ABSTRACTS OF DIPLOMA WORKS AT THE MATHEMATICAL ANALYSIS CHAIR, DEPT. OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY 1999

Separation of singularities of harmonic functions and embedding theorems

Abramenko K.
adviser: Victor Havin

This work is a continuation of a recent work [5] by Victor Havin, the thesis advisor of the author. The paper [5] concerns analytic functions of the complex plane, and we deal with similar problems for harmonic functions in $\mathbb{R}^n, n > 1$.

For an open set O in $\mathbb{R}^n$, let $H(O)$ be some class of harmonic functions in O. The choice of a concrete class will be one of free pararmeters of our problem.

The problem of separation of singularities can be formulated as follows. Let G_1, G_2 be open subsets of F, and let G be an open subset of $G_1 \cap G_2$. Some additional conditions are imposed on G. For instance, G may be a component of $G_1 \cap G_2$ or it is assumed that the boundary of $G_1 \cap G_2$ is contained in the boundary of G.

When for any function $f \in H(G)$ there exist functions f_1 and f_2 in $H(G_1)$ and $H(G_2)$, respectively, such that $f = f_1 + f_2$ in G?

The boundary of G for interesting classes of functions is the singularity set of f_1 and f_2. The representation $G = G_1 \cap G_2$ decomposes the boundary of G into two parts in a natural way, or, more precesely, induces the covering of ∂G by two its subsets $\overline{G}_1 \cap \partial G$ and $\overline{G}_2 \cap \partial G$. This is why this problem is called a problem of singularity separation.

For the simplest choice of H, when H is the class of all harmonic functions in the domain under consideration, the answer to our problem is "yes".

This result is rather old. For a special position of domains and for analytic functions in the complex plane this has been proved by Poincaré [6] and for domains in general position by Aronshain in 1936 [3]. In 1958, V. P. Havin independently found a very simple and clear proof by means of functional analysis, which was based on the Hahn–Banach theorem.

The author has extended this proof to harmonic functions in space of arbitrary dimension.

The narrower is the functional class H, the more restrictions are imposed on the summands f_i and the more difficult is to find them (or it turns out to be easier to prove their non-existence). We consider only one special choice of H, namely, the class of bounded harmonic functions.

The existence of such "bounded" singularity separation turns out to depend not only on the domains G_1 and G_2, but also on the choice of the subset G of their intersection. For instance, if G_1 and G_2 are the complements of two halves of the segment $[-\frac{1}{2}, \frac{1}{2}]$ and G is one of the half-planes, then the "bounded" singularity separation is possible for any bounded harmonic funciton in the half-plane. This follows from the main positive sesult of the work (see below). But if G is the entire complement of the segment, then the answer is negative. An explicit counterexample is constructed:

$$f(z) := \int_{-1/2}^{1/2} \frac{\log|z-t|}{t \cdot \log^2|t|}\, dt.$$

Here we have the same effect as in relation with the class H: the extension of G with the same G_1 and G_2 means the strengthening of the condition $f = f_1 + f_2$ on f_1 and f_2. The wider G is, the more difficult is to find f_1 and f_2.

We were able to prove the positive result for the following configuration of domains: G is a "good" bounded domain (it can be any Lipshchitz domain, in particular), $K_1 \cup K_2 = \partial G$ is an arbitrary covering of ∂G by closed sets, and $G_i = \mathbb{R}^n \setminus K_i$.

The idea of the proof is very simple: to represent f by an integral over ∂G, to split the set of integration into two parts, so one obtains f_1 and f_2. Then, obviously, f_1 and f_2 are harmonic and $f_1 + f_2 = f$. One has only to prove boundedness of f_1 and f_2.

In [5], Havin has used the representation of analytic functions by the Cauchy integral. For harmonic functions, it turned out to be more convenient to use the double lay potential representation.

The theory of the double lay potential is well-developed for domains with $C^{1+\alpha}$ boundary. This restriction is not not good for us: G appears as an intersection of two domains, and need not in general to have a $C^{1+\alpha}$ boundary. This is not so even for the complements of two disks.

Rather recently, a potential theory for domains with irregular boundary was developed. For instance, in 1969 in the work by Burago and Maz'ya [1]. Their requirements on the domain mean, in fact, the following.

(1) The existence ot the double lay potential; (2) This operator acts from $C(\partial G)$ to $C(\partial G)$; (3) The Fredholm radius of this operator is greater than one.

These conditions are suitable for us, but the condition on the function (the continuity on $\overline{G}$) is too sharp.

On the other hand, more recently (in particular, by Dahlberg [4] and Verchota [7]) the existence of the representation have been proved for Lipschitz domains for functions in L^p, $p < \infty$. The potential density is also in L^p. We cannot apply this result, because for such density it is difficult to prove the boundedness of the resulting functions f_i. It is natural to expect that a function, bounded inside the domain is represented by a double lay potential with bounded density.

We were unable to find a proof of this fact in the literature. We have proved this, following a classical scheme. Most of the techniques were borrowed in [1], [4], [7], and the resting facts are proved.

So we obtained a sufficient condition on the domain. Its necessity is an open question.

References

[1]. Yu. D. Burago, V.G. Maz'ya, "Some questions of the potential theory and the function theory for odmains with irregular boundary". Zapiski Nauchn. Semim. LOMI, 1967.

[2]. V. P. Havin, "On the singularity separation for analytic functions", Doklady AN SSSR, vol. 121, No. 2, p. 239, 1958.

[3]. N.Aronszajn. "Sur les decompositions des fonctions analutiques uniformes et sur leur applications". Acta Math. 65(1935), 1–156.

[4]. B.E.J. Dahlberg, "Estimates of harmonic measure", Arch. Rat. Mech. Anal. 65 (1977), 275–288.

[5]. V.Havin, A.Nersessyan, "Bounded separation of singularities of analytic functions",to appear in "Proceedings of the Conference dedicated to the memory of B.Ya.Levin", Dec. 1997,Tel Aviv

[6]. H. Poincaré. "Sur les fonctions aux espaces lacunaires". Amer.J.of Math. 14, 1892, 201–221.

[7]. G. Verchota, "Layer potentials and regularity for the Dirichlet problem for Laplace's equation on Lipschitz domains", J. Func. Anal. 59 (1984), 572–611.

Differentiation in de Branges spaces and embedding theorems

A. Baranov
adviser: Victor Havin

Abstract

The conditions of boundedness of differentiation operator in de Branges spaces of entire functions $\mathcal{H}(E)$ are considered. It is proved that the differentiation operator in $\mathcal{H}(E)$ is bounded if $E'/E \in H^\infty(\mathbf{C}^+)$; under certain restrictions on the function E the same condition turns out to be necessary. The convex structure of the set of the measures μ, such that $\int\limits_{\mathbb{R}} \left|\frac{F}{E}\right|^2 d\mu = \|F\|_E^2$ whenever $F \in \mathcal{H}(E)$ (de Branges measures), is studied also. Certain classes of measures, which are extreme points of the set of de Branges measures are described.

Summary

In the present work Hilbert spaces of entire functions $\mathcal{H}(E)$ introduced by L. de Branges [1] are studied. Let E be an entire function satisfying $|E(z)| > |E(\overline{z})|$ whenever z is in the upper half-plane $\mathbf{C}^+$. Then $\mathcal{H}(E)$ will denote the set of all entire functions F such that F/E and F^*/E belong to Hardy class $H^2(\mathbf{C}^+)$ (here $F^*(z) = \overline{F}(\overline{z})$). We begin with relationship between de Branges theory and another classical theme of harmoic analysis — coinvariant spaces of the shift operator in the upper half-plane (spaces K_Θ). If Θ is an inner function, we put $K_\Theta = H^2 \ominus \Theta H^2$. The spaces $\mathcal{H}(E)$ turn out to be a particular case of the spaces K_Θ in the following way:

Theorem 1. Let $\Theta = \frac{E^*}{E}$. Then the mapping $F \mapsto \frac{F}{E}$ is an unitary operator from $\mathcal{H}(E)$ to K_Θ.

In the second part of the work we are concerned with the conditions of boundedness of the differentiation operator in the spaces $\mathcal{H}(E)$. In the case $E(z) = \exp(-iaz)$, $a > 0$ the corresponding space $\mathcal{H}(E)$ coincides with Paley-Wiener space W_a, which consists of all entire functions of exponential type at most a, square integrable on the real axis. It is well known that $F' \in W_a$ and $\|F'\|_2 \le a\|F\|_2$ for all $F \in W_a$. Similar estimates (so-called Bernstein inequalities) for the spaces K_Θ were obtained by K.M. Dyakonov [2]. He has proved that the estimate $\|f'\|_2 \le C\|F\|_2$, where C is a constant, takes place if and only if $\Theta' \in L^\infty(\mathbb{R})$ (in the sense of boundary values).In the present work the following sufficient condition was obtained:

Theorem 2. If $\frac{E}{E} \in H^\infty(\mathbf{C}^+)$ then the differentiation operator $D : F \mapsto F'$ is bounded in $\mathcal{H}(E)$ and $\|D\| \le C \left\| \frac{E'}{E} \right\|_\infty$, where C is an absolute constant.

By $H^\infty(\mathbf{C}^+)$we denote the space of bounded analytic functions in the upper half-plane.

Under the certain restrictions on the function E the condition of the Theorem 2 proves to be necessary. But there is a counterexample, which shows that this condition is not necessary in the general case.

The last part of the work is devoted to de Branges measures that is the measures μ on $\mathbb{R}$ such that $\int\limits_{\mathbb{R}} \left|\frac{F}{E}\right|^2 d\mu = \|F\|_E^2$ for all $F \in \mathcal{H}(E)$. It is shown by L. de Branges that all these measures μ satisfy the equation

$$\Re\left(\frac{E(z) + E^*(z)W(z)}{E(z) - E^*(z)W(z)}\right) = py + \frac{y}{\pi}\int\limits_{\mathbb{R}} \frac{d\mu(t)}{(t-x)^2 + y^2}, \qquad z \in \mathbf{C}^+,$$

for some function $W \in H^\infty(\mathbf{C}^+)$, $\|W\|_\infty \le 1$. We study the convex structure of the set of all such measures in terms of the parameter W. For certain classes of functions W (for example, for finite Blaschke products in the case of Paley-Wiener spaces) it is proved that they generate measures μ, which are extreme points in the set of de Branges measures.

[1] L. De Branges, Hilbert spaces of entire functions. Prentice Hall, Englewood Cliffs (NJ), 1968.

[2] K.M. Dyakonov, Entire functions of exponential type and model subspaces in H^p. - Zapiski Nauchn. Semin. LOMI, v. 199, Leningrad, 1991, p. 81-100.

A nonuniqueness example for the Riesz potentials

Belyaev Dmitry
adviser: Victor Havin

This work continues the study, which was begun by V.P. Havin, who proved that if a function of class $C^{\alpha+\epsilon}$ vanishes on a set of positive measure together with its potential, then it is identically zero. However, it have been unknown whether this result can be generalized to all continuous functions. In this work we answer negatively to this question by constructing an explicit counterexample. We apply the techniques by T. Wolff, which was refined by J. Bourgain, and also some results by A.B. Alexandrov and P.P. Kargaev.

The local spectral multiplicity of contractions
Dmitry Galaktionov

adviser: D. V. Yakubovich

The spectral multiplicity of a linear operator, which is studied in many papers, is one of its invariants. We use a definition of the local spectral multiplicity of an arbitrary, not necessarily normal linear operator, which was given by D. Yakubovich in [1]. The local multiplicity of a linear operator with respect to a measure μ on the plane is introduced as a measurable nonnegative function defined μ-a.e. Let H be a Hilbert space and $T : H \to H$ be a contraction, which means simply that $\|T\| \leq 1$. The nonegative operators $(I - T^*T)^{1/2}$ and $(I - TT^*)^{1/2}$ are called the defect operators. In this work we calculate the local spectral multiplicity of a contraction with defects of finite rank with respect to the Lebesgue measure on the unit circle. The proof of this result uses the Nagy–Foias model of contractions.

[1] D.V. Yakubovich, The local spectral multiplicity of a linear operator with respect to a measure, Zapiski Nauchn. Seminarov POMI vol. 222 (1995), p. 293-306; Issledovaniya po Linejnym Operatoram i Teorii Funkcij 23; transl. in J. of Soviet Mathematics (to appear)

[2] Sz.-Nagy, B. and C. Foias, Harmonic analysis of operators on a Hilbert space North-Holland, Amsterdam, and Akad. Kiadó, Budapest (1970)

About linear means of spherical Fourier series
Yakov Sirotkin

adviser: A. N. Podkorytov

In this work, the behavior of Fourier series of continuous periodic functions of many variables is studied. It is well known that even in the one-dimensional case the Fourier series is not always convergent. Let μ be an auxiliary continuous function. Consider linear means of Fourier series

$$S_R^{\mu}(f, x) = \sum_{k \in \mathbb{Z}^n} \mu\left(\frac{k}{R}\right) c_k(f)\, e^{2\pi i(k,x)},$$

where $c_k(f)$ is the kth Fourier coefficient of the function f. If μ converges to zero at infinity fast enough, $S_R^{\mu}(f, x)$ converges absolutely and we can consider linear operators S_R^{μ} on the space of continuous periodic functions.

It turns out that the convergence of operators S_R^μ to the identical operator is equivalent to two conditions: $\mu(0) = 1$, and the Fourier transform of the function μ is to be integrable.

The case of the most interest is a radial one, where $\mu(x) = l(|x|)$, and integrability of Fourier transform is equivalent to the convergence of the integral

$$H_n(l) = \int_0^\infty \left| \int_0^\infty (tr)^{\frac{n}{2}} l(t) J_{\frac{n}{2}-1}(2\pi r t)\, dt \right| dr.$$

And besides this, the convergence of the integral $H_n(l)$ is equivalent to convergence of a simpler integral

$$F_n(l) = \int_0^\infty \left| \int_0^\infty (tr)^{\frac{n-1}{2}} l(t) \cos\left(2\pi r t - (n-1)\frac{\pi}{4}\right) dt \right| dr.$$