", for instance, to the Jordan canonical form, although reduces the degree of interrelationship of the equations, does not eliminate it completely [11]. This article suggests a method for achieving a complete diagonalization of all matrices included in the system of ordinary differential equations (1). In the existing literature [9, 10], diagonalization of matrices in matrix ordinary differential equations is considered only in relation to the first-order equations, while methods for diagonalizing matrices included in matrix ordinary differential equations of the second and higher orders are not suggested at all.

It's also worth noting that to reduce the solution of matrix ordinary differential equations of various orders, including with numerical methods, one strives, first of all, to get rid of the matrix A at the highest-order derivative, for instance, by multiplying both parts of original equation (1) by its inverse matrix A^{-1}. However, the matrix A at the highest-order derivative of the equation does not always have an inverse matrix, which can be due to the matrix singularity, semidefiniteness, or the presence of the rank of matrix of the lesser dimension. The method developed in the article allows reducing the original system of coupled equations (1) to a system of uncoupled equations and obtaining their analytical solutions for both singular and semidefinite matrices at the highest derivative.

One of the most powerful methods for obtaining analytical solutions of the systems of differential equations is to reduce the matrices included in the system of equations to a diagonal form. In this case, the system of coupled equations of differential equations is decomposed into m independent equations for each unknown $x_i(t)$, $i = 1,2,\dots,m$, in the vector $x(t)$, the solutions of which are easily determined analytically. This approach is used to develop analytical solutions of matrix differential equations with one or two matrices, namely, such matrix equations as $\dot{x}(t) + Cx(t) = 0$ [9] and $A\ddot{x}(t) + Cx(t) = 0$ [10] ($\dot{x}(t)$, where $\ddot{x}(t)$ – shorthand notation for $dx(t)/dt$ and d^2/dt^2, respectively.

For instance, in the equation $\dot{x}(t) + Cx(t) = 0$, $x(0) = x_0$ with one matrix, the matrix C is subjected to the spectral decomposition $U^TCU = \Lambda_C$ with the diagonal matrix $\Lambda_C = diag\{\lambda_{C1}, \lambda_{C2}, \dots, \lambda_{Cm}\}$, consisting of the eigenvalues λ_{Ci}, $i = 1,2,\dots,m$, and the orthonormal matrix U, composed of the eigenvectors of the matrix C. The spectral decomposition of the matrix C allows reducing the matrix equation $\dot{x}(t) + Cx(t) = 0$ to a system of m independent equations $\dot{y}_i(t) + \lambda_{Ci}y_i(t) = 0$, $y_{0i}(0) = y_{0i}$, $i = 1,2,\dots,m$, with respect to the new transformed vector

$y(t) = (y_1(t), y_2(t), \dots, y_m(t))^T$, the solutions of which are easily found analytically and are $y_i(t) = \exp(-\lambda_{Ci} t) y_{0i}$ [9].

As for the equation $A\ddot{x}(t) + Cx(t) = 0$ with two matrices, the application of the method under consideration is significantly complicated by the fact that in this case it is necessary to simultaneously diagonalize the matrices A and C. This complexity can be overcome using the theorem [9, 10] on the reduction of two real non-singular symmetric matrices A and C (moreover, the matrix A is a positive definite one) with one non-singular transformation of the similarity T, which transforms the matrix A to a diagonal identity matrix, and the matrix C - to a diagonal matrix, consisting of eigenvalues of some specially defined matrix [9, 10]. As a result of the simultaneous diagonalization of the matrices A and C, the matrix equation $\ddot{x}(t) + Cx(t) = 0$ is decomposed into a system of m independent equations $\ddot{y}_i(t) + \lambda_i y_i(t) = 0$, $i = 1,2, \dots, m$, with respect to the transformed vector $y(t)$. The solutions of these equations are easily found analytically: $y_i(t) = \alpha_i \cos(\lambda_i t + \varepsilon_i)$, where α_i and ε_i – constants, determined from the initial conditions [10].

At the same time, there are no methods that allow ontaining explicit solutions of matrix differential equations, which include more than two matrices and require simultaneous reduction to the diagonal form of more than two matrices.

This article suggests a method that allows reducing matrix differential equations (1) to a system of independent equations, each of which is easily solved analytically. The method is based on the spectral decomposition of matrices included in the equation and the use of Kronecker matrix algebra. Therewith, simultaneous diagonalization is applied to two of the three matrices from the equation, in which one of the matrices is reduced to a diagonal identity matrix, and the other to a diagonal form of some specially constructed matrix. The diagonalization of the third remaining matrix is carried out by moving from the usual matrix space to the Kronecker matrix space, in which the rules of Kronecker matrix algebra are applied. To apply the method, it is sufficient that only one of the matrices in the equation is a positive definite one, while the other two matrices can be nonsymmetric and positive semidefinite. The application of the developed method is considered in a specific context.

2. Diagonalizing the matrix differential equation with three matrices

Let's consider the matrix differential equation in second-order ordinary derivatives (1) and bring all three matrices of the equation A, B and C to a diagonal form. It is assumed that A, B, C are square, real, time-independent and not necessarily symmetric $m \times m$ matrices. One of the matrices is positive definite, the other two can be positive semidefinite. In equation (1), first, the positive definite matrix that can be reduced with some non-singular similarity transformation to a diagonal identity matrix is diagonalized, and then another matrix of the equation is diagonalized. To diagonalize the third matrix, the entire equation with two previously diagonalized matrices is transformed, whereby the transition is made from the space with ordinary matrix algebra to the Kronecker space, in which Kronecker matrix algebra is applied.

For definiteness, let's further assume that the matrix A at the second derivative in equation (1) is a real positive definite (and symmetric) $m \times m$-matrix, and the other two $m \times m$-matrices B, C can be positive and semidefinite.

2.1. Diagonalizing the matrix A in matrix equation (1)

Let the spectral decomposition of the symmetric positive definite matrix A have the form $U^{-1}AU = \Lambda_A$, where U is the transforming $m \times m$-similarity matrix [10, 11], consisting of the eigenvectors of the matrix A; $\Lambda_A = diag\{\lambda_{A1}, \lambda_{A2}, \dots, \lambda_{Am}\}$ is the diagonal matrix of the eigenvalues λ_{Ai}, $i = 1,2, \dots, m$, of the matrix A, where all its eigenvalues $\lambda_{A1}, \lambda_{A2}, \dots, \lambda_{Am}$ are positive due to the positive definiteness of the matrix A.

Let's introduce into equation (1) and the initial conditions a new vector variable x^* according to the equation $x = U\Lambda_A^{-\frac{1}{2}}x^*$ and multiply the resulting equation and the initial conditions on the left by the matrix $\Lambda_A^{-\frac{1}{2}}U^{-1}$. The following will be obtained:

$$\Lambda_A^{-\frac{1}{2}}U^{-1}AU\Lambda_A^{-\frac{1}{2}}\frac{d^2x^*(t)}{dt^2} + \Lambda_A^{-\frac{1}{2}}U^{-1}BU\Lambda_A^{-\frac{1}{2}}\frac{dx^*(t)}{dt} + \Lambda_A^{-\frac{1}{2}}U^{-1}CU\Lambda_A^{-\frac{1}{2}}x^*(t) = \Lambda_A^{-\frac{1}{2}}U^{-1}f(t),$$
$$\frac{dx^*(0)}{dt} = \Lambda_A^{\frac{1}{2}}U^{-1}x_0', \;\; x^*(0) = \Lambda_A^{\frac{1}{2}}U^{-1}x_0 .$$

Taking into account that the matrix at the second-order derivative in the resulting equation is $\Lambda_A^{-\frac{1}{2}}U^{-1}AU\Lambda_A^{-\frac{1}{2}} = \Lambda_A^{-\frac{1}{2}}\Lambda_A\Lambda_A^{-\frac{1}{2}} = I$ and introducing notation for the matrices $D = \Lambda_A^{-\frac{1}{2}}U^{-1}BU\Lambda_A^{-\frac{1}{2}}$ and $E = \Lambda_A^{-\frac{1}{2}}U^{-1}CU\Lambda_A^{-\frac{1}{2}}$, the following will be obtained:

$$\frac{d^2x^*(t)}{dt^2} + D\frac{dx^*(t)}{dt} + Ex^*(t) = \Lambda_A^{-\frac{1}{2}}U^{-1}f(t),$$
$$\frac{dx^*(0)}{dt} = \Lambda_A^{\frac{1}{2}}U^{-1}x_0', \;\; x^*(0) = \Lambda_A^{\frac{1}{2}}U^{-1}x_0 . \qquad (2)$$

2.2. Diagonalizing the matrix D in transformed equation (2)

Let's diagonalize the matrix $D = \Lambda_A^{-\frac{1}{2}}U^{-1}BU\Lambda_A^{-\frac{1}{2}}$ at the first-order derivative in equation (2). If the spectral decomposition of the matrix D has the form $V^{-1}DV = \Lambda_D$, then V – transforming $m \times m -$ similarity matrix, the columns of which are eigenvectors of the matrix D, and $\Lambda_D = diag\{\lambda_{D1}, \lambda_{D2}, \dots, \lambda_{Dm}\}$ – diagonal $m \times m$-matrix of eigenvalues of the matrix D.

Let's implement a new change of variables in equation (2), namely, introduce $m\, y(t)$-vector using equation $x^*(t) = Vy(t)$ and multiply the resulting equation on the left by the matrix V^{-1}. Then indicating in equation (2) $m \times m$-matrices $F = V^{-1}EV = V^{-1}\Lambda_A^{-\frac{1}{2}}U^{-1}CU\Lambda_A^{-\frac{1}{2}}V$, $G = V^{-1}\Lambda_A^{\frac{1}{2}}U^{-1}$, $H = V^{-1}\Lambda_A^{-\frac{1}{2}}U^{-1}$, the following vector equation will be obtained:

$$\frac{d^2y(t)}{dt^2} + \Lambda_D\frac{dy(t)}{dt} + Fy(t) = Hf(t),$$
$$\frac{dy(0)}{dt} = Gx_0', \;\; y(0) = Gx_0 . \qquad (3)$$

Thus, equation (1) with three matrices is reduced to equation (3) with two diagonal matrices - a diagonal identity matrix at the second derivative and a diagonal matrix Λ_D at the first-order derivative in the equation.

Let's diagonalize matrix F.

3. Diagonalizing the matrix F in transformed equation (3)

Let's diagonalize $m \times m$-matrix $F = V^{-1}\Lambda_A^{-\frac{1}{2}}U^{-1}CU\Lambda_A^{-\frac{1}{2}}V$ in equation (3). For this, it is necessary to make the transition from the matrix space with the usual rules for operations with matrices to the Kronecker matrix space in which Kronecker matrix algebra, which is significantly different from the usual one. Below is the brief information from Kronecker matrix algebra [6, 9, 10, 12, 13], which is of interest for the future:

3.1. Some facts from Kronecker matrix algebra

In Kronecker matrix algebra, the Kronecker product of two rectangular matrices $k \times l$ $A = \|a_{ij}\| \in \mathcal{F}_{k\times l}$ $(i = 1,2,\dots,k;\ j = 1,2,\dots,l)$ and $m \times n$ $B = \|b_{ij}\| \in \mathcal{F}_{m\times n}$ $(i = 1,2,\dots,m;\ j = 1,2,\dots,n)$ is defined as the block matrix $C = A \otimes B$ in the matrix space $\mathcal{F}_{km\times ln}$, which is composed according to the following rule:

$$C = A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1l}B \\ a_{21}B & a_{22}B & \dots & a_{2l}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1}B & a_{k2}B & \dots & a_{kl}B \end{pmatrix} \in \mathcal{F}_{km\times ln}.$$

Let's introduce some useful statements from Kronecker matrix algebra:

(1) In the Kronecker space, multiple solutions of the matrix equation $AXB = C$ with respect to the unknown required matrix X and square real $m \times m -$ matrices $A, X, B, C \in \mathcal{F}_{m\times m}$ coincides with multiple solutions of the equation $\mathcal{G}x = c$, in which the matrix $\mathcal{G} \in \mathcal{F}_{m^2\times m^2}$ is $\mathcal{G} = A \otimes B^T$, and the vectors $x \in \mathcal{F}_{m^2}$ and $c \in \mathcal{F}_{m^2}$ are defined using the expressions:

$$x = \begin{pmatrix} X_{1*}^T \\ X_{2*}^T \\ \vdots \\ X_{m*}^T \end{pmatrix}, \quad c = \begin{pmatrix} C_{1*}^T \\ C_{2*}^T \\ \vdots \\ C_{m*}^T \end{pmatrix}, \tag{4}$$

where X_{i*}, X_{*j} и C_{i*}, C_{*j} – the i-th row $(i = 1,2,\dots,m)$ and j-th column $(j = 1,2,\dots,m)$ of the matrices X and C, respectively, * – notation of a set of elements in the i-th row X_{i*} и C_{i*} and a set of elements in the j-th column X_{*j} and C_{*j} of the matrices X and C.

(2) The general linear matrix equation $A_1XB_1 + A_2XB_2 + \dots + A_kXB_k = C$, with respect to the unknown matrix $X \in \mathcal{F}_{m\times m}$, with the matrices $A_k, X, B_k, C \in \mathcal{F}_{m\times m}$ and vectors x and c defined with expressions (4), is equivalent to the equation $\mathcal{G} \in \mathcal{F}_{m^2\times m^2}$, in which the matrix $\mathcal{G} \in \mathcal{F}_{m^2\times m^2}$ (2) looks like $\mathcal{G} = A_1 \otimes B_1^T + A_2 \otimes B_2^T + \dots + A_k \otimes B_k^T$.

(3) If $A \in \mathcal{F}_{m\times m}$ and $B \in \mathcal{F}_{n\times n}$, with $\lambda_1, \lambda_2, \dots, \lambda_m$ ad $\mu_1, \mu_2, \dots, \mu_n$ are the eigenvalues of the matrices A and B, respectively, then the eigenvalues of the matrix function in $\mathcal{F}_{mn\times mn}$ in the form of $\varphi(A, B) = \sum_{i,j=0}^{p} c_{ij}\,(A^i \otimes B^j)$, will be mn of the values $\varphi(\lambda_r, \mu_s)$, where $r = 1,2,\dots,m$ and $s = 1,2,\dots,n$.

(4) Basic rules of matrix Kronecker algebra:

(a) $(\mu A) \otimes B = A \otimes (\mu B) = \mu(A \otimes B)$, where μ is an arbitrary number, $A \in \mathcal{F}_{m\times m}$, $B \in \mathcal{F}_{n\times n}$;

(b) $(A + C) \otimes B = A \otimes B + C \otimes B$, where $A, C \in \mathcal{F}_{m\times m}$, $B \in \mathcal{F}_{n\times n}$;
(c) $A \otimes (B + C) = A \otimes B + A \otimes C$, where $A \in \mathcal{F}_{m\times m}$, $B, C \in \mathcal{F}_{n\times n}$;
(d) $(A \otimes B)^T = A^T \otimes B^T$, где $(\cdot)^T$ – operation of transposition;
(e) $(A \otimes B)(C \otimes D) = AC \otimes BD$, where $A, C \in \mathcal{F}_{m\times m}$, $B, D \in \mathcal{F}_{n\times n}$, $(\cdot)(\cdot)$ – normal matrix multiplication;
(f) $(A_1 \otimes B_1)(A_2 \otimes B_2) \dots (A_k \otimes B_k) = (A_1 A_2 \dots A_k) \otimes (B_1 B_2 \dots B_k)$, where $A_1, A_2, \dots, A_k \in \mathcal{F}_{m\times m}$ and $B_1, B_2, \dots, B_k \in \mathcal{F}_{n\times n}$.

3.2. Transformation of equations (3) from the matrix space with ordinary matrix algebra to the matrix space with Kronecker matrix algebra

To apply Kronecker matrix algebra, it is necessary to reduce matrix equation (3) with respect to the vector of unknowns to a matrix equation with respect to the matrix of unknowns. For this, let's apply the following approach.

Let's introduce the m-vector $y(t) = (y_1(t), y_2(t), \dots, y_m(t))^T$ in equation (3) as the product of $y(t) = Y(t)\mathcal{I}$ of the diagonal $m \times m$-matrix of variables $Y(t)$ and the unit m-vector $\mathcal{I}$, namely,

$$Y(t) = \begin{pmatrix} y_1(t) & 0 & \dots & 0 \\ 0 & y_2(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & y_m(t) \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Similarly, let's represent the m-vector $f(t) = (f_1(t), f_2(t), \dots, f_m(t))^T$ on the right side of equation (3) as the product of $f(t) = \Phi(t)\mathcal{I}$ of the diagonal $m \times m$-matrix $\Phi(t)$ and the unit m-vector $\mathcal{I}$

$$\Phi(t) = \begin{pmatrix} f_1(t) & 0 & \dots & 0 \\ 0 & f_2(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_m(t) \end{pmatrix}.$$

Similarly, let's represent the m-vectors in the initial conditions $x_0 = (x_{01}, x_{02}, \dots, x_{0m})^T$ and $x'_0 = (x'_{01}, x'_{02}, \dots, x'_{0m})^T$ of equation (3) in the form of products of $x_0 = X_0\mathcal{I}$ and $x'_0 = X'_0\mathcal{I}$ of diagonal $m \times m$-matrices X_0 and X'_0 and the m-vector $\mathcal{I} = (11 \dots 1)^T$:

$$X_0 = \begin{pmatrix} x_{01} & 0 & \dots & 0 \\ 0 & x_{02} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_{0m} \end{pmatrix}, \quad X'_0 = \begin{pmatrix} x'_{01} & 0 & \dots & 0 \\ 0 & x'_{02} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x'_{0m} \end{pmatrix}.$$

Equation (3) may be written as follows:

$$\frac{d^2Y(t)}{dt^2}\mathcal{I} + \Lambda_D \frac{dY(t)}{dt}\mathcal{I} + FY(t)\mathcal{I} = H\Phi(t)\mathcal{I}, \tag{5}$$

$$\frac{dY(0)}{dt}\mathcal{I} = GX_0'\mathcal{I},\ \ Y(0)\mathcal{I} = GX_0\mathcal{I}\,,$$

wherein vector equation (3) with respect to the vector of unknowns $y(t)$ is reduced to matrix equation (5) with respect to the matrix of unknowns $Y(t)$.

To move to the Kronecker space with Kronecker matrix algebra, let's transform the matrices $Y(t), \Phi(t), X_0', X_0 \in \mathcal{F}_{m\times m}$ in equation (5) into the corresponding Kronecker vectors $\mathscr{y}, \phi, \mathscr{x}_0, \mathscr{x}_0' \in \mathcal{F}_{m^2}$ according to expressions of the form (4), namely,

$$\mathscr{y}(t) = \begin{pmatrix} Y_{1*}^T(t) \\ Y_{2*}^T(t) \\ \vdots \\ Y_{m*}^T(t) \end{pmatrix},\ \ \phi(t) = \begin{pmatrix} \Phi_{1*}^T(t) \\ \Phi_{2*}^T(t) \\ \vdots \\ \Phi_{m*}^T(t) \end{pmatrix},\ \ \mathscr{x}_0 = \begin{pmatrix} X_{01*}^T \\ X_{02*}^T \\ \vdots \\ X_{0m*}^T \end{pmatrix},\ \ \mathscr{x}_0' = \begin{pmatrix} X_{01*}'^T \\ X_{02*}'^T \\ \vdots \\ X_{0m*}'^T \end{pmatrix}, \tag{6}$$

where $Y_{i*}(t)$, $\Phi_{i*}(t)$, X_{0i*}, X_{0i*}' – i-th rows ($i = 1,2,\dots,m$) of the matrices $Y(t), \Phi(t), X_0,\ X_0' \in \mathcal{F}_{m\times m}$, respectively. Therewith, in the vectors $\phi(t)$, $\mathscr{x}$, $\mathscr{x}_0'$ (9) the respective columns are equal to $\Phi_{i*}^T(t) = (0,\dots,0, f_i(t), 0,\dots,0)^T$, $X_{0i*} = (0,\dots,0, x_{0i}, 0,\dots,0)^T$, $X_{0i*}' = (0,\dots,0, x_{0i}', 0,\dots,0)^T$.

Then, in accordance with statement (1) (Section 3.1), the set of solutions of equation (5) with respect to the required matrix $Y(t) \in \mathcal{F}_{m\times m}$ in equation (5) coincides with the set of solutions of the following equation in the Kronecker space with respect to the m^2-vector $\mathscr{y}(t)$ (see (6)):

$$\begin{aligned} &(I\otimes \mathcal{I}^T)\frac{d^2\mathscr{y}(t)}{dt^2} + (\Lambda_D\otimes \mathcal{I}^T)\frac{d\mathscr{y}(t)}{dt} + (F\otimes \mathcal{I}^T)\mathscr{y}(t) = (H\otimes \mathcal{I}^T)\phi(t), \\ &(I\otimes \mathcal{I}^T)\frac{d\mathscr{y}(0)}{dt} = (G\otimes \mathcal{I}^T)\mathscr{x}_0', \qquad (I\otimes \mathcal{I}^T)\mathscr{y}(0) = (G\otimes \mathcal{I}^T)\mathscr{x}_0\,. \end{aligned} \tag{7}$$

By introducing in the last equation a new vector variable $\mathscr{p}(t) \in \mathcal{F}_m$ according to the equation $\mathscr{y}(t) = (I\otimes \mathcal{I})\mathscr{p}(t)$ and taking into consideration that $\mathcal{I}^T\mathcal{I} = m$, the following equation will be obtained:

$$\begin{aligned} &(I\otimes m)\frac{d^2\mathscr{p}(t)}{dt^2} + (\Lambda_D\otimes m)\frac{d\mathscr{p}(t)}{dt} + (F\otimes m)\mathscr{p}(t) = (H\otimes \mathcal{I}^T)\phi(t), \\ &(I\otimes m)\frac{d\mathscr{p}(0)}{dt} = (G\otimes \mathcal{I}^T)\mathscr{x}_0', \qquad (I\otimes m)\mathscr{p}(0) = (G\otimes \mathcal{I}^T)\mathscr{x}_0\,. \end{aligned} \tag{8}$$

Note. Let's consider in more detail the question of the sets of solutions of the matrix vector equation $Ax = b$ with respect to the unknown vector x and the transformed matrix equation $AX\mathcal{I} = B\mathcal{I}$ with respect to the unknown diagonal matrix X (B - is the diagonal matrix on the right side), and the matrix equation $\mathcal{G}\mathscr{x} = \mathscr{b}$, to which the equation $AX\mathcal{I} = B\mathcal{I}$ is reduced in the transition from the ordinary matrix space to the Kronecker space.

Let's show that the set of solutions of the matrix algebraic equation $Ax = b$ (or $AX\mathcal{I} = B\mathcal{I}$) with the non-singular matrix $A \in \mathcal{F}_{m\times m}$ coincides with the set of solutions of the equation $A \in \mathcal{F}_{m\times m}\mathcal{P}\mathscr{x} = \mathcal{Q}\mathscr{b}$ with the matrices $\mathcal{P} = (A\otimes \mathcal{I}^T) \in \mathcal{F}_{m\times m^2}$ and $\mathcal{Q} = (I\otimes \mathcal{I}^T) \in \mathcal{F}_{m\times m^2}$) and vectors $\mathscr{x} \in \mathcal{F}_{m^2}$ и $\mathscr{b} \in \mathcal{F}_{m^2}$ that are defined by equations:

$$\mathscr{x} = \begin{pmatrix} X_{1*}^T \\ X_{2*}^T \\ \vdots \\ X_{m*}^T \end{pmatrix}, \quad \mathscr{b} = \begin{pmatrix} B_{1*}^T \\ B_{2*}^T \\ \vdots \\ B_{m*}^T \end{pmatrix}$$

where $X_{i*} = (0, \dots, 0, x_i, 0, \dots, 0)^T$ and $B_{i*} = (0, \dots, 0, b_i, 0, \dots, 0)^T$.

In other words, the equation $Ax = b$ (or the same equation, but written as $AX\mathcal{J} = B\mathcal{J}$) is the equation $\mathcal{P}\mathscr{x} = \mathcal{Q}\mathscr{b}$, that is, the equation $(A \otimes \mathcal{J}^T)\mathscr{x} = (I \otimes \mathcal{J}^T)\mathscr{b}$.

Let's express in the original equation $Ax = b$ the required vector of unknowns $x = (x_1, x_2 \dots, x_m)^T \in \mathcal{F}_m$ and the vector on the right side $b = (b_1, b_2 \dots, b_m)^T \in \mathcal{F}_m$ in terms of the diagonal matrices $X \in \mathcal{F}_{m\times m}$ and $B \in \mathcal{F}_{m\times m}$ in the form $x = X\mathcal{J}$ and $b = B\mathcal{J}$, where the matrices X, B and the vector $\mathcal{J} \in \mathcal{F}_m$ are equal

$$X = \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_m \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_m \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Then the equation $Ax = b$ can be represented as $AX\mathcal{J} = B\mathcal{J}$. Note that the transformation of the equation $Ax = b$ for the vector of unknowns x to the equation $AX\mathcal{J} = B\mathcal{J}$ for the matrix of unknowns X, is necessary to move from the usual matrix space to the Kronecker space and apply Kronecker matrix algebra (Section 3.1).

When describing in detail the Kronecker multiplication of the matrices, it is easy to see that

$$(A \otimes \mathcal{J}^T)\mathscr{x} = \begin{pmatrix} a_{11}\mathcal{J}^T & a_{12}\mathcal{J}^T & \dots & a_{1m}\mathcal{J}^T \\ a_{21}\mathcal{J}^T & a_{22}\mathcal{J}^T & \dots & a_{2m}\mathcal{J}^T \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathcal{J}^T & a_{m2}\mathcal{J}^T & \cdots & a_{mm}\mathcal{J}^T \end{pmatrix} \begin{pmatrix} X_{1*}^T \\ X_{2*}^T \\ \vdots \\ X_{m*}^T \end{pmatrix} = Ax,$$

$$(I \otimes \mathcal{J}^T)\mathscr{b} = \begin{pmatrix} \mathcal{J}^T & O^T & \dots & O^T \\ O^T & \mathcal{J}^T & \dots & O^T \\ \vdots & \vdots & \ddots & \vdots \\ O^T & O^T & \cdots & \mathcal{J}^T \end{pmatrix} \begin{pmatrix} X_{1*}^T \\ X_{2*}^T \\ \vdots \\ X_{m*}^T \end{pmatrix} = b,$$

where $\mathcal{J}^T = (1\ 1 \dots 1) \in \mathcal{F}_m$, $O^T = (0\ 0 \dots 0) \in \mathcal{F}_m$.

In this way, the equations $Ax = b$ и $\mathcal{P}\mathscr{x} = \mathcal{Q}\mathscr{b}$ are equivalent, where $\mathcal{P} = (A \otimes \mathcal{J}^T) \in \mathcal{F}_{m\times m^2}$ and $\mathcal{Q} = (I \otimes \mathcal{J}^T) \in \mathcal{F}_{m\times m^2}$.

Let's consider the equation $\mathcal{P}\mathscr{x} = \mathcal{Q}\mathscr{b}$, or in the expansion the equation $(A \otimes \mathcal{J}^T)\mathscr{x} = (I \otimes \mathcal{J}^T)\mathscr{b}$, and show that it has the same solutions as $Ax = b$.

Let's introduce the change of variables $\mathscr{x} = (A^{-1} \otimes \mathcal{J})z$ in the equation $\mathcal{P}\mathscr{x} = \mathcal{Q}\mathscr{b}$. Then $(A \otimes \mathcal{J}^T)(A^{-1} \otimes \mathcal{J})z = (I \otimes \mathcal{J}^T)\mathscr{b}$, i.e. $(I \otimes m)z = (I \otimes \mathcal{J}^T)\mathscr{b}$ or $mz = (I \otimes \mathcal{J}^T)\mathscr{b}$. Whence, taking into account that $(I \otimes \mathcal{J}^T)\mathscr{b} = b$ (see above), $\mathscr{x} = \frac{1}{m}(A^{-1} \otimes \mathcal{J})(I \otimes \mathcal{J}^T)\mathscr{b} = \frac{1}{m}(A^{-1} \otimes \mathcal{J})b$.

By multiplying both parts of the last equation on the left by $(I \otimes \mathcal{I}^T)$, one can find that $(I \otimes \mathcal{I}^T)x = \frac{1}{m}(I \otimes \mathcal{I}^T)(A^{-1} \otimes \mathcal{I})b$, and, taking into account that $(I \otimes \mathcal{I}^T)x = x$, this results in the equation $x = \frac{1}{m}(A^{-1} \otimes m)b = A^{-1}b$.

From the above, it follows that the set of solutions of the matrix equation $Ax = b$ in the ordinary matrix space coincides with the set of solutions of the equation $\mathcal{P}\mathcal{x} = \mathcal{Q}\mathcal{b}$, where $\mathcal{P} = (A \otimes \mathcal{I}^T)$ and $\mathcal{Q} = (I \otimes \mathcal{I}^T)$ in the Kronecker matrix space. ◄

3.3. Diagonalizing the matrix F in equation (8)

Let's represent equation (8) in the following operator form $\mathcal{G}(t)\mathcal{p}(t) = (H \otimes \mathcal{I}^T)\phi(t)$ ($\mathcal{D} = d/dt, \mathcal{D}^2 = d^2/dt^2$ [4]) or in the expanded form:

$$\big((I \otimes m)\mathcal{D}^2 + (\Lambda_D \otimes m)\mathcal{D} + (F \otimes m)\big)\mathcal{p}(t) = (H \otimes \mathcal{I}^T)\phi(t). \tag{9}$$

Let's determine the eigenvalues $\lambda(t)$ and eigenvectors $v \in \mathcal{F}_m$ of the operator matrix $\mathcal{G}(t)$ that satisfy the equation $\mathcal{G}(t)v = \lambda(t)v$. It's worth noting that from the below it will become clear that the eigenvectors v are time-independent.

Let $\lambda_{F,k}$ and $w_{F,k}$, $k = 1,2,\dots,m$ be the eigenvalues and the corresponding eigenvectors of the non-singular $m \times m$-matrix F that satisfy the equation $Fw_{F,k} = \lambda_{F,k}w_{F,k}$, and since the elements of the matrix F are time-independent, its eigenvalues and eigenvectors $\lambda_{F,k}$ and $w_{F,k}$ are time-independent as well.

The eigenvalues $\lambda_k(t)$ and the eigenvectors v_k of the matrix $\mathcal{G}(t) \in \mathcal{F}_m$ are obtained in the matrix Kronecker space (Section 3.1) as follows. Assuming $v_k = (w_{F,k} \otimes 1)$ and taking into account the commutativity of the independent variables and the differential operator $\mathcal{D}$, the following is obtained:

$$\mathcal{G}(t)v_k = \mathcal{G}(t)\big(w_{F,k} \otimes 1\big) = \big((I \otimes m)\mathcal{D}^2 + (\Lambda_D \otimes m)\mathcal{D} + (F \otimes m)\big)\big(w_{F,k} \otimes 1\big) =$$

$$= (I \otimes m)\big(w_{F,k} \otimes 1\big)\mathcal{D}^2 + (\Lambda_D \otimes m)\big(w_{F,k} \otimes 1\big)\mathcal{D} + (F \otimes m)\big(w_{F,k} \otimes 1\big).$$

As $(F \otimes m)\big(w_{F,k} \otimes 1\big) = \big(Fw_{F,k} \otimes m\big) = \big(\lambda_{F,k}w_{F,k} \otimes m\big) = \lambda_{F,k}m\big(w_{F,k} \otimes 1\big)$, the following is written:

$$\mathcal{G}(t)v_k = (I \otimes m)\big(w_{F,k} \otimes 1\big)\mathcal{D}^2 + (\Lambda_D \otimes m)\big(w_{F,k} \otimes 1\big)\mathcal{D} + \lambda_{F,k}m\big(w_{F,k} \otimes 1\big) =$$

$$= ((I \otimes m)\mathcal{D}^2 + (\Lambda_D \otimes m)\mathcal{D} + \lambda_{F,k}(I \otimes m))\big(w_{F,k} \otimes 1\big)).$$

For each value k ($k = 1,2,\dots,m$), the eigenvalue$\lambda_k(t)$ and the corresponding eigenvector v_k of the matrix $\mathcal{G}(t) \in \mathcal{F}_m$ is $\lambda_k(t) = m\mathcal{D}^2 + \lambda_{D,k}m\mathcal{D} + \lambda_{F,k}m$ and $v_k = (w_{F,k} \otimes 1)$, respectively.

With the found eigenvalues $\lambda_k(t)$ and the eigenvectors $v_k = (w_{F,k} \otimes 1)$ of the matrix $\mathcal{G}(t) \in \mathcal{F}_m$, the spectral decomposition of the matrix $\mathcal{G}(t) \in \mathcal{F}_m$ is done, namely:

$$\mathcal{G}(t) = W\Lambda(t)W^{-1}, \tag{10}$$

where $\Lambda(t) = diag\{\lambda_1(t), \lambda_2, (t) \dots, \lambda_m(t)\}$, $\lambda_k(t) = m\mathcal{D}^2 + \lambda_{D,k}m\mathcal{D} + \lambda_{F,k}m$, $k = 1,2,\dots,m$; $\Lambda_F = diag\{\lambda_{F1}, \lambda_{F2}, \dots, \lambda_{Fm}\}$ – diagonal $m \times m$-matrix that consists of the eigenvalues $\lambda_{F,k}$, $k = 1,2,\dots,m$, of the matrix F; W – transforming $m \times m$-similarity matrix that consists of the

eigencolumns of the matrix $\mathcal{G}(t) \in \mathcal{F}_m$ and is $W = (w_{F,1}, w_{F,2}, \dots, w_{F,m}) \in \mathcal{F}_m$. In the matrix form, the diagonal $m \times m$-matrix of eigenvalues of the matrix $\mathcal{G}(t) \in \mathcal{F}_m$ is
$\Lambda(t) = mI\mathcal{D}^2 + m\Lambda_D\mathcal{D} + m\Lambda_F$.

It's worth noting that the eigenvectors of the matrix $\mathcal{G}(t)$ and the transforming similarity matrix W are time-independent, and the eigenvalues $\lambda_k(t) = m\mathcal{D}^2 + \lambda_{D,k}m\mathcal{D} + \lambda_{F,k}m$, being the sum of time-independent eigenvalues $\lambda_{D,k}$ and $\lambda_{F,k}$ of the matrices D and F, depend on time only through the differential operator $\mathcal{D}$.

4. Obtaining a matrix system of equations with diagonal matrices

By substituting the spectral decomposition (10) of the matrix $\mathcal{G}(t)$ into equation (9) and taking into account that the Kronecker product of an arbitrary matrix by one is equal to the matrix itself, the following equation is obtained, in which only diagonal matrices appear, namely:

$$
\begin{aligned}
&W(mI\mathcal{D}^2 + m\Lambda_D\mathcal{D} + m\Lambda_F)W^{-1}\wp(t) = (H \otimes \mathcal{I}^T)\phi(t),\\
&m\frac{d\wp(0)}{dt} = (G \otimes \mathcal{I}^T)x_0', \qquad m\wp(0) = (G \otimes \mathcal{I}^T)x_0 .
\end{aligned}
\tag{11}
$$

By introducing a new vector variable $z(t) = W^{-1}\wp(t)$ into the last equation and multiplying the resulting equation on the left by the matrix W^{-1}, the following system of equations is obtained:

$$
\begin{aligned}
&mI\frac{d^2z(t)}{dt^2} + m\Lambda_D\frac{dz(t)}{dt} + m\Lambda_F z(t) = W^{-1}(H \otimes \mathcal{I}^T)\phi(t),\\
&m\frac{dz(0)}{dt} = W^{-1}(G \otimes \mathcal{I}^T)x_0', \qquad mz(0) = W^{-1}(G \otimes \mathcal{I}^T)x_0 ,
\end{aligned}
\tag{12}
$$

which is decomposed into m independent equations for each independent variable $z_i(t)$, $i = 1,2, \dots, m$

$$
\begin{aligned}
&\frac{d^2z_i(t)}{dt^2} + \lambda_{Di}\frac{dz_i(t)}{dt} + \lambda_{Fi}z_i(t) = \frac{1}{m}\{W^{-1}(H \otimes \mathcal{I}^T)\phi(t)\}_i,\\
&\frac{dz_i(0)}{dt} = \frac{1}{m}\{W^{-1}(G \otimes \mathcal{I}^T)x_0'\}_i, \qquad z_i(0) = \frac{1}{m}\{W^{-1}(G \otimes \mathcal{I}^T)x_0\}_i,
\end{aligned}
\tag{13}
$$

where $\{W^{-1}(H \otimes \mathcal{I}^T)\phi(t)\}_i$, $\{W^{-1}(G \otimes \mathcal{I}^T)x_0'\}_i$, $\{W^{-1}(G \otimes \mathcal{I}^T)x_0\}_i$ – i-th elements of vectors $W^{-1}(H \otimes \mathcal{I}^T)\phi(t)$, $W^{-1}(G \otimes \mathcal{I}^T)x_0'$, $W^{-1}(G \otimes \mathcal{I}^T)x_0$.

Each of equations (13) has an analytical solution, which, in most cases, is known and contained in numerous reference literature [14]. The vector $z(t)$ is determined using the found solution elements $z_i(t)$, $i = 1,2, \dots, m$.

The required vector of solutions $x(t)$ of original equation (1) is related to the matrix of unknowns $Y(t)$ by the equation $x(t) = U\Lambda_A^{-\frac{1}{2}}VY(t)\mathcal{I}$. In turn, the matrix $Y(t)$ upon moving from the ordinary matrix space to the Kronecker matrix space, is transformed into the Kronecker vector $\mathscr{y}(t)$, which is transformed, first, into the vector $\mathscr{p}(t)$, and then into the vector $\mathscr{z}(t)$, determined by equations (13), that is, $Y(t) \Rightarrow \mathscr{y}(t) = (I \otimes \mathcal{I})\mathscr{p}(t) = (I \otimes \mathcal{I})W\mathscr{z}(t)$, where the matrices U, V, W are the transforming m × m-similarity matrices.

It's worth noting that if one of the matrices turns out to be diagonal when diagonalizing matrices in equation (1), then its spectral decomposition is not required

5. The example of application

The application of the developed method is disclosed in this section by the example of a matrix of differential equation in the second-order ordinary derivatives (e.g., the Lagrange equation for generalized coordinates [4])

$$\begin{aligned} &A\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Cx(t) = 0, \\ &\frac{dx(0)}{dt} = x_0', \;\; x(0) = x_0, \end{aligned} \tag{14}$$

where A, B, C – non-singular matrices ($\det A = 3$, $\det B = 5$, $\det C = 10$)

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \;\; B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \;\; C = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix},$$
$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \;\; x_0 = \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix}, \;\; x_0' = \begin{pmatrix} x_{01}' \\ x_{02}' \end{pmatrix}.$$

Using the method suggested in the article, system of matrix equations (14) is reduced to a system of two independent equations with respect to the variables $z_i(t)$, $i = 1{,}2$,

$$\begin{aligned} &\frac{d^2z_i(t)}{dt^2} + \lambda_{Di}\frac{dz_i(t)}{dt} + \lambda_{Fi}z_i(t) = 0, \\ &\frac{dz_i(0)}{dt} = z_{01}' = \frac{1}{2}\{W^{-1}(G \otimes \mathcal{I}^T)x_0'\}_i, \qquad z_i(0) = z_{01} = \frac{1}{2}\{W^{-1}(G \otimes \mathcal{I}^T)x_0\}_i. \end{aligned} \tag{15}$$

Let's find the numerical values of the variables from equations (15), i.e. the eigenvalues (λ_{A1}, λ_{A2}), (λ_{D1}, λ_{D2}), (λ_{F1}, λ_{F2}) of the matrices A, D, F, the transforming similarity matrices U, V, W, the matrix G in the initial conditions, and also obtain the necessary spectral decompositions of all matrices.

The eigenvalues (λ_{A1}, λ_{A2}), (λ_{D1}, λ_{D2}), (λ_{F1}, λ_{F2}) of matrices A, D, F and the transforming similarity matrices U, V, W are obtained by spectral decomposition of matrices A, $D = \Lambda_A^{-\frac{1}{2}}U^{-1}BU\Lambda_A^{-\frac{1}{2}}$ and $F = V^{-1}\Lambda_A^{-\frac{1}{2}}U^{-1}CU\Lambda_A^{-\frac{1}{2}}V$, namely,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = U\Lambda_A\,U^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix};$$

$$D = \Lambda_A^{-\frac{1}{2}} U^{-1} B U \Lambda_A^{-\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{-\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ \frac{2\sqrt{3}}{3} & \frac{5}{3} \end{pmatrix};$$

$$D = \begin{pmatrix} 1 & 0 \\ \frac{2\sqrt{3}}{3} & \frac{5}{3} \end{pmatrix} = V \Lambda_D V^{-1} = \begin{pmatrix} \frac{-\sqrt{3}}{3} & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{3} \end{pmatrix} \begin{pmatrix} -\sqrt{3} & 0 \\ \sqrt{3} & 1 \end{pmatrix};$$

$$F = V^{-1} \Lambda_A^{-\frac{1}{2}} U^{-1} C U \Lambda_A^{-\frac{1}{2}} V =$$

$$= \begin{pmatrix} -\sqrt{3} & 0 \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} \frac{-\sqrt{3}}{3} & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{5}{3} \end{pmatrix};$$

$$F = \begin{pmatrix} 2 & 0 \\ 0 & \frac{5}{3} \end{pmatrix} = W \Lambda_F W^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & \frac{5}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

whence it follows that the eigenvalues of the matrices A, D and F are $(\lambda_{A1}, \lambda_{A2}) = (1, 3)$, $(\lambda_{D1}, \lambda_{D2}) = (1, \frac{5}{3})$, $(\lambda_{F1}, \lambda_{F2}) = (2, \frac{5}{3})$, respectively.

The matrices in the initial conditions of equations (15) are

$$G = V^{-1} \Lambda_A^{\frac{1}{2}} U^{-1} = \begin{pmatrix} -\sqrt{3} & 0 \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \sqrt{3} \end{pmatrix}$$

$$(G \otimes \mathcal{J}^T) = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \sqrt{3} \end{pmatrix} \otimes (1 \quad 1) = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \sqrt{3} & \sqrt{3} \end{pmatrix},$$

and the initial conditions in equation (15) can be written as follows

$$\begin{pmatrix} z'_{01} \\ z'_{02} \end{pmatrix} = \frac{1}{2} \{W^{-1} (G \otimes \mathcal{J}^T) x'_0\}_i = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \sqrt{3} & \sqrt{3} \end{pmatrix} \begin{pmatrix} x'_{01} \\ x'_{02} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{4} x'_{01} - \frac{\sqrt{3}}{4} x'_{02} \\ \frac{\sqrt{3}}{2} x'_{02} \end{pmatrix},$$

$$\begin{pmatrix} z_{01} \\ z_{02} \end{pmatrix} = \frac{1}{2} \{W^{-1} (G \otimes \mathcal{J}^T) x_0\}_i = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \sqrt{3} & \sqrt{3} \end{pmatrix} \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{4} x_{01} - \frac{\sqrt{3}}{4} x_{02} \\ \frac{\sqrt{3}}{2} x_{02} \end{pmatrix}.$$

Equations (15) define the vector of independent solutions $z(t) = (z_1(t), z_2(t))^T$, the components of which $z_1(t)$ and $z_2(t)$ are found by solving the following equations:

– component $z_1(t)$

$$\frac{d^2 z_1(t)}{dt^2} + \frac{dz_1(t)}{dt} + 2z_1(t) = 0, \tag{16}$$

$$\frac{dz_1(0)}{dt} = z'_{01} = \frac{\sqrt{3}}{4}x'_{01} - \frac{\sqrt{3}}{4}x'_{02}, \quad z_1(0) = z_{01} = \frac{\sqrt{3}}{4}x_{01} - \frac{\sqrt{3}}{4}x_{02},$$

– component $z_2(t)$

$$\frac{d^2 z_2(t)}{dt^2} + \frac{5}{3}\frac{dz_2(t)}{dt} + \frac{5}{3}z_2(t) = 0\,,$$
$$\frac{dz_2(0)}{dt} = z'_{02} = \frac{\sqrt{3}}{2}x'_{02}, \qquad z_2(0) = z_{02} = \frac{\sqrt{3}}{2}x_{02}. \tag{17}$$

The solutions of equations (16) and (17) are known and can be expressed analytically [14]

$$z_1(t) = z_{01}e^{\alpha t}\left(\cos\omega\, t - \frac{\alpha}{\omega}\sin\omega\, t\right) + \frac{1}{\omega}z'_{01}e^{\alpha t}\sin\omega t, \tag{18}$$

at $\alpha = -0{,}5,\ \omega = 1{,}322876;$

$$z_2(t) = z_{02}e^{\alpha t}\left(\cos\omega\, t - \frac{\alpha}{\omega}\sin\omega\, t\right) + \frac{1}{\omega}z'_{02}e^{\alpha t}\sin\omega t. \tag{19}$$

at $\alpha = -0{,}833,\ \omega = 0{,}986.$

The required solution $x(t)$ of original equation (14) is expressed through the independent solutions $z_1(t)$ and $z_2(t)$ of equations (16) and (17)

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = U\Lambda_A^{-\frac{1}{2}}VY(t)\mathcal{J} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{-\frac{1}{2}}\begin{pmatrix} \frac{-\sqrt{3}}{3} & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} z_1(t) & z_1(t) \\ z_2(t) & z_2(t) \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} =$$
$$= \begin{pmatrix} \frac{4\sqrt{3}}{3}z_1(t) + \frac{2\sqrt{3}}{3}z_2(t) \\ \frac{2\sqrt{3}}{3}z_2(t) \end{pmatrix}.$$

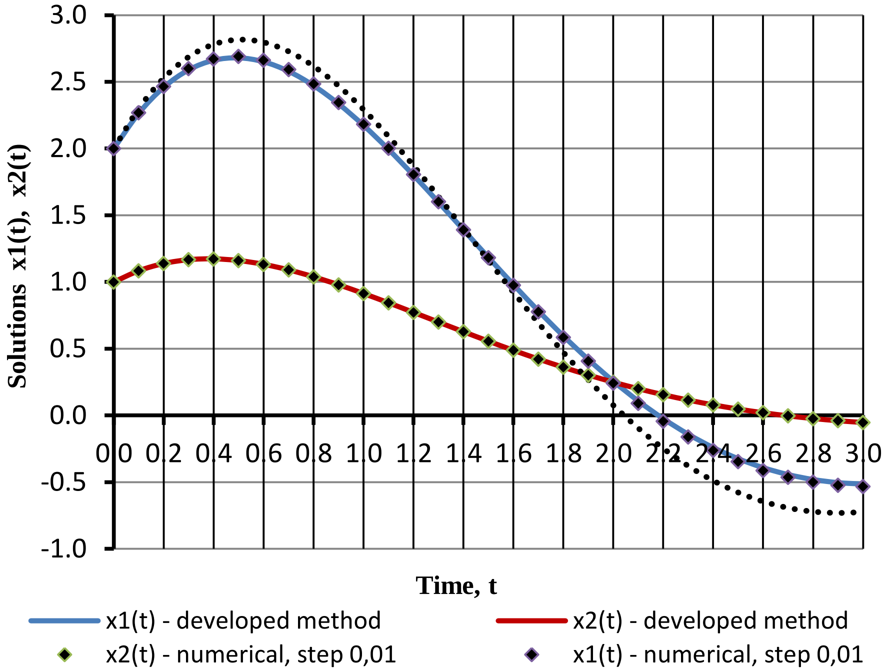

Fig. 1. Solutions $x_1(t)$ and $x_2(t)$ for the example of the matrix differential equation in the second-order ordinary derivatives calculated using the developed method and numerical methods on a computer

The required solutions $x_1(t)$ and $x_2(t)$ of original equation (14), obtained using the developed method developed, are shown in Fig. 1.

Fig. 1 also presents solutions obtained using a numerical method on a computer. For comparison, solutions are also given calculated with a time step of 0.1 and 0.01. Comparison of the solutions obtained using the developed method with the solutions calculated using the numerical method shows their complete coincidence, which is a consequence of the fact that the developed method presented in the article does not contain approximating conditions and assumptions and is accurate.

6. Conclusion

The existing methods of reducing matrix systems of coupled differential equations in ordinary derivatives to a system of decoupled differential equations are based on the simultaneous diagonalization of two symmetric matrices of the equation based on the theorem on their simultaneous diagonalization and reduction of one of them to a diagonal identity one. Since the number of simultaneously diagonalized matrices does not exceed two, the initial matrix equations are considered, as a rule, in a truncated form, without any equation term, so that the total number of matrices does not exceed two. At the same time, in many applications, matrix differential equations have three or more matrices, which essentially motivated the development.

This article suggests a method that allows one to diagonalize three matrices in a second-order matrix differential equation and thereby obtain a system of independent equations, the solution of each of which are easily found in an explicit analytical form.

For this, one of the matrices of the equation (positive definite) is reduced to a diagonal identity form, and the other is subjected to spectral decomposition using a general similarity transformation. In the method proposed in the article, the third matrix in the matrix equation is reduced to a diagonal form using the transition from the usual matrix space with the usual matrix algebra to the Kronecker matrix space in which the rules of the Kronecker matrix algebra are used.

The developed method is equally applicable to matrix differential equations of the second order $A\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Cx(t) = f(t)$ with three matrices A, B and C, and matrix differential equations of higher orders, but with three matrices in the equation, i.e.

$$A\frac{d^n x(t)}{dt^n} + B\frac{d^m x(t)}{dt^m} + Cx(t) = f(t),\ \ n > m \geq 1.$$

The main requirement for the matrices A, B, C is the positive definiteness of one of them, which is to be diagonalized first and reduced to the diagonal identity form.

It's worth noting that the fundamental possibilities inherent in the method proposed herein allow, under certain assumptions and additional studies, to consider matrix differential equations in ordinary derivatives of a higher order (n≥3) and with the number of matrices in the equation greater than three, i.e. equations ($D^s x(t) = d^s x(t)/dt^s$ - differential operator)

$$A_n D^n x(t) + A_{n-1} D^{n-1} x(t) + \cdots + A_0 x(t) = f(t).$$

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