

COMPLEMENTED INVARIANT SUBSPACES AND INTERPOLATION SEQUENCES

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ABSTRACT. It is shown that the invariant subspace of the Bergman space A^p of the unit disc, generated by a finite union of Hardy interpolation sequences, is complemented in A^p .

Let \mathbb{D} be the open unit disc of the complex plane \mathbb{C} , and let $H^p = H^p(\mathbb{D})$ denote the usual Hardy space. For $0 < p < \infty$ the Bergman space A^p consists of analytic functions f on \mathbb{D} such that

$$\|f\|_p^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty,$$

where dA is area measure on \mathbb{C} normalized so that $A(\mathbb{D}) = 1$.

A closed subspace I of A^p is said to be invariant if $zf(z) \in I$ for all $f \in I$.

A sequence $Z = \{a_j\}$ of points in \mathbb{D} is called an A^p zero set if there exists a non-zero function $f \in A^p$ such that f vanishes on Z , counting multiplicities. Every A^p zero set generates an invariant subspace I_Z^p consisting of all functions in A^p that vanish on Z , counting multiplicities.

Recently, Korenblum and Zhu [4] exhibited several classes of *complemented* invariant subspaces I in A^p . By definition, I is complemented in A^p if there exists a bounded projection Q from A^p onto I . In particular, Korenblum and Zhu considered the subspaces I_Z^p for a special class of zero sequences Z . Recall that a sequence $\{a_j\}$ of *distinct* points in the disc is an A^p (H^p) interpolation sequence if for every sequence $\{w_j\} \subset \mathbb{C}$ with

$$\sum (1 - |a_j|^2)^2 |w_j|^p < \infty \quad \left(\sum (1 - |a_j|^2) |w_j|^p < \infty \right)$$

there exists a function $f \in A^p$ ($f \in H^p$) such that $f(a_j) = w_j$ for all j . Remark that every H^p interpolation sequence is an A^p interpolation set (see, e.g., [5]);

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the converse implication does not hold. Further results about the interpolation sequences can be found in the monographs [3] and [2].

Theorem 1. (Korenblum and Zhu [4].) *Suppose that $0 < p < \infty$ and Z is an A^p interpolation sequence. Then the invariant subspace I_Z^p is complemented in A^p .*

In the present paper we consider a different class of zero sets Z (with possible multiplicities).

Theorem 2. *Suppose that $0 < p < \infty$ and Z is a finite union of H^p interpolation sequences. Then the invariant subspace I_Z^p is complemented in A^p .*

Proof. Let $Z = \bigcup_{k=1}^n Z_k$, where Z_k is an H^p interpolation sequence, $k = 1, 2, \dots, n$. Observe that Z_k is an A^p interpolation sequence. Therefore, by Theorem 1, there exists a bounded linear operator (a projection) Q_k from A^p onto $I_{Z_k}^p$ such that $Q_k^2 = Q_k$.

Consider the Blaschke products

$$B_m(z) = \prod_{a_j \in Z(m)} \frac{|a_j|}{a_j} \frac{a_j - z}{1 - z\bar{a}_j}, \quad m = 1, 2, \dots, n,$$

where $Z(m) = \bigcup_{k=1}^m Z_k$ (as usual, if $a_j = 0$ then the quotient $|a_j|/a_j$ is replaced by -1). Let $B_m A^p$ denote the set of all multiples of B_m by functions in A^p . Since $Z(m)$ is a finite union of H^p interpolation sequences, we have

$$(1) \quad B_m A^p = I_{Z(m)}^p;$$

$$(2) \quad \|f/B_m\|_p \leq \text{const}(m) \|f\|_p \text{ for every function } f \in I_{Z(m)}^p$$

(see, e.g., [2], Sections 4.5 and 4.6).

Given $f \in I_{Z(m-1)}^p$, $m = 2, 3, \dots, n$, put

$$T_m f = B_{m-1} Q_m (f/B_{m-1})$$

(cf. [1], where such operators are used in the H^p setting). Finally, define

$$Q = T_n T_{n-1} \dots T_2 Q_1.$$

By (1) and (2), Q is a bounded linear operator from A^p onto $I_{Z(n)}^p = I_Z^p$ such that $Q^2 = Q$. In other words, I_Z^p is complemented in A^p . \square

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