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Filtration and identification

Optimal dynamic measurement method using the Savitsky - Golay digital filter

Keller Alevtina Viktorovna
Voronezh State Technical University, Russia
alevtinak@inbox.ru

Abstract. We consider one of the mathematical models of the theory of optimal dynamic measurements to solve the problem of recovering a dynamically distorted signal in the presence of noise. The measuring device is simulated by a Leontief-type system which is a finite-dimensional analogue of a Sobolev-type equation, and its initial state is given by the Showalter – Sidorov condition. In order to find the input signal from the known observed signal, an optimal control problem, namely the minimization of the penalty functional in which the simulated and observed output signals are compared should be solved. The solution of this problem is called the optimal dynamic measurement. The theorem on the existence of a unique exact solution of the problem posed and the algorithm of the spline method for finding an approximate solution are given.

At the same time, the presence of noise at the output of the measuring device does not give a possibility to solve the problem of recovering a dynamically distorted signal satisfactorily. In the article we propose to use in the numerical algorithm the Savitsky-Golay digital filter for the observed signal. As a result, we obtain an observation smoothed by the filter, which is then used in the penalty functional. The choice of parameters for the Savitsky-Golay digital filter is discussed, and the results of computational experiments on the data of bench tests are presented.

Keywords: virtual model, optimal dynamic measurements, Leontief type system, computational experiments.

1 Introduction

The main problem of dynamic measurements is recovery of a dynamically distorted signal (input signal) by the observed one (output signal) with known parameters of the mathematical model of the measuring device [1]. Various mathematical methods and models [2]–[4] can be used as the basis for the analytical implementation of the virtual model for recovering a dynamically distorted signal. In this paper, we use the methods of the recently created [5] and actively developing [6],[7] theory of optimal dynamic measurements. This theory accumulates methods of the theory of optimal control, equations of Sobolev and Leontief types, dynamic measurements and automatic control.

The work [8] proposes to use the differential-algebraic system

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + D\eta \end{cases} \tag{1}$$

as a mathematical model of the measuring device. In addition to (1), the model also contains the initial Cauchy condition

$$x(0) = x_0, \tag{2}$$

which defines the initial state of the measuring system, $x_0 \in \mathbb{R}^n$. In (1), $x(t)$, $y(t)$, $u(t)$ are the vector functions of the states of the measuring system, the observed signal and the input signal, respectively, A is a square matrix of the order n characterizing the measuring device, the matrix B characterizes the relationship between the input of the system and its state, the matrices C and D characterize the relationship between the state of the system and observations, $\eta(t)$ is a vector function of noise at the output of the measuring system.

In the general case, when a measuring device is a complex system, for example, consisting of several sensors, it is necessary to use the Leontief type system, which is a finite-dimensional analogue of the Sobolev type equation [9]

$$\begin{cases} L\dot{x} = Ax + Bu + G\xi, \\ y = Cx + D\eta, \end{cases} \tag{3}$$

where $\det L = 0$, L is a square matrix of the order n characterizing the measuring system, $\xi(t)$ is a vector function of noise at the input and in the circuits of the measuring system, the matrix G characterizes the relationship between the state of the system and the corresponding noise.

Therefore, the first equation of system (3) is a Leontief type system, i.e. a degenerate system of ordinary differential equations. At the same time, the Leontief type system is a finite-dimensional analogue of the Sobolev type equation, therefore, the Showalter – Sidorov condition

$$\left[(\alpha L - A)^{-1} L\right]^{p+1} (x(0) - x_0) = 0 \tag{4}$$

is taken as the initial one. In problem (3), (4), the matrix A is $(L; p)$-regular, $\alpha \in \rho^L(M)$ [10]. The results of the solvability of the Showalter - Sidorov problem for the Sobolev type equation give a possibility to develop numerical methods for solving initial and optimal control problems, both for the equations mentioned above and Leontief type systems [10]. At present, methods and models of the theory of optimal dynamic measurements are designed which are based on the development of the theory of Sobolev type equations [11, 12] and numerical methods [13]. Such methods allow investigating the problem of optimal dynamic measurements in the presence of "white noise" and to increase the efficiency of numerical methods.

The article presents a new algorithmic implementation of a virtual model of optimal dynamic measurements. We propose to apply the Savitsky - Golay digital filter [14] to the observation data with the subsequent application of the spline method of optimal dynamic measurements. Basing on the results of computational experiments with various parameters of the digital filter, we give recommendations on the use of the filter and draw conclusions about the effectiveness of the new algorithmic implementation of the virtual model under study.

2 Analytical implementation of the model of optimal dynamic measurements

Let us consider the spaces of states

$$\aleph = \{x \in L_2((0,\tau), R^n) : \dot{x} \in L_2((0,\tau), R^n)\},$$

observations $\Upsilon = C[\aleph]$ and measurements

$$\mathfrak{A} = \left\{u \in L_2((0,\tau), R^n) : u^{(p+1)} \in L_2((0,\tau), R^n)\right\}.$$

Suppose that the measuring device is simulated by the system

$$\begin{cases} L\dot{x} = Ax + Bu, \\ y = Cx \end{cases} \tag{5}$$

and the Showalter - Sidorov initial condition (4). In $\mathfrak{A}$ consider a closed convex set of admissible measurements $\mathfrak{A}_\partial \subset \mathfrak{A}$ of the form

$$\mathfrak{A}_\partial = \left\{ u \in \mathfrak{A} : \sum_{q=0}^{\theta} \int_0^\tau \left\| u^{(q)}(t) \right\|^2 dt \leq d \right\}. \tag{6}$$

The parameter d is characterized based on the physical properties of the measured process. It is required to find the optimal dynamic measurement$v \in \mathfrak{A}_\partial$ at which the minimum value

$$J(v) = \min_{u \in \mathfrak{A}_\partial} J(u) \tag{7}$$

of the functional

$$J(u) = \sum_{q=0}^{1} \int_0^\tau \left\| y^{(q)}(u,t) - y_0^{(q)}(t) \right\|^2 dt \tag{8}$$

is achieved, where $y_0(t)$, $t \in [0, \tau]$ is a continuously differentiable function (we consider the function as a "real observation") constructed on the basis of the values Y_{0i} observed at the output of the measuring system. The penalty functional reflects the assessment of the proximity of the real observation $y_0(t)$ and the observation $y(t)$ obtained on the basis of the mathematical model of the measuring device. The presented problem (4) – (8) is called the main problem of optimal dynamic measurements. Note that in the absence of noise, the distortion of the input signal is due to the inertia of the measuring device.

In the presence of deterministic noises, for example, resonances, only at the output of the measuring device, in order to restore a dynamically distorted signal, it is permissible to use the penalty functional of the form (8). In this case, problem (1), (4), (6) – (8) is considered. In the presence of deterministic noise both at the output and at the input of the measuring device, the quality functional has the form

$$J(u) = \beta \sum_{q=0}^{1} \int_0^\tau \left\| y^{(q)}(u,\varsigma,\eta,t) - \left(y_0^{(q)}(u,\varsigma,\eta,t) - \bar{y}_0^{(q)}(\varsigma,\eta,t) \right) \right\|^2 dt +$$

$$+(1-\beta)\sum_{q=0}^{1}\int_0^{\tau}\left\langle N_q(u+\varsigma)^{(q)}(t),(u+\varsigma)^{(q)}(t)\right\rangle dt, \qquad (9)$$

where $\bar{y}_0(\varsigma,\eta,t)$ are real observations in the absence of measurement, $y_0(u,\varsigma,\eta,t)$ are real observations during measurement, $y(u,\varsigma,\eta,t)$ are observations obtained in the course of mathematical modelling of dynamic measurement. Therefore, in this case, we consider problem (3), (4), (6), (7), (9). Note that the bracketed expression acts as an observation filter. This approach is acceptable only in the case of determinism of the noise, which is reproduced in each experiment. However, this situation should be characterized as a model one, since the situation is hardly possible in real experiments. Moreover, with noise in the form of "white noise", such a situation is not possible, since each observation is one realization of a random process.

Theorem 1 [5]. *Let L and A be square matrices of the order n, the matrix A be $(L; p)$- regular, and det$A$? 0. Then for any $x_0 \in R^n$ there exists a unique solution $v \in \mathfrak{A}_\partial$ to problem (4) – (8), which is an optimal dynamic measurement, and $x(v) \in \aleph$ satisfies system (5) under initial condition (4) and has the form*

$$x(t)=\lim_{k\to\infty}x_k(t)=\lim_{k\to\infty}\left[\sum_{q=0}^{p}\left(A^{-1}\left(\left(kL_k^L(A)\right)^{p+1}-\mathbb{I}_n\right)L\right)\times\right.$$
$$\times A^{-1}\left(\mathbb{I}_n-\left(kL_k^L(A)\right)^{p+1}\right)(Bu)^{(q)}+\left(\left(L-\tfrac{t}{k}A\right)^{-1}L\right)^k x_0+ \qquad (10)$$
$$\left.+\int_0^t\left(\left(L-\tfrac{t-s}{k}A\right)^{-1}L\right)^k\left(L-\tfrac{t-s}{k}A\right)^{-1}\times\left(kL_k^L(A)\right)^{p+1}Bu(s)\,ds\right],$$

where $\lim_{k\to\infty}\left(kL_k^L(A)\right)^{p+1}$ is the projector, $L^L(A)$ is the left resolvent of A .

The work [15] presents a numerical method, which uses a digital filter of a moving average applied to the observation data and a spline method of optimal dynamic measurements.

3 Algorithmic implementation of the model of optimal dynamic measurements

Let us search for the optimal dynamic measurement in the form

$$u^\ell=col\left(\sum_{j=1}^{\ell}a_{1j}t^j,\sum_{j=1}^{\ell}a_{2j}t^j,\ldots,\sum_{j=1}^{\ell}a_{nj}t^j\right), \qquad (11)$$

and taking into account (10), we get

$$J_k(v_k^\ell) = \min J_k(u^\ell) = \min \sum_{q=1}^{1} \int_0^\tau \left\| !x_k^{(q)}(u^\ell, t) - y_0^{(q)}(t) \right\|^2 dt. \qquad (12)$$

Therefore, the pair $(v_k^\ell, x_k^\ell) = (v_k^\ell, x_k(v^\ell))$ denotes an approximate solution to the problem of optimal dynamic measurement if, in addition to the optimal dynamic measurement, we also search for an approximate state of the system. Consider the initial values of the coefficients$a_{ij}$to be zeroes and use the coordinate descent method with memory to find the valuesa_{ij}, which provide the minimum of the penalty functional [5]. At the same time, the convergence of the approximate solution to the problem on optimal dynamic measurement to the exact one in the norm is proved. The following theorem is true.

Theorem 2 [16]. *Let the matrix A be $(L;\ p)$-regular, and $detA0$, $p \in \{0\} \cup N$, functional* (5) *be a strongly convex function on a compact and convex set $\mathfrak{A}_\partial \subset \mathfrak{A}$. Then $J_k(v_k^\ell) \to J(v)$, $v_k^\ell \to v$ for $k \geq K$and $\ell > p$ and there exists T>0 such that the following inequality holds:*

$$T \left\| v_k^\ell - v \right\|^2 \leq J_k(v_k^\ell) - J(v).$$

However, the original numerical method required a lot of computing power and time. Therefore, a spline method was proposed for finding the optimal dynamic measurements.

Let us describe the spline method for solving the problem of optimal dynamic measurement (4) – (8).

Suppose that the following components are given: the matrices included in system (5); the initial value $x_0 \in R^n$; the array of observed values Y_{0i} at the nodal points $t_i = 0, 1, \ldots, n$ of the output signal, and $t_{i+1} - t_i = \delta$, $t_0 = 0$, $t_n = \tau$.

Step 1. Divide the interval $[0, \tau]$into M intervals $[\tau_{m-1}, \tau_m]$, where $m = 1, 2, \ldots, M$, and $t_0 = \tau_0 = 0$, $t_n = \tau_M$.

Step 2. At each interval $[\tau_{m-1}, \tau_m]$, construct the interpolation function $y_{0m}^\ell(t)$ in the form of a polynomial of the degree $\ell \leq (n-1)/M$.

Step 3. For m = 1, 2,. . . , M at $[\tau_{m-1}, \tau_m]$, consecutively solve the optimal dynamic measurement problem (4)-(8) for $u \in \mathfrak{A}_{\partial m}$, where $\mathfrak{A}_{\partial m} \subset \mathfrak{A}_\partial$ is a closed convex subset of $\mathfrak{A}_\partial$, by the method described in [5]. We find the approximate value of the optimal measurement $v_{km}^\ell(t)$ in the form of a polynomial of the degree ℓ imposing the continuity condition

$$v^{\ell}_{km}(\tau_m) = v^{\ell}_{k,m+1}(\tau_m). \quad (13)$$

Step 4. As a result, we get a spline function

$$\tilde{v}^{\ell}_{k}(t) = \bigcup_{m} v^{\ell}_{km}(t)$$

continuous on $[0, \tau]$.

Let us consider the result of the implementation of the spline method algorithm (without using a digital filter) in the presence of noise at the output of the measuring device. Therefore, the spline method solves the following optimal dynamic measurement problem. The measuring device is simulated by the system

$$\begin{cases} L\dot{x} = Ax + Bu, \\ y = Cx + D\eta, \end{cases} \quad (14)$$

and condition (4). It is required to find an optimal dynamic measurement $v \in \mathfrak{A}_{\partial}$ (6) at which the minimum value (7) of the penalty functional (8) is achieved.

At the stand, the Metran-43 sensor with an analog electronic converter was tested, a test signal and an output signal of the tested sensor were obtained. Fig. 1 shows

$\hat{u}(t_i)$, i.e. a test signal recorded by the control sensor (black line),

$\hat{v}^{\ell}(t_i)$, i.e. an approximate optimal dynamic measurement or an input signal reconstructed by the spline method (red line).

In order to smooth the output signal, we use the Savitsky - Golay digital filter [14], which is a noise filtering technique based on the least squares method. The idea is to construct an s-th degree approximating polynomial by 2μ+1 sequential equidistant points and use the polynomial value at the μ+1-th point as a smoothed value.

Hence, we obtain the following algorithm for the numerical solution of the optimal dynamic measurement problem (14), (4), (6) – (8) with the combined use of the Savitsky - Golay digital filter and the spline method.

Step 0. Determine the parameters μ and s of the Savitsky - Golay digital filter and apply the filter to the array of the values Y_{0i}. As a result, we get the smoothed values y_{0i}, i=0, 1,. . . , n.

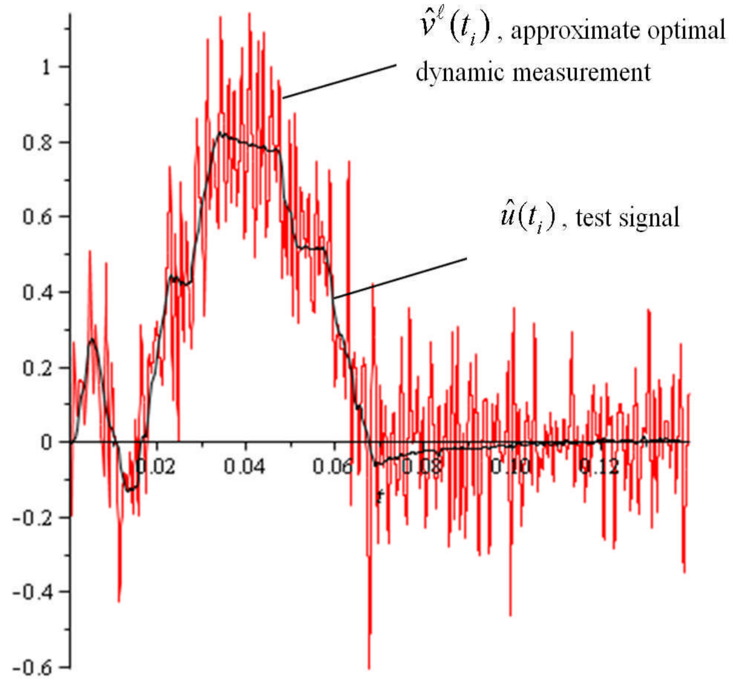

Fig. 1. Result of application of the spline method without ursing a digital filter

Assuming that the effect of noise at the output is eliminated by smoothing by the filter, we proceed to the implementation of the spline method to solution of problem (4) – (8) taking the smoothed valuesy_{0i} as the array of observations.

Steps 1 – 4 are implemented.

4 Computational experiments

In computational experiments, we apply the Savitsky - Golay digital filter at various values of the parameters

– μ=5, s=3;

– μ=5, s=1;

– μ=10, s=3;

– μ=10, s=1;

– μ=15, s=3

to the array of observations (s-th degree approximating polynomial by 2μ+1 sequential equidistant points and use the polynomial value at the μ+1-th point as a smoothed value).

Fig. 2 shows the result of a computational experiment with a smoothing window containing 11 counts (μ=5) and an approximating polynomial of the third degree.

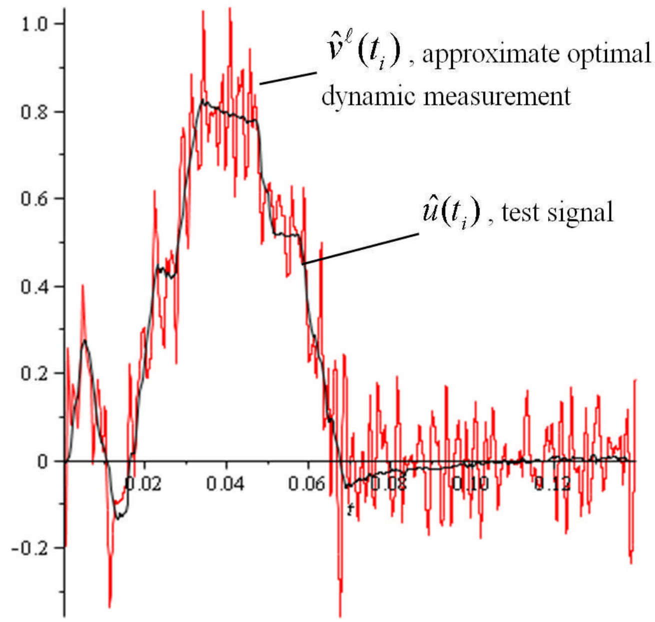

Fig. 2. The result of applying the spline method and the Savitsky - Golay digital filter with a smoothing window of 11 counts and an approximating polynomial of the third degree

Fig. 3 shows the result of a computational experiment with a smoothing window containing 11 counts (μ=5) and an approximating polynomial of the first degree. An analysis of the results of these two computational experiments shows that the use of an approximating polynomial of the first degree allows to achieve a smaller error in the discrepancy between the test and simulated signals.

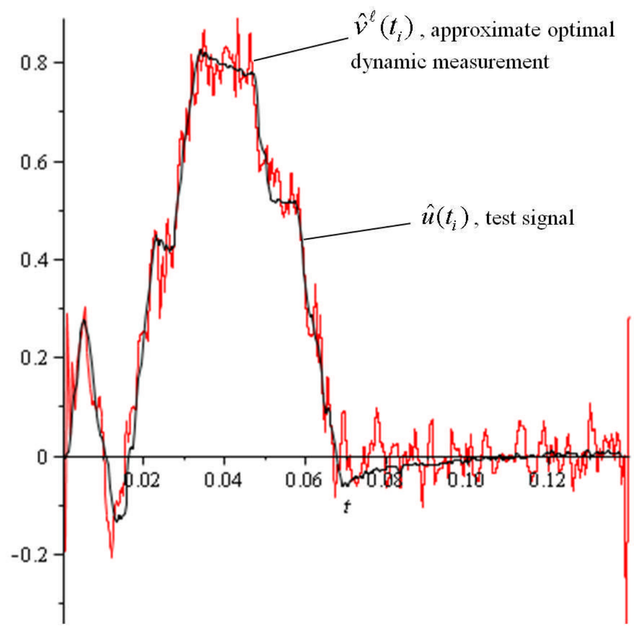

Fig. 3. The result of applying the spline method and the Savitsky - Golay digital filter with a smoothing window of 11 counts and an approximating polynomial of the first degreee

Note that with a smoothing window containing 11 counts, the lag effect is completely eliminated. Significant deviations at the end and beginning of the time interval are due to the failure to apply a digital filter to the first five and last five counts.

Fig. 4 shows the result of a computational experiment with a smoothing window containing 21 counts (μ=10) and an approximating polynomial of the third degree.

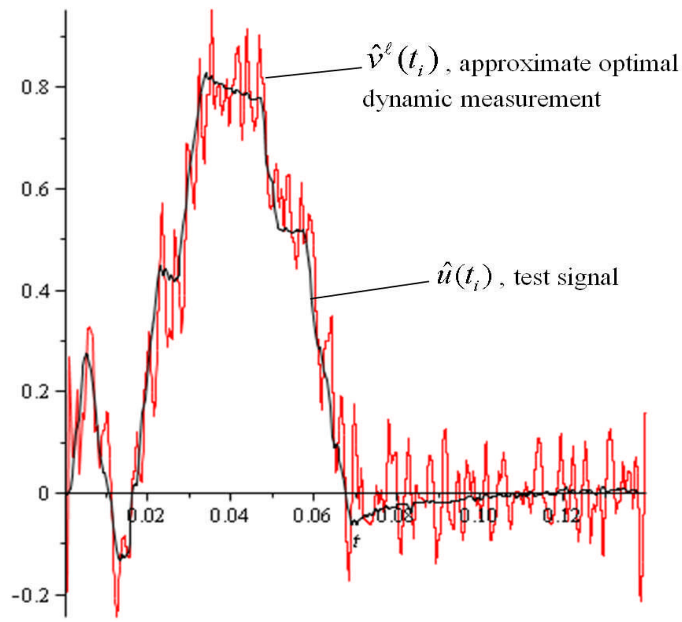

Fig. 4. The result of applying the spline method and the Savitsky - Golay digital filter with a smoothing window of 21 counts and an approximating polynomial of the third degree

Fig. 5 shows the result of a computational experiment with a smoothing window containing 21 counts (μ=10) and an approximating polynomial of the first degree.

Analysis of the results of these two computational experiments shows that the use of an approximating polynomial of the first degree allows to achieve a smaller dynamic error in comparison with a polynomial of the third degree. In both cases, the shift is required to be insignificant, namely, to be equal to 7δ to the left along the time axis t.

Fig. 6 shows the result of a computational experiment with a smoothing window containing 21 counts, an approximating polynomial of the third degree and a shift of 7 counts to the left along the axis t.

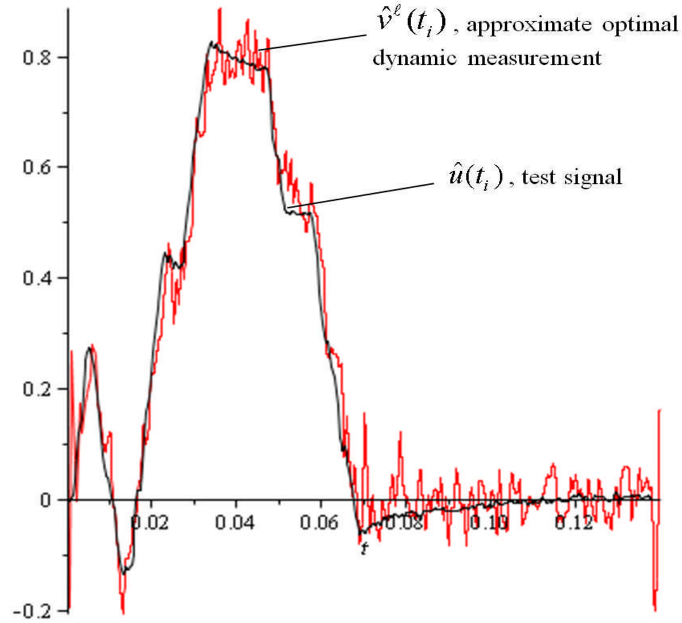

Fig. 5. The result of applying the spline method and the Savitsky - Golay digital filter with a smoothing window of 21 counts and an approximating polynomial of the first degree

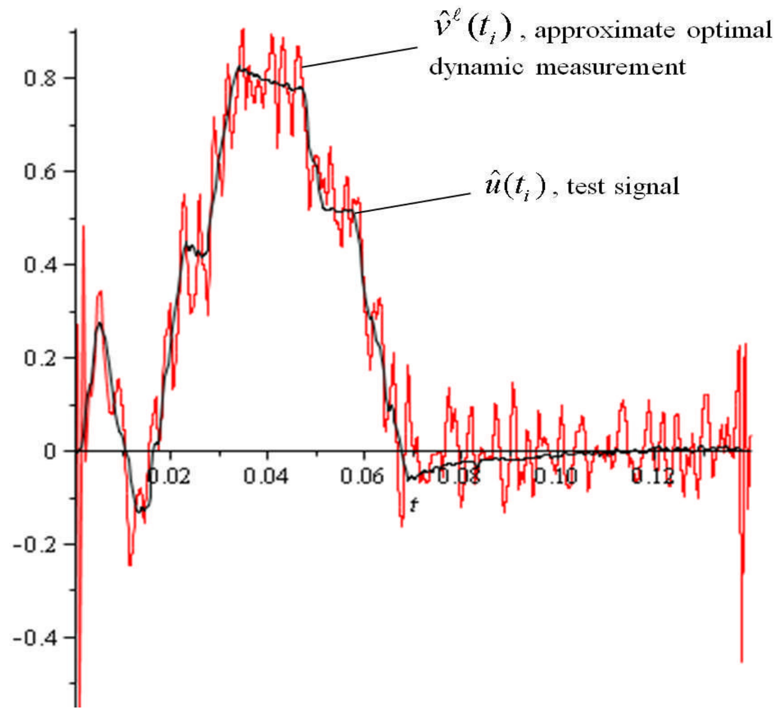

Fig. 6. The result of applying the spline method and the Savitsky - Golay digital filter with a smoothing window of 21 counts, an approximating polynomial of the third degree and a shift 7δ to the left along the time axis t

Fig. 7 shows the result of a computational experiment with a smoothing window containing 21 counts, an approximating polynomial of the first degree and a shift of 7 counts to the left along the axis t.

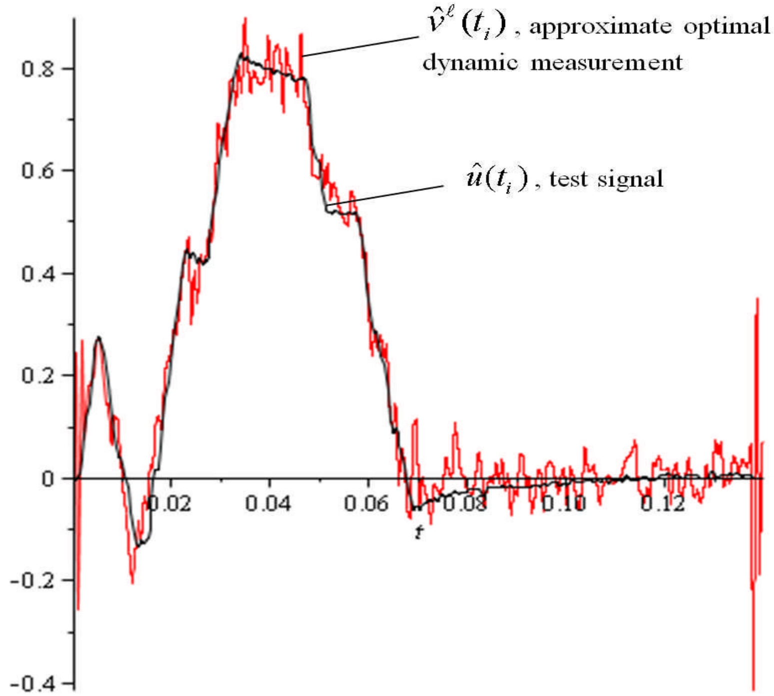

Fig. 7. The result of applying the spline method and the Savitsky - Golay digital filter with a smoothing window of 21 counts, an approximating polynomial of the first degree and a shift 7δ to the left along the time axis t

Comparison of the results shows that the implementation of a shift reduces the dynamic error, and a visual analysis of the corresponding graphs of $\hat{v}^{\ell}(t_i)$, for example, Fig. 4 and Fig. 6, at local extrema $\hat{u}(t_i)$ shows a more accurate restoration of the signal in the presence of a shift (Fig. 6).

Computational experiments with a larger smoothing window (more than 21 counts) lead to an increase in the dynamic error. Similar results with respect to the size of the smoothing window were obtained using the moving average filter [15].

In addition, on the basis of the Kotelnikov theorem, the connection between the length of the interval of the smoothing window and the value of the maximum frequency f_c , which limits the spectrum of the desired input signal, is shown.

Fig. 8 shows the result of a computational experiment with a smoothing window containing 31 counts (μ=15) and an approximating polynomial of the third degree.

Despite the fact that the error in this case is smaller compared to the smoothing window containing 21 counts, we emphasize that in the neiborhood of a number of local extrema, for example, at t=0,0225 s, the values of the optimal dynamic measurement are significantly less than the test signal (smoothing effect). In solving certain applied problems (for example, design), such a situation may turn out to be unacceptable.

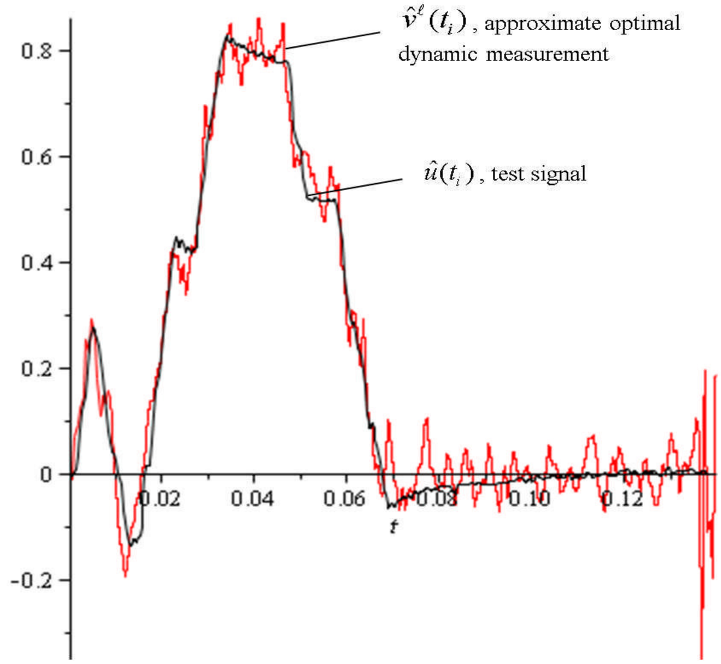

Fig. 8. The result of applying the spline method and the Savitsky - Golay digital filter with a smoothing window of 31 counts and an approximating polynomial of the third degree

Conclusions and recommendations

The application of the Savitsky - Golay digital filter to the observed signal when recovering a dynamically distorted signal by the method of optimal dynamic measurement allows to draw the following conclusions. The counts contained in the smoothing window must belong to an interval equal to 2Δ, where $0 < \triangle \leq 1/2f_c$, f_c is the maximum frequency that limits the spectrum of the desired input signal.

A smaller error is achieved when using an approximating polynomial of the first degree. The elimination of the lag effect is proposed by shift of m counts to the left such that $m\delta \approx 2\triangle/3$. In the future, with the similar statement of the problem, other types of digital filters will be considered.

References

[1] Granovsky V.A. Dynamic measurements: theory and metrological assurance at yesterday and tomorrow, Datchiki and Systemi (Sensors and Sys-

tems) 3(201) (2016) pp. 57–72. (in Russian)

[2] Tikhonov A.N. and Arsenin V.Ya., Solutions of ill-posed problems, Nauka. 1979. (in Russian)

[3] Ruhm K.H. Dynamic and stability: A proposal for related terms in metrology from a mathematical point of view, Measurement 79 (2016) p. 276.

[4] Eichstadt S., Elster C., Smith I.M. and Esward T.J. Evaluation of dynamic measurement uncertainty: an open-source software package to bridge theory and practice, Journal of Sensors and Sensor Systems 6 (2017) p. 97.

[5] Shestakov A.L. and Sviridyuk G.A. A new approach to measuring dynamically distorted signals, Bulletin of the South Ural state university. Series: mathematical modelling, programming and computer software 16(192) (2010) pp. 116–120.

[6] Shestakov A.L., Keller A.V. and Nazarova E.I. Numerical solution of the optimal measurement problem, Automation and Remote Control 73(1) (2012) pp. 97–104, DOI: 10.1134/S0005117912010079

[7] Shestakov A.L., Keller A.V., Zamyshlyaeva A.A., Manakova N.A., Zagrebina S.A. and Sviridyuk G.A. The optimal measurements theory as a new paradigm in the metrology, Journal of Computational and Engineering Mathematics 7(1) (2020) pp. 3-23 doi: 10.14529/$jcem$200101

[8] Shestakov A.L. Analysis of dynamic error and selection of a measuring transducer's parameters at stepwise, linear an hierarchic segments, Izmeritelnaya tekhnika [Measuring Engineering] 6(13) (1992) pp. 13 (in Russian)

[9] Khudyakov Yu.V. On mathematical modeling of the measurement transducers, Journal of Computational and Engineering Mathematics 3(3) (2016) pp. 68-73.

[10] Keller A.V. On the computational efficiency of the algorithm of the numerical solution of optimal control problems for models of leontieff type, Journal of Computational and Engineering Mathematics 2(2) (2015) pp. 39–59 DOI: 10.14529/$jcem$150205

[11] Favini A., Zagrebina S.A. and Sviridyuk G.A. Multipoint initial-final value problems for dynamical Sobolev-type equations in the space of noises, Electronic Journal of Differential Equations 2018 (2018) P. 128.

[12] Manakova N.A., Gavrilova O.V. and Perevozchikova K.V. Numerical investigation of the optimal measurement for a semilinear descriptor system with the Showalter-Sidorov condition: algorithm and computational experiment, Differencialnie Uravnenia i Protcesy Upravlenia, 2020 (4) P. 115–126.

[13] Shestakov A.L., Sviridyuk G.A., Keller A.V., Khudyakov Yu.V. and Zamyshlyaeva A.A. Numerical investigation of optimal dynamic measurements, Acta IMEKO 7(2) (2018). pp. 65 – 72

[14] Savitzky A. and Golay M.J.E. Smoothing and Differentiation of Data by Simplified Least Squares Procedures, Anal. Chem. 1964 vol. 36 pp. 1627–1639. doi:10.1021/ac60214a047.

[15] Shestakov A.L. and Keller A.V. Optimal dynamic measurement method using a digital sliding medium filter, to appear

[16] Sviridyuk G.A. and Keller A.V. On the numerical solution convergence of optimal control problems for leontief type system, Journal of Samara State Technical University. Series Physical and Mathematical Sciences 2(23) (2011) pp. 24–33. (in Russian)