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Delay differential equations

The Problem of Convergence of Solutions of Certain Third-Order Nonlinear Delay Differential Equations

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Abstract. We present in this paper the problem of convergence behavior of solutions of certain third-order nonlinear delay differential equations and obtained the sufficient conditions involved under which the solutions of the delay differential equation are convergent. An example is also given to demonstrate the correctness of the proposed approach which improves earlier results on delay differential equations.

Keywords: Convergence of solutions, complete Lyapunov function, nonlinear delay differential equations of third-order.

1 Introduction

This paper considers the problem of convergence of solutions of third-order nonlinear delay differential equation

$$x''' + ax'' + bx' + h(x(t - r(t))) \;=\; p(t, x, x', x'') \tag{1}$$

where a, b are positive constants and functions h, p are continuous in their respective arguments. Also, $0 \leq r(t) \leq \gamma$, $r'(t) \leq \beta$, $0 < \beta < 1$, β and γ are some positive constants, γ will be determined later.

Several authors have investigated the qualitative and other properties of solutions of various forms of equation (1) in [17] where the Lyapunov second method was used. In cases where a, b are nonlinear or continuous, several results have been obtained on the more general form of (1) in one way or another involving the use of generalized Routh-Hurwitz conditions on the nonlinear terms and h in some form or the other, see [3 - 9] and [12], [13], [14], [16], [19], [20]. The Routh-Hurwitz conditions on h when specialized to equation (1) and its various forms, usually take the form $h'(x)$ and $\frac{h(x)}{x}$, $x \neq 0$ to lie in an open Routh-Hurwitz interval $(0, ab)$. Almost all the results mentioned above hold good for h not depending on the deviating arguments or delay being zero, but there are some results [1], [2], [10], [11], [15], [18], [21], [22] who investigated the qualitative behavior of solutions on stability, uniform boundedness and so on except of course the convergence of solutions where h actually depend on some deviating arguments. Analysis of the convergence behaviour of solutions for nonlinear delay differential equations is quite complicated. The difficulties of the convergence of solutions of nonlinear delay system increases depending on the assumptions made on h and the requirement for a complete Lyapunov function.(See also [6]).
Our motivation come from the above mentioned papers. To the best of our knowledge in the relevant literature, till now, the convergence of solutions of (1) and its various forms has not been discussed. We established sufficient conditions for the convergence (when $p \neq 0$) of solutions of (1) which extend and improves some well known results in the literature. Results obtained are not only new but also for the development of more general formulations.

Definition 1 *Any two solutions $x_1(t)$, $x_2(t)$ of (1) are said to converge if*

$$x_1(t) - x_2(t) \to 0, \;\; x_1'(t) - x_2'(t) \to 0, \;\; x_1''(t) - x_2''(t) \to 0 \;\; as \;\; t \to \infty.$$

If the relations above are true of each other(arbitrary) pair of solutions of (1), we shall describe this saying that all solutions of (1) converge.

Now, we will state the stability criteria for the general non-autonomous delay differential system. We consider:

$$\dot{x} = f(t, x), x_t = x(t+\theta) \quad -r \leq \theta \leq 0, t \geq 0, \tag{2}$$

where $f : \mathbf{I} \times C_H \longrightarrow \mathbb{R}^n$ is a continuous mapping,

$$f(t, 0) = 0, C_H := \{\phi \in (C[-r, 0], \mathbb{R}^n) : \|\phi\| \leq H\}$$

and for $H_1 \leq H$, there exists $L(H_1) > 0$, with

$$|f(\phi)| \leq L(H_1) \;\; when \;\; \|\phi\| \leq H_1.$$

Definition 2 ***([2],[19])*** *An element $\psi \in C$ is in the ω-limit set of ϕ, say, $\Omega(\phi)$, if $x(t,0,\phi)$ is defined on $[0,\infty)$ and there is a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, with $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$ where*

$$x_{tn}(\phi) = x(t_n + \theta, 0, \phi) \;\; for \;\; -r \leq \theta \leq 0.$$

Definition 3 ***([2],[19])*** *A set $Q \in C_H$ is an invariant set if for any $\phi \in Q$, the solution of (2), $x(t,0,\phi)$, is defined on $[0,\infty)$ and $x_t(\phi) \in Q$ for $t \in [0,\infty)$.*

Lemma 1 ***([2],[19])*** *An element $\phi \in C_H$ is such that the solution $x_t(\phi)$ of (2) with $x_o(\phi) = \phi$ is defined on $[0,\infty)$ and $\|x_t(\phi)\| \leq H_1 < H$ for $t \in [0,\infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and*

$$dist(x_t(\phi), \Omega(\phi)) \to 0 \;\; as \;\; t \to \infty.$$

Lemma 2 ***([2],[19])*** *Let $V(t,\phi) : I \times C_H \longrightarrow \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(t,\phi) = 0$, and such that:*

(i) *$W_1|\phi(0)| \leq V(t,\phi) \leq W_2\|\phi\|$ where $W_1(r)$, $W_2(r)$ are wedges*

(ii) *$\dot{V}_{(2)}(t,\phi) \leq 0$ for $\phi \leq C_H$.*

Then the zero solution of (2) is uniformly stable. If we define $Z = \{\phi \in C_H : V_{(2)}(t,\phi) = 0\}$, then the zero solution of (2) is asymptotically stable provided that the largest invariant set in Z is $Q = \{0\}$.

Lemma 3 ***([19])*** *Let $V(t,\phi) : \mathbb{R} \times C_{\mathbf{H}} \longrightarrow \mathbb{R}$ be continuous and locally Lipschitz in ϕ. If*

(i) *$W(|x(t)|) \leq V(t,x_t) \leq W_1(|x(t)|) + W_2\big(\int_{t-r(t)}^{t} W_3(|x(s)|)ds\big)$ and*

(ii) *$\dot{V}_{(2)} \leq -W_3((|x(s)|) + M$,*

for some $M > 0$, where $W(r)$, $W_i (i = 1,2,3)$ are wedges, then the solutions of (2) are uniformly bounded and uniformly ultimately bounded for bound $\mathbf{B}$.

Now, we write the equation (1) as the following equivalent system:

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= -az - by - h(x) + \int_{t-r(t)}^{t} h'(x(s))y(s)ds + p(t,x,y,z), \end{aligned} \qquad (3)$$

where $h'(x)$ is continuous for all x, y. It is also assumed that the function $h(x)$ satisfy a Lipschitz condition in x.

2 Statement of result

Theorem 1 *Let $a, b, L, \gamma, \Delta_1$ and δ_o be positive constants, $h(0) = 0$ and suppose that*

(i) *the incremental ratio for h satisfies*

$$\delta_o \le \frac{h(x_1) - h(x_2)}{x_1 - x_2} \le kab, \quad x_1 \ne x_2 \ \text{with} \ k < 1;$$

(ii) $|h'(x)| \le L$

(iii) γ *satisfies*

$$\gamma < \min\left\{\frac{\delta_o}{L}; \frac{b(1+a)(1-\beta)}{L[(1+a)+b+(1+a)+(\frac{2}{a}+1)]}; \frac{1}{L(\frac{2}{a}+1)}\right\}$$

and

(iv) $p(t,x,y,z)$ *satisfies*

$$|p(t,x,y,z)| \le \phi(t)\{|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\},$$

holds for for $t, x_1, y_1, z_1, x_2, y_2$ and z_2.
Then, there exists a constant δ_1 such that any two solutions $x_1(t), x_2(t)$ of (1) necessarily converge if

$$\int_0^t \phi^\alpha(s)ds \le \delta_1 t$$

for some α in the range $1 \le \alpha \le 2$ and the solutions of (1) satisfying

$$[x^2 + \dot{x}^2 + \ddot{x}^2] \le \Delta_1.$$

3 The function V

Our main tool in the proof of the Theorem 1 will be the following scalar function defined as

$$\begin{aligned} 2V(x_t, y_t, z_t) &= \frac{1}{2}b^2(x_1-x_2)^2 + \left(a + \frac{1}{2}b + \frac{2b}{a} + a^2\right)(y_1-y_2)^2 + \left(\frac{2}{a}+1\right)(z_1-z_2)^2 \\ &+ ab(x_1-x_2)(y_1-y_2) + 2(a+1)(y_1-y_2)(z_1-z_2) \\ &+ b(x_1-x_2)(z_1-z_2) + \lambda\int_{-r(t)}^{0}\int_{t+s}^{t}[(y_1-y_2)(\theta)]^2 d\theta ds, \end{aligned} \tag{4}$$

where λ positive constant which will be determined later.

The following result is immediate from (4).

Lemma 4 *Suppose conditions of Theorem 1 hold, then there exists positive constants δ_2, δ_3 such that*

$$\delta_2\Big((x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2\Big) \le V(x_t, y_t, z_t) \le \delta_3\Big((x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2\Big) \tag{5}$$

Proof: (4) can be arranged as

$$\begin{aligned} 2V(x_t, y_t, z_t) &= \frac{1}{4}b^2(x_1-x_2)^2 + \left(\frac{1}{2}b + \frac{2b}{a}\right)(y_1-y_2)^2 + \frac{1}{a}(z_1-z_2)^2 \\ &+ a[(y_1-y_2) + a^{-1}(z_1-z_2)]^2 + [(z_1-z_2) + a(y_1-y_2) + \frac{1}{2}b(x_1-x_2)]^2 \\ &+ \lambda\int_{-r(t)}^{0}\int_{t+s}^{t}[(y_1-y_2)(\theta)]^2 d\theta ds. \end{aligned}$$

It follows that

$$2V(x_t, y_t, z_t) \ge \frac{1}{4}b^2(x_1-x_2)^2 + \left(\frac{1}{2}b + \frac{2b}{a}\right)(y_1-y_2)^2 + a[(y_1-y_2) + a^{-1}(z_1-z_2)]^2,$$

since $a > 0$, $b > 0$ by Theorem 1 and the integral $\lambda\int_{-r(t)}^{0}\int_{t+s}^{t}[(y_1-y_2)(\theta)]^2 d\theta ds$ is non-negative.
So that

$$2V(x_t, y_t, z_t) \ge \xi_1\Big((x_1-x_2)^2 + (y_1-y_2)^2\Big) + a[(y_1-y_2) + a^{-1}(z_1-z_2)]^2,$$

where $\xi_1 = \min\left\{\frac{1}{4}b^2, (\frac{1}{2}b + \frac{2b}{a})\right\}$.
Thus, it is evident from the terms contained in the above inequality that exists a constant $\delta_2 > 0$ small enough such that

$$V(x_t, y_t, z_t) \geq \delta_2\bigg((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\bigg).$$

To prove the right side of inequality (5), by the assumptions of Theorem 1 and using the fact that

$$2|x_1 - x_2||y_1 - y_2| \leq (x_1 - x_2)^2 + (y_1 - y_2)^2$$

yields from $2V$, term by term

$$|(x_1 - x_2)(y_1 - y_2)| \leq |x_1 - x_2||y_1 - y_2| \leq (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$|(y_1 - y_2)(z_1 - z_2)| \leq |y_1 - y_2||z_1 - z_2| \leq (y_1 - y_2)^2 + (z_1 - z_2)^2$$

$$|(x_1 - x_2)(z_1 - z_2)| \leq |x_1 - x_2||z_1 - z_2| \leq (x_1 - x_2)^2 + (z_1 - z_2)^2$$

and

$$\begin{aligned}\lambda\int_{-r(t)}^{0}\int_{t+s}^{t}[(y_1 - y_2)(\theta)]^2 d\theta ds &= \frac{1}{2}\lambda r^2(t)(y_1 - y_2)^2\\ &\leq \frac{1}{2}\lambda\gamma^2(y_1 - y_2)^2.\end{aligned}$$

$$\begin{aligned}2V(x_t, y_t, z_t) &\leq \frac{1}{2}b^2(x_1 - x_2)^2 + \left(a + \frac{1}{2}b + \frac{2b}{a} + a^2\right)(y_1 - y_2)^2 + \left(\frac{2}{a} + 1\right)(z_1 - z_2)^2\\ &+ \frac{1}{2}ab(x_1 - x_2)^2 + \frac{1}{2}ab(y_1 - y_2)^2 + (a+1)(y_1 - y_2)^2 + (a+1)(z_1 - z_2)^2\\ &+ \frac{1}{2}b(x_1 - x_2)^2 + \frac{1}{2}b(z_1 - z_2)^2 + \frac{1}{2}\lambda\gamma^2(y_1 - y_2)^2.\\ &= \left(\frac{1}{2}b^2 + (a+1) + \frac{1}{2}b\right)(x_1 - x_2)^2\\ &+ \left(a + \frac{1}{2}b + \frac{2b}{a} + a^2 + \frac{1}{2}ab + (a+1) + \frac{1}{2}\lambda\gamma^2\right)(y_1 - y_2)^2\\ &+ \left((\frac{2}{a} + 1) + (a+1) + \frac{1}{2}b\right)(z_1 - z_2)^2.\\ &\leq \xi_2\bigg((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\bigg),\end{aligned}$$

where $\xi_2 = \max\left\{(\frac{1}{2}b^2+(a+1)+\frac{1}{2}b), (a+\frac{1}{2}b+\frac{2b}{a}+a^2+\frac{1}{2}ab+(a+1)+\frac{1}{2}\lambda\gamma^2), ((\frac{2}{a}+1)+(a+1)+\frac{1}{2}b)\right\}$.
If we choose a positive constant δ_3, then we have

$$V(x_t, y_t, z_t) \leq \delta_3\bigg((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\bigg).$$

Thus, (5) of Lemma 4 is established where δ_2, δ_3 are finite constants.

Consider the function

$$W(t) = V\bigg((x_1(t)-x_2(t), y_1(t)-y_2(t), z_1(t)-z_2(t)\bigg),$$

where V is the function defined in (4). Then, by (5) we have positive constants δ_4 *and* δ_5 such that

$$\delta_4 S(t) \leq W(t) \leq \delta_5 S(t) \tag{6}$$

and

$$S(t) = \bigg\{|x_1(t)-x_2(t)|^2+|y_1(t)-y_2(t)|^2+|z_1(t)-z_2(t)|^2\bigg\}.$$

The time derivative of (4) along the system (3) defined as

$$\frac{dV(x_t, y_t, z_t)}{dt} = \frac{\partial V}{\partial x}\frac{dx_t}{dt}+\frac{\partial V}{\partial y}\frac{dy_t}{dt}+\frac{\partial V}{\partial z}\frac{dz_t}{dt},$$

hence we prove the following result.

Lemma 5 *Let the hypotheses (i)-(iv) of Theorem 1 hold. Then, there exists positive constants δ_6, δ_7 such that*

$$\frac{dW(t)}{dt} \leq -\delta_6 S(t) + \delta_7 S^{\frac{1}{2}}(t)|\theta|,$$

where $\theta = p(t, x_1, y_1, z_1) - p(t, x_2, y_2, z_2)$.

Proof: Thus, from (4) and (3), we have

$$\frac{dW(t)}{dt} = -U_1 + U_2 + U_3,$$

where

$$U_1 = \frac{1}{2}b(x_1 - x_2)(h(x_1) - h(x_2)) + (a+1)(y_1 - y_2)(h(x_1) - h(x_2)) + \frac{b}{2}(1+a)(y_1 - y_2)^2 + \left(\frac{2}{a} + 1\right)(z_1 - z_2)(h(x_1) - h(x_2)),$$

$$U_2 = \left[\frac{1}{2}b(x_1 - x_2) + (a+1)(y_1 - y_2) + \left(\frac{2}{a} + 1\right)(z_1 - z_2)\right]\int_{t-r(t)}^{t} h'(x_1 - x_2)(s), (y_1 - y_2)(s)ds + \lambda r(t)(y_1 - y_2)^2 - \lambda \int_{t-r(t)}^{t}(y_1 - y_2)^2(s)ds + \lambda r'(t)\int_{t-r(t)}^{t}(y_1 - y_2)^2(s)ds$$

and

$$U_3 = \left(b(x_1 - x_2) + (a+1)(y_1 - y_2) + \left(\frac{2}{a} + 1\right)(z_1 - z_2)\right)|\theta|.$$

Let

$$U_1 = U_{11} + U_{12} + U_{13},$$

where

$$U_{11} = \tfrac{1}{4}b(x_1 - x_2)^2\left\{\tfrac{h(x_1)-h(x_2)}{x_1-x_2}\right\} + \tfrac{b}{2}(1+a)(y_1 - y_2)^2 + \tfrac{1}{2}(z_1 - z_2)^2.$$

By the hypotheses of Theorem 1, we have that

$$U_{11} \geq \frac{1}{4}b\delta_o(x_1 - x_2)^2 + \frac{b}{2}(1+a)(y_1 - y_2)^2 + \frac{1}{2}(z_1 - z_2)^2,$$

$$U_{12} = \tfrac{1}{8}b(x_1 - x_2)^2\left\{\tfrac{h(x_1)-h(x_2)}{x_1-x_2}\right\} + (1+a)(y_1 - y_2)(h(x_1) - h(x_2)) + \tfrac{1}{2}b(y_1 - y_2)^2$$

and

$$U_{13} = \frac{1}{8}b(x_1 - x_2)^2\left\{\frac{h(x_1) - h(x_2)}{x_1 - x_2}\right\} + \left(\frac{2}{a} + 1\right)(z_1 - z_2)(h(x_1) - h(x_2)) + \frac{1}{2}(z_1 - z_2)^2.$$

Next, we give estimates for

$$
\begin{aligned}
(1+a)(y_1-y_2)(h(x_1)-h(x_2)) &= \left\{(1+a)^{\frac{1}{2}}(y_1-y_2)+2^{-1}(1+a)^{\frac{1}{2}}(h(x_1)-h(x_2)\right\}^2 \\
&- (1+a)(y_1-y_2)^2-\frac{1}{4}(1+a)(h(x_1)-h(x_2))^2
\end{aligned}
$$

Thus,

$$
\begin{aligned}
U_{12} &= \left\{(1+a)^{\frac{1}{2}}(y_1-y_2)+2^{-1}(1+a)^{\frac{1}{2}}(h(x_1)-h(x_2))\right\}^2 \\
&+ \frac{1}{8}b(x_1-x_2)^2\left\{\frac{h(x_1)-h(x_2)}{x_1-x_2}\right\}-\frac{1}{4}(1+a)(h(x_1)-h(x_2))^2 \\
&+ \left(\frac{1}{2}b-(1+a)\right)(y_1-y_2)^2.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\frac{1}{8}b(x_1-x_2)^2\left\{\frac{h(x_1)-h(x_2)}{x_1-x_2}\right\}-\frac{1}{4}(1+a)(h(x_1)-h(x_2))^2 \\
&\qquad = \frac{1}{4}\frac{h(x_1)-h(x_2)}{x_1-x_2}\left\{\frac{1}{2}b-(1+a)\frac{h(x_1)-h(x_2)}{x_1-x_2}\right\}(x_1-x_2)^2
\end{aligned}
$$

and if

$$
\frac{h(x_1)-h(x_2)}{x_1-x_2} \le kab \le \frac{b}{2(1+a)},
$$

with

$$
k=\min\left\{\frac{1}{2a(1+a)};\frac{1}{2a(2a^{-1}+1)}\right\}<1,
$$

then,

$$
U_{12}\ge 0.
$$

Similarly,

$$
\begin{aligned}
\left(\frac{2}{a}+1\right)(z_1-z_2)(h(x_1)-h(x_2)) &= \left\{\left(\frac{2}{a}+1\right)^{\frac{1}{2}}(z_1-z_2)\right. \\
&+ \left. 2^{-1}\left(\frac{2}{a}+1\right)^{\frac{1}{2}}(h(x_1)-h(x_2))\right\}^2 \\
&- \left(\frac{2}{a}+1\right)(z_1-z_2)^2 \\
&- \frac{1}{4}\left(\frac{2}{a}+1\right)(h(x_1)-h(x_2))^2.
\end{aligned}
$$

Thus,

$$\begin{aligned} U_{13} &= \left\{ \left(\frac{2}{a}+1\right)^{\frac{1}{2}}(z_1-z_2) + 2^{-1}\left(\frac{2}{a}+1\right)^{\frac{1}{2}}(h(x_1)-h(x_2)) \right\}^2 \\ &+ \frac{1}{8}b(x_1-x_2)^2\left\{\frac{h(x_1)-h(x_2)}{x_1-x_2}\right\} - \frac{1}{4}\left(\frac{2}{a}+1\right)(h(x_1)-h(x_2))^2 \\ &+ \left(\frac{1}{2} - \left(\frac{2}{a}+1\right)\right)(z_1-z_2)^2. \end{aligned}$$

Since

$$\begin{aligned} &\frac{1}{8}b(x_1-x_2)^2\left\{\frac{h(x_1)-h(x_2)}{x_1-x_2}\right\} - \frac{1}{4}\left(\frac{2}{a}+1\right)(h(x_1)-h(x_2))^2 \\ &\qquad = \frac{1}{4}\frac{h(x_1)-h(x_2)}{x_1-x_2}\left\{\frac{1}{2} - \left(\frac{2}{a}+1\right)\frac{h(x_1)-h(x_2)}{x_1-x_2}\right\}(x_1-x_2)^2 \end{aligned}$$

and if

$$\frac{h(x_1)-h(x_2)}{x_1-x_2} \le kab \le \frac{b}{2(2a^{-1}+1)},$$

with

$$k = \min\left\{\frac{1}{2a(1+a)}; \frac{1}{2a(2a^{-1}+1)}\right\} < 1,$$

then,

$$U_{13} \ge 0.$$

Hence,

$$U_1 \ge \frac{1}{4}b\delta_o(x_1-x_2)^2 + \frac{b}{2}(1+a)(y_1-y_2)^2 + \frac{1}{2}(z_1-z_2)^2.$$

In U_2, we give estimates for the following and using the fact that $2uv = u^2+v^2$,

$$\begin{aligned} &\frac{1}{2}b(x_1-x_2)\int_{t-r(t)}^{t} h'(x_1-x_2)(s)(y_1-y_2)(s)ds \\ &\qquad \le \frac{1}{4}Lbr(t)(x_1-x_2)^2 + \frac{1}{2}bL\int_{t-r(t)}^{t}(y_1-y_2)^2(s)ds, \end{aligned}$$

$$(1+a)(y_1-y_2)\int_{t-r(t)}^{t} h'(x_1-x_2)(s)(y_1-y_2)(s)ds$$
$$\leq \frac{1}{2}L(1+a)r(t)(y_1-y_2)^2+\frac{1}{2}(1+a)L\int_{t-r(t)}^{t}(y_1-y_2)^2(s)ds,$$

and

$$(\frac{2}{a}+1)(z_1-z_2)\int_{t-r(t)}^{t} h'(x_1-x_2)(s)(y_1-y_2)(s)ds$$
$$\leq \frac{1}{2}(\frac{2}{a}+1)r(t)(z_1-z_2)^2+\frac{1}{2}(\frac{2}{a}+1)L\int_{t-r(t)}^{t}(y_1-y_2)^2(s)ds.$$

Thus,

$$\begin{aligned} U_2 &= \frac{1}{4}bLr(t)(x_1-x_2)^2+\frac{1}{2}((1+a)L+2\lambda)r(t)(y_1-y_2)^2 \\ &+ \frac{1}{2}L\left(\frac{2}{a}+1\right)r(t)(z_1-z_2)^2 \\ &+ \left\{\frac{1}{2}L(b+(1+a)+\left(\frac{2}{a}+1\right)-\lambda(1-r'(t))\right\}\times\int_{t-r(t)}^{t}(y_1-y_2)^2(s)ds. \end{aligned}$$

Using $r(t)$ and $r'(t)$, we obtain

$$\begin{aligned} U_2 &= \frac{1}{4}bL\gamma(x_1-x_2)^2+\frac{1}{2}((1+a)L+2\lambda)\gamma(y_1-y_2)^2+\frac{1}{2}L\left(\frac{2}{a}+1\right)\gamma(z_1-z_2)^2 \\ &+ \left\{\frac{1}{2}L(b+(1+a)+\left(\frac{2}{a}+1\right)-\lambda(1-\beta)\right\}\times\int_{t-r(t)}^{t}(y_1-y_2)^2(s)ds. \end{aligned}$$

If we choose,

$$\lambda=\frac{L(b+(1+a)+(\frac{2}{a}+1))}{2(1-\beta)},$$

$$\begin{aligned} U_2 &\geq \frac{1}{4}bL\gamma(x_1-x_2)^2+\frac{1}{2}L\left((1+a)+\frac{L(b+(1+a)+(\frac{2}{a}+1)}{2(1-\beta)}\right)\gamma(y_1-y_2)^2 \\ &+ \frac{1}{2}\gamma L(\frac{2}{a}+1)(z_1-z_2)^2 \end{aligned}$$

and

$$U_3 \;\geq\; \left\{b(x_1-x_2)+(1+a)(y_1-y_2)+\left(\frac{2}{a}+1\right)(z_1-z_2)\right\}|\theta|.$$

Combining all the estimates for U_1, U_2 and U_3, we obtain

$$\begin{aligned}\dot{W}(t) \leq & - \frac{1}{4}(b\delta_o - \gamma bL)(x_1 - x_2)^2 \\ & - \frac{1}{2}\Big\{b(1+a) - \gamma L\Big(\frac{(1+a) + (b + (1+a) + (\frac{2}{a}+1)}{(1-\beta)}\Big)\Big\}(y_1 - y_2)^2 \\ & - \frac{1}{2}\Big\{1 - \gamma L\Big(\frac{2}{a}+1\Big)\Big\}(z_1 - z_2)^2 \\ & + \Big\{b(x_1 - x_2) + (1+a)(y_1 - y_2) + \Big(\frac{2}{a}+1\Big)(z_1 - z_2)\Big\}|\theta|.\end{aligned}$$

Choosing

$$\gamma < \min\left\{\frac{\delta_o}{L}; \frac{b(1+a)(1-\beta)}{L\Big[(1+a) + b + (1+a) + (\frac{2}{a}+1)\Big]}; \frac{1}{L(\frac{2}{a}+1)}\right\}.$$

We have that

$$\begin{aligned}\dot{W}(t) \leq & - \delta_8\big((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\big) \\ & + \delta_9\big((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\big)^{\frac{1}{2}}|\theta|,\end{aligned}$$

where δ_8 and δ_9 are finite positive constants.

Using (6), it follows that

$$\frac{dW(t)}{dt} \leq -\delta_8 S(t) + \delta_9 S^{\frac{1}{2}}|\theta|. \tag{7}$$

The conditions of Lemma 2 and Lemma 3 is immediate if provided γ satisfy (iv) of Theorem 1.
This completes the proof of Lemma 5.

4 Proof of Theorem 1

Let α be any constant such that $1 \leq \alpha \leq 2$ and set $\upsilon = 1 - \frac{1}{2}\alpha$, so that $0 \leq \upsilon \leq \frac{1}{2}$.
Then, (7) becomes

$$\frac{dW(t)}{dt} + \delta_8 S(t) \leq \delta_9 S^{\upsilon} W^* \tag{8}$$

and $W^* = S^{(\frac{1}{2}-v)}(|\theta| - \delta_8\delta_8^{-1}S^{\frac{1}{2}}(t))$.

Considering these two cases:

(i) $|\theta| \leq \delta_8\delta_9^{-1}S^{\frac{1}{2}}$ and

(ii) $|\theta| > \delta_8\delta_9^{-1}S^{\frac{1}{2}}$

separately, we can find that in either case, there exists some constants $\delta_{10} > 0$ such that

$$W^* \leq \delta_{10}|\theta|^{2(1-v)}.$$

By using (iv) of Theorem 1, inequality (8) becomes

$$\frac{dW(t)}{dt} + \delta_8 S(t) \leq \delta_{11} S^v \phi^{(2(1-v)} S^{(1-v)}$$

where $\delta_{11} \geq 2\delta_9\delta_{10}$. On using (6) on W, it follows that

$$\frac{dW(t)}{dt} + \big(\delta_{12} - \delta_{13}\big)W(t) \leq 0, \tag{9}$$

where δ_{12} and δ_{13} are positive constants.
Integrating (9) from t_o to t, $(t \geq t_o)$, we get

$$W(t) \leq W(t_o)\exp\Big\{ - \delta_{12}(t_2 - t_1) + \delta_{13}\int_{t_o}^{t} \phi^{\alpha}(s)ds\Big\}.$$

If

$$\int_{t_o}^{t} \phi^{\alpha}(s)ds < \delta_1(t - t_o),$$

where $\delta_1 = \delta_{12}\delta_{13}^{-1}$. Then, the exponential index remains negative for all $(t - t_o) \geq 0$. As $(t_2 - t_1) \to \infty$, we have that

$$W(t) \leq 0 \quad for \;\; any \;\; t.$$

Again, by (6), we have that

$$S(t) \to 0.$$

Thus,

$$x_1(t) - x_2(t) \to 0, \;\; y_1(t) - y_2(t) \to 0, \;\; z_1(t) - z_2(t) \to 0 \;\; as \;\; t \to \infty$$

or in (1) as,

$$x_1(t) - x_2(t) \to 0, \;\; x_1'(t) - x_2'(t) \to 0, \;\; x_1''(t) - x_2''(t) \to 0 \;\; as \;\; t \to \infty.$$

Example 4.1 *Consider equation (1) in the form*

$$x''' + 3x'' + x' + [x(t - r(t))]^2 \;=\; e^{-t} \tag{10}$$

Comparing (1) with (10), it is obvious that $a = 3$, $b = 1$ *and* $h(x(t - r(t)) = [x(t - r(t))]^2$, *where* e^{-t} *is a bounded continuous function of* t *only on* $[0, \infty)$.
With the earlier notations, gives

$$k = \min\left\{\frac{1}{24}; \frac{3}{10}\right\} < 1,$$

thus $k = \frac{1}{24}$, *we have that*

$$\delta_o \leq \frac{h(x_1) - h(x_2)}{x_1 - x_2} \leq \frac{1}{8}.$$

We choose $\delta_o = \frac{2}{25}$.
If we take $r(t) = \frac{1}{45+t^2}$, *then* $0 \leq \frac{1}{45+t^2} \leq \gamma$ *and that* $r'(t) = \frac{-2t}{45+t^2} \leq \beta$, $0 < \beta < 1$.
$|h'(x)| = |2||x(t)||y(t)||1 - r'(t)| \leq 2$, *i.e* $L = 2$.
If we choose $\beta = \frac{1}{2}$, *we must have that*

$$\gamma < \min\left\{\frac{1}{25}; \frac{2}{32}, \frac{1}{5}\right\},$$

$$\gamma < \frac{1}{25}$$

that is,

$$\gamma = \frac{1}{40}.$$

Hence, we can choose

$$r(t) = \frac{1}{100}.$$

Thus, all the conditions of Theorem 1 are satisfied and so for every solution of (10) is such that

$$x_1(t) - x_2(t) \to 0, \;\; x_1'(t) - x_2'(t) \to 0, \;\; x_1''(t) - x_2''(t) \to 0 \;\; as \;\; t \to \infty.$$

The plot of $x(t), x'(t), x''(t)$ *or equivalently* $x(t)$, $y(t)$ *and* $z(t)$ *of equation (10) which are the solutions characterizing the system (10) is shown in Fig. 1, Fig. 2 respectively above and Fig.3 below. It is very clear from Fig. 1, Fig. 2 and Fig. 3 that all conditions of Theorem 1 are satisfied along the graphs and* $x(t), x'(t), x''(t)$ *of (10) converges to zero as* $t \to \infty$.

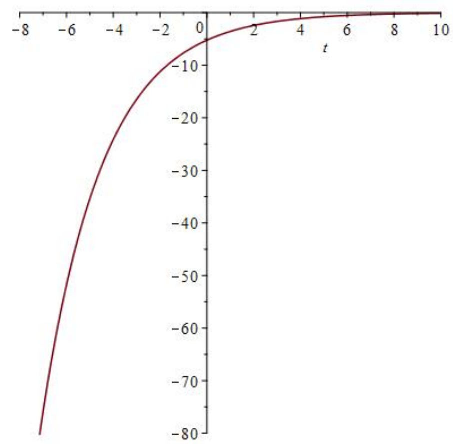

Figure 1: *The plot of* $x(t)$, ***r(t)=0.01***

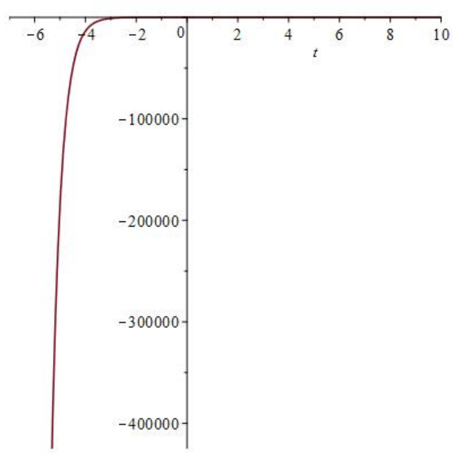

Figure 2: *The plot of* $x'(t)$, ***r(t)=0.01***

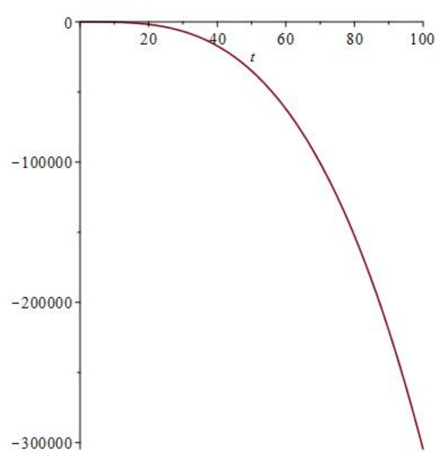

Figure 3: *The plot of* $x''(t)$, ***r(t)=0.01***

Conclusion So, we can formulate the CONVERGENCE CRITERIA OF NONLINEAR DELAY SYSTEM (1): the solutions of the third order nonlinear system are convergent according to Lyapunov's theory if the conditions of Theorem 1 hold.

Analysis of nonlinear systems literary shows that Lyapunov's theory in convergence of solutions of nonlinear delay differential equations is rarely scarce. The second Lyapunov's method allows to predict the convergence of solutions of nonlinear delay physical system.

References

[1] Adams, D.O., Olutimo, A.L.; *Some results on the boundedness of solutions of a certain third order non-autonomous differential equations with delay.* Advanced Studies in Contemporary Mathematics, **29(2)**, 237–249, 2019. http://dx.doi.org/10.107777/ascm2019.29.2.237.

[2] Afuwape, A.U., Omeike, M.O.; *On the stability and boundedness of solutions of a kind of third order delay differential equations.* Appl. Math. Comput., **200**, 444–451, 2008.

[3] Afuwape, A.U.; *Convergence of solutions of certain non-homogeneous third order differential equations.* Kragujevac J. Math., **31**, 1–12, 2008.

[4] Andres, J.; *Boundedness results of solutions to the equation* $\dddot{x} + a\ddot{x} + g(x)\dot{x} + h(x) = p(t)$ *without the hypothesis* $h(x)sgn(x) \geq 0$ *for* $|x| > R$. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. math. Natur, **80(8)**, 533–539, 1987.

[5] Bereketoglu, H., Gyori, I.; *On the boundedness of the solutions of a third-order nonlinear differential equation.* Dynam. Systems Appl., **6(2)**, 263–270, 1977.

[6] Chukwu, E.N.; *On the boundedness of solutions of third order differential equations.* Ann. Mat. Pura Appl., **155(4)**, 123–149, 1975.

[7] Ezeilo, J.O.C.; *Further result on the existence of periodic solutions of the equation* $\dddot{x} + \psi(\dot{x})\ddot{x} + \phi(x)\dot{x} + v(x, \dot{x}, \ddot{x}) = p(t, x, \dot{x}, \ddot{x})$ *with a bound* v. Atti.Accad.Naz.Lincei Rend.Cl.Sci.Fis.Mat.Natur., **55**, 51–57, 1978.

[8] Ezeilo, J.O.C.; *Actes Conference Internat. :New properties of the equation* $\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x})$ *for certain special values of the incrementary ratio* $y^{-1}h(x + y) - h(x)$. Quations differentielles et functionelles non

linaires. "Equa-Dff. 73", Brussels/Louvain-la-Neuve, 447–462, Herman, Paris, 1973.

[9] Li, Q.; *On the construction of globally asymptotically stable Liapunov's functions of a nonlinear third order systems.* Ann. Differential Equations, **7(1)**, 39–51, 1991.

[10] Olutimo, A.L.; *On the stability and ultimate boundedness of solutions of certain third order non-autonomous delay differential equations:* In Proceedings of the 14th International Conference: Dynamical Systems-Theory and Applications, Lodz, Poland. Vibration, control and stability of dynamical systems. ISBN 978-83-935312-5-7. (Department of Automation, Biomechanics and Mechatronics), 389–400, 2018.

[11] Olutimo, A.L., Adams, D.O.; *On the stability and boundedness of solutions of certain non-autonomous delay differential equation of third order.* Applied Mathematics, **7**, 457–467, 2016.

[12] Omeike, M.O.; *New result in the ultimate boundedness of solutions of a third-order nonlinear ordinary differential equation.* J. of Inequalities in Pure and Applied mathematics, **9**, Article 15.8, 2008.

[13] Omeike, M.O.; *Further results for the solutions of certain third-order differential equations.* Nonlinear Analysis. **67**, 3394–3400, 2007.

[14] Omeike, M.O., Olutimo, A.L., Oyetunde, O.O.; *The boundedness of solutions of certain nonlinear third order ordinary differential equations.* J. Nig. Math. Soc., **31**, 49-54, 2012.

[15] Oudjedi, L.D., Remili, M.; *Boundedness and stability in third order nonlinear vector differential equations with bounded delay.* African mathematical Unimand Springer- Verlag GmbH Deutchland cin Teil Von Springer nature 2018.

[16] Qian, C.; *On global stability of third order nonlinear differential equations.* Nonlinear Anal., Ser. A: Theory methods, **42(4)**, 651–661, 2000.

[17] Reissig, R., Sansone, G., Conti, R.; *Nonlinear differential equations of higher order.* Translated from the German, Noordhoff International Publishing, Leyden, 1974.

[18] Sadek, A.I.; *On the stability of solutions of some non-autonomous delay differential equations of third order.* Asymptotic Analysis, 1-7, **43**, 2005.

[19] Swick, K.E.; *Boundedness and stability for a nonlinear third order differential equation.* Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. mat. Natur., **56(6)**, 859-865, 1974.

[20] Tunc, C.; *The boundedness of solutions of nonlinear third order differential equations.* Nonlinear Dynamics and Systems Theory, **10**, 97-102, 2010.

[21] Zhu, Y.F.; *On stability, boundedness and existence of periodic solution of a kind of third order nonlinear delay differential system.* Ann Differential Equation, **8**, 249-259, 1992.

[22] Yao, H., Meng, W.; *On the stability of solutions of certain non-linear third order delay differential equations.* International Journal of non-linear science, **6**, 230-237, 2008.